CU-CSSC-91-5　　CENTER FOR SPACE STRUCTURES AND CONTROLS

SECOND-ORDER DISCRETE KALMAN FILTERING EQUATIONS FOR CONTROL-STRUCTURE INTERACTION SIMULATIONS

by

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Second-Order Discrete Kalman Filtering Equations
for
Control-Structure Interaction Simulations[+]

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Abstract

A general form for the first-order representation of the continuous, second-order linear structural dynamics equations is introduced in order to derive a corresponding form of first-order continuous Kalman filtering equations. Time integration of the resulting first-order Kalman filtering equations is carried out via a set of linear multistep integration formulas. It is shown that a judicious combined selection of computational paths and the undetermined matrices introduced in the general form of the first-order linear structural systems leads to a class of second-order discrete Kalman filtering equations involving only symmetric, sparse $N \times N$ solution matrices. The present integration procedure thus overcomes the difficulty in resolving the difference between the time derivative of the estimated displacement vector ($\frac{d}{dt}\hat{x}$) and the estimated velocity vector ($\hat{\dot{x}}$) that are encountered when one attempts first to eliminate ($\hat{\dot{x}}$) in order to form an equivalent set of second-order filtering equations in terms of ($\frac{d}{dt}\hat{x}$). A partitioned solution procedure is then employed to exploit matrix symmetry and sparsity of the original second-order structural systems, thus realizing substantial computational simplicity heretofore thought difficult to achieve.

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Introduction

Current practice in the design, modeling and analysis of flexible large space structures is by and large based on the finite element method and the associated software. The resulting discrete equations of motion for structures, both in terms of physical coordinates and of modal coordinates, are expressed in a second-order form. As a result, the structural engineering community has been investing a considerable amount of research and development resources to develop computer-oriented discrete modeling tools, analysis methods and interface capabilities with design synthesis procedures; all of these exploiting the characteristics of second-order models.

On the other hand, modern linear control theory has its roots firmly in a first-order form of the governing differential equations, e.g., (Kwakernaak and Sivan, 1972). Thus, several investigators have addressed the issues of interfacing second-order structural systems and control theory based on the first-order form (Hughes and Skelton, 1980; Arnold and Laub, 1984; Bender and Laub, 1985; Oshman, Inman and Laub, 1987; Belvin and Park, 1989, 1990). As a result of these studies, it has become straightforward for one to synthesize non-observer based control laws within the framework of a first-order control theory and then to recast the resulting control laws in terms of the second-order structural systems.

Unfortunately, controllers based on a first-order observer are difficult to express in a pure second-order form because the first-order observer implicitly incorporates an additional filter equation (Belvin and Park, 1989). However a recent work (Juang and Maghami, 1990) has enabled the first-order observer gain matrices to be synthesized using only second-order equations. To complement the second-order gain synthesis, the objective of the present paper is to develop a second-order based simulation procedure for first-order observers. The particular class of first-order observers chosen for study are the Kalman Filter based state estimators as applied to second-order structural systems. The procedure permits simulation of first-order observers with nearly the same solution procedure used for treating the structural dynamics equation. Hence, the reduced size of system matrices and the computational techniques that are tailored to sparse second-order structural systems may be employed. As will be shown, the procedure hinges on discrete time integration formulas to effectively reduce the continuous time Kalman Filter to a set of second-order difference equations.

The paper first reviews of the conventional first-order representation of the continuous second-order structural equations of motion. An examination of the corresponding first-order Kalman filtering equations indicates that, due to the difference in the derivative of the estimated displacement ($\frac{d}{dt}\hat{x}$) and the estimated velocity ($\hat{\dot{x}}$), transformation of the first-order observer into an equivalent second-order observer requires the time derivative of measurement data, a process not recommended for practical implementation.

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Next, a transformation via a generalized momentum is introduced to recast the structural equations of motion in a general first-order setting. It is shown that discrete time numerical integration followed by reduction of the resulting difference equations circumvents the need for the time derivative of measurements to solve Kalman filtering equations in a second-order framework. Hence, the Kalman filter equations can be solved using a second-order solution software package.

Subsequently, computer implementation aspects of the present second-order observer are presented. Several computational paths are discussed in the context of discrete and continuous time simulation. For continuous time simulation, an equation augmentation is introduced to exploit the symmetry and sparcity of the attendant matrices by maintaining state dependant control and observer terms on the right-hand-side (RHS) of the filter equations. In addition, the computational efficiency of the present second order observer as compared to the first order observer is presented.

Continuous Formulation of Observers
for Structural Systems

Linear, second-order discrete structural models can be expressed as

$$M\ddot{x} + D\dot{x} + Kx = Bu + Gw \ , \qquad x(0) = x_0 \ , \quad \dot{x}(0) = \dot{x}_0 \qquad (1)$$

$$u = -Z_1 x - Z_2 \dot{x}$$

with the associated measurements

$$z = H_1 x + H_2 \dot{x} + \nu \qquad (2)$$

where M, D, K are the mass damping and stiffness matrices of size $(N \times N)$; x is the structural displacement vector, $(N \times 1)$; u is the active control force $(m \times 1)$; B is a constant force distribution matrix $(N \times m)$; z is a set of measurements $(r \times 1)$; H_1 and H_2 are the measurement distribution matrices $(r \times N)$; Z_1 and Z_2 are the control feedback gain matrices $(m \times N)$; w and ν are zero-mean, white Gaussian processes with their respective covariances Q and R; and the superscript dot designates time differentiation. In the present study, we will restrict ourselves to the case wherein Q and R are uncorrelated with each other and the initial conditions x_0 and $\dot{x}_0$ are also themselves jointly Gaussian with known means and covariances.

The conventional representation of (1) in a first-order form is facilitated by

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} = \dot{x}_1 \\ M\dot{x}_2 = M\ddot{x} = Bu + Gw - Dx_2 - Kx_1 \end{cases} \qquad (3)$$

which, when cast in a first-order form, can be expressed as

$$\begin{cases} E\dot{q} = Fq + \bar{B}u + \bar{G}w, & q = \langle x_1 \quad x_2 \rangle^T \\ z = Hq + \nu \end{cases} \tag{4}$$

where

$$E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \quad F = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix},$$

$$\bar{B} = \begin{Bmatrix} 0 \\ B \end{Bmatrix}, \quad \bar{G} = \begin{Bmatrix} 0 \\ G \end{Bmatrix} \tag{5}$$

It is well-known that the Kalman filtering equations (Kalman, 1961; Kalman and Bucy, 1963) for (4) can be shown to be (Arnold and Laub, 1984):

$$E\dot{\hat{q}} = F\hat{q} + \bar{B}u + EPH^TR^{-1}\bar{z} \tag{6}$$

where

$$\bar{z} = z - H\hat{q}, \quad P = \begin{bmatrix} U & S^T \\ S & L \end{bmatrix}, \quad \hat{q} = \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{Bmatrix} = \begin{Bmatrix} \hat{x} \\ \hat{\dot{x}} \end{Bmatrix} \tag{7}$$

in which U and L are positive definite matrices and the matrix P is determined by the Riccati equation (Kwakernaak and Sivan, 1972; Arnold and Laub, 1984)

$$E\dot{P}E^T = FPE^T + EPF^T - EPH^TR^{-1}HPE^T + \bar{G}Q\bar{G}^T \tag{8}$$

The inherent difficulty of reducing the first-order Kalman filtering equations given by (6) to second order form can be appreciated if one attempts to write (6) in a form introduced in (3):

$$\begin{cases} a) & \hat{x}_1 = \hat{x} \\ b) & \hat{x}_2 = \hat{\dot{x}} = \dot{\hat{x}}_1 - L_1\bar{z} \\ c) & M\dot{\hat{x}}_2 = -D\hat{x}_2 - K\hat{x}_1 + Bu + ML_2\bar{z} \end{cases} \tag{9}$$

where

$$L_1 = (H_1U + H_2S)^TR^{-1}, \qquad L_2 = (H_1S^T + H_2L)^TR^{-1}$$

Note from (9b) that $\hat{x}_2 \neq \dot{\hat{x}}_1$. In other words, the time derivative of the estimated displacement ($\dot{\hat{x}}$) is not the same as the estimated velocity ($\hat{\dot{x}}$); hence, $\hat{x}_1$ and $\hat{x}_2$ must be treated as two independent variables, an important observation somehow overlooked in Hashemipour and Laub (1988).

Of course, although not practical, one can eliminate $\hat{x}_2$ from (9). Assuming $\hat{x}_1$ and $\hat{x}_2$ are differentiable, differentiate (9b) and multiply both sides by M to obtain

$$M\ddot{\hat{x}}_1 = M\dot{\hat{x}}_2 + ML_1\dot{\bar{z}} \tag{10}$$

Substituting $M\dot{\hat{x}}_2$ from (9c) and $\hat{x}_2$ from (9b) in (10) yields

$$M\ddot{\hat{x}}_1 = -D(\dot{\hat{x}}_1 - L_1\bar{z}) - K\hat{x}_1 + Bu + ML_2\bar{z} + ML_1\dot{\bar{z}} \qquad (11)$$

which, upon rearrangements, becomes

$$M\ddot{\hat{x}}_1 + D\dot{\hat{x}}_1 + K\hat{x}_1 = Bu + ML_2\bar{z} + ML_1\dot{\bar{z}} + DL_1\bar{z} \qquad (12)$$

There are two difficulties with the above second-order observer. First, the numerical solution of (12) involves the computation of $\ddot{\hat{x}}_1$ when rate measurements are made. The accuracy of this computation is in general very susceptible to errors caused in numerical differentiation of $\dot{\hat{x}}_1$. Second, and most important, the numerical evaluation of $\dot{\bar{z}}$ that is required in (12) assumes that the derivative of measurement information is available which should be avoided in practice. We now present a computational procedure that circumvents the need for computing measurement derivatives and that enables one to construct observers based on the second-order models.

Second-Order Transformation of Continuous Kalman Filtering Equations

This section presents a transformation of the continuous time first-order Kalman filter to a discrete time set of second-order difference equations for digital implementation. The procedure avoids the need for measurement derivative information. In addition, the sparsity and symmetry of the original mass, damping and stiffness matrices can be maintained. Prior to describing the numerical integration procedure, a transformation based on generalized momenta is presented which is later used to improve computational efficiency of the equation solution.

Generalized Momenta

Instead of the conventional transformation (3) of the second-order structural system (1) into a first-order form, let us consider the following generalized momenta (Jensen, 1974; Felippa and Park, 1978):

$$\left\{ \begin{array}{ll} a) & x_1 = x \\ b) & x_2 = AM\dot{x}_1 + Cx_1 \end{array} \right. \qquad (13)$$

where A and C are constant matrices to be determined. Time differentiation of (13b) yields

$$\dot{x}_2 = AM\ddot{x}_1 + C\dot{x}_1 \qquad (14)$$

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Substituting (1) via (13a) into (14), one obtains

$$\dot{x}_2 = A(Bu + Gw) - (AD - C)\dot{x}_1 - AKx_1 \qquad (15)$$

Finally, pairing of (13b) and (15) gives the following first-order form:

$$\begin{bmatrix} AM & 0 \\ AD - C & I \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} C & -I \\ AK & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} =$$

$$\begin{bmatrix} 0 \\ A(Bu + Gw) \end{bmatrix} \qquad (16)$$

The associated Kalman filtering equation can be shown to be of the following form:

$$\begin{bmatrix} AM & 0 \\ AD - C & I \end{bmatrix} \begin{Bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{Bmatrix} + \begin{bmatrix} C & -I \\ AK & 0 \end{bmatrix} \begin{Bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ ABu \end{Bmatrix} +$$

$$\begin{bmatrix} AM & 0 \\ AD - C & I \end{bmatrix} \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \end{bmatrix} \bar{z} \qquad (17)$$

where

$$\bar{L}_1 = (\bar{H}_1 U + \bar{H}_2 S)^T R^{-1}, \qquad \bar{L}_2 = (\bar{H}_1 S^T + \bar{H}_2 L)^T R^{-1}$$

and $\bar{H}_1$ and $\bar{H}_2$ correspond to a modified form of measurements expressed as

$$z = H_1 x + H_2 \dot{x} = \bar{H}_1 x_1 + \bar{H}_2 x_2 \qquad (18)$$

where

$$\bar{H}_1 = H_1 - H_2 M^{-1} A^{-1} C, \qquad \bar{H}_2 = H_2 M^{-1} A^{-1}$$

Clearly, as in the conventional first-order form (9), $\hat{x}_1$ and $\hat{x}_2$ in (17) are now two independent variables. Specifically, the case of $A = M^{-1}$ and $C = 0$ corresponds to (3) with $x_2 = \dot{x}_1$. However, as we shall see below, the Kalman filtering equations based on the generalized momenta (13) offer several computational advantages over (3).

Numerical Integration

At this juncture it is noted that in the previous section one first performs the elimination of $\hat{x}_1$ in order to obtain a second-order observer, then performs the numerical solution of the resulting second-order observer. This approach has the disadvantage of having to deal with the time derivative of measurement data. To avoid this, we will first integrate numerically the associated Kalman filtering equation (17).

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The direct time integration formula we propose to employ is a mid-point version of the trapezoidal rule:

$$
\begin{cases}
a) & \left\{ \begin{matrix} \hat{x}_1 \\ \hat{x}_2 \end{matrix} \right\}^{n+1/2} = \left\{ \begin{matrix} \hat{x}_1 \\ \hat{x}_2 \end{matrix} \right\}^{n} + \delta \left\{ \begin{matrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{matrix} \right\}^{n+1/2} \\
b) & \left\{ \begin{matrix} \hat{x}_1 \\ \hat{x}_2 \end{matrix} \right\}^{n+1} = 2 \left\{ \begin{matrix} \hat{x}_1 \\ \hat{x}_2 \end{matrix} \right\}^{n+1/2} - \left\{ \begin{matrix} \hat{x}_1 \\ \hat{x}_2 \end{matrix} \right\}^{n}
\end{cases}
\tag{19}
$$

where the superscript n denotes the discrete time interval $t^n = nh$, h is the time increment and $\delta = h/2$.

Time discretization of (17) by (19a) at the $n + 1/2$ time step yields

$$
\begin{bmatrix} AM & 0 \\ AD - C & I \end{bmatrix} \left\{ \begin{matrix} \hat{x}_1^{n+1/2} - \hat{x}_1^n \\ \hat{x}_2^{n+1/2} - \hat{x}_2^n \end{matrix} \right\} + \delta \begin{bmatrix} C & -I \\ AK & 0 \end{bmatrix} \left\{ \begin{matrix} \hat{x}_1^{n+1/2} \\ \hat{x}_2^{n+1/2} \end{matrix} \right\}
$$

$$
= \delta \begin{bmatrix} AM & 0 \\ AD - C & I \end{bmatrix} \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \end{bmatrix} \bar{z}^{n+1/2} + \delta \left\{ \begin{matrix} 0 \\ ABu^{n+1/2} \end{matrix} \right\}
\tag{20}
$$

The above difference equations require the solution of matrix equations of $2N$ variables, namely, in terms of the two variables $\hat{x}_2^{n+1/2}$ and $\hat{x}_1^{n+1/2}$, each with a size of N. To reduce the above coupled equations of order $2N$ into the corresponding ones of order N, we proceed in the following way by exploiting the nature of parametric matrices of A and C as introduced in (13). To this end, we write out (20) as two coupled difference equations as follows:

$$
AM(\hat{x}_1^{n+1/2} - \hat{x}_1^n) + \delta(C\hat{x}_1^{n+1/2} - \hat{x}_2^{n+1/2})
$$

$$
= \delta AM \bar{L}_1 \bar{z}^{n+1/2}
\tag{21}
$$

$$
(AD - C)(\hat{x}_1^{n+1/2} - \hat{x}_1^n) + (\hat{x}_2^{n+1/2} - \hat{x}_2^n) + \delta AK \hat{x}_1^{n+1/2}
$$

$$
= \delta(AD - C)\bar{L}_1 \bar{z}^{n+1/2} + \delta \bar{L}_2 \bar{z}^{n+1/2} + \delta ABu^{n+1/2}
\tag{22}
$$

Multiplying (22) by δ and adding the resulting equation to (21) yields

$$
A(M + \delta D + \delta^2 K)\hat{x}_1^{n+1/2} = (AM + \delta(AD - C))\hat{x}_1^n + \delta \hat{x}_2^n
$$

$$
+ \{\delta AM \bar{L}_1 + \delta^2(AD - C)\bar{L}_1 + \delta^2 \bar{L}_2\}\bar{z}^{n+1/2} + \delta^2 ABu^{n+1/2}
\tag{23}
$$

Of several possible choices for matrices A and B, we will examine

$$
\begin{cases}
a) & A = I, \quad C = D \\
b) & A = M^{-1}, \quad C = 0
\end{cases}
\tag{24}
$$

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The choice of (24a) reduces (23) to:

$$(M + \delta D + \delta^2 K)\hat{x}_1^{n+1/2} = M\hat{x}_1^n + \delta\hat{x}_2^n + \delta^2 Bu^{n+1/2}$$

$$+\delta\{M\bar{L}_1 + \delta\bar{L}_2\}\bar{z}^{n+1/2} \tag{25}$$

so that once $\hat{x}_1^{n+1/2}$ is computed, $\hat{x}_2^{n+1/2}$ is obtained from (22) rewritten as

$$\hat{x}_2^{n+1/2} = \hat{x}_2^n + \delta\hat{g}^n - \delta K\hat{x}_1^{n+1/2} \tag{26}$$

where

$$\hat{g}^n = Bu^{n+1/2} + \bar{L}_2\bar{z}^{n+1/2} \tag{27}$$

which is already computed in order to construct the right-hand side of (25). Hence, $K\hat{x}_1^{n+1/2}$ is the only additional computation needed to obtain $\hat{x}_2^{n+1/2}$. It is noted that neither any numerical differentiation nor matrix inversion is required in computing $\hat{x}_2^{n+1/2}$. This has been achieved through the introduction of the general transformation (13) and the particular choice of the parameter matrices given by (24a).

On the other hand, if one chooses the conventional representation (24b), the solution of $\hat{x}_1^{n+1/2}$ is obtained from (23)

$$(M + \delta D + \delta^2 K)\hat{x}_1^{n+1/2} = (M + \delta D)\hat{x}_1^n + \delta M\hat{x}_2^n$$

$$+\delta\{(M + \delta D)\bar{L}_1 + \delta M\bar{L}_2\}\bar{z}^{n+1/2} + \delta^2 Bu^{n+1/2} \tag{28}$$

Once $\hat{x}_1^{n+1/2}$ is obtained, $\hat{x}_2^{n+1/2}$ can be computed either by

$$\hat{x}_2^{n+1/2} = (\hat{x}_1^{n+1/2} - \hat{x}_1^n)/\delta - \bar{L}_1\bar{z}^{n+1/2} \tag{29}$$

which is not accurate due to the numerical differentiation to obtain $\dot{\hat{x}}_1^{n+1/2}$, or by (22)

$$\hat{x}_2^{n+1/2} = \hat{x}_2^n + \delta\hat{g}^n - \delta M^{-1}K\hat{x}_1^{n+1/2} -$$

$$M^{-1}D(\hat{x}_1^{n+1/2} - \hat{x}_1^n) + \delta M^{-1}D\bar{L}_1\bar{z}^{n+1/2} \tag{30}$$

which involves two additional matrix-vector multiplications, when $D \neq 0$, as compared with the choice of $A = I$ and $C = D$. Thus (24a) is the preferred representation in a first-order form of the second-order structural dynamics equations (1) and is used in the remainder of this work.

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Decoupling Of Difference Equations

We have seen in the previous section, instead of solving the first-order Kalman filtering equations of $2n$ variables for the structural dynamics systems (1), the solution of the implicit time-discrete observer equation (25) of n variables can potentially offer a substantial computational saving by exploiting the reduced size and sparsity of M, D and K. This assumes that $\bar{z}^{n+1/2}$ and $u^{n+1/2}$ are available, which is not the case since at the nth time step

$$u^{n+1/2} = -\bar{Z}_1 \hat{x}_1^{n+1/2} - \bar{Z}_2 \hat{x}_2^{n+1/2} \tag{31}$$

$$\bar{z}^{n+1/2} = z^{n+1/2} - \bar{H}_1 \hat{x}_1^{n+1/2} - \bar{H}_2 \hat{x}_2^{n+1/2} \tag{32}$$

requires both $\hat{x}_1^{n+1/2}$ and $\hat{x}_2^{n+1/2}$ even if $z^{n+1/2}$ is assumed to be known from measurements or by solution of (1). Note in (32), the control gain matrices are transformed by

$$\bar{Z}_1 = Z_1 - Z_2 M^{-1} A^{-1} C, \qquad \bar{Z}_2 = Z_2 M^{-1} A^{-1}$$

There are two distinct approaches to uncouple (25) and (26) as described in the following sections.

Discrete Time Update

Equations (31) and (32) can be approximated using

$$\bar{z}^{n+1/2} \simeq z^n - \bar{H}_1 \hat{x}_1^n - \bar{H}_2 \hat{x}_2^n \tag{33}$$

$$u^{n+1/2} \simeq -\bar{Z}_1 \hat{x}_1^n - \bar{Z}_2 \hat{x}_2^n \tag{34}$$

This approximation leads to a discrete time update of the control force and state correction terms which is analogous to that which exists in experiments where a finite bandwidth of measurement updates occurs. For discrete time approximation, the step size $h = t^{n+1} - t^n$ should be chosen to match the time required to acquire, process and output a control update.

Discrete time simulation is quite simple to implement as the control force and state corrections are treated with no approximation on the right-hand-side (RHS) of (25) and (26). Should continuous time simulation be required, a different approach is necessary.

Continuous Time Update

To simulate the system given in (25) and (26) in continuous time, strictly speaking, one must rearrange (25) and (26) so that the terms involving $\hat{x}_1^{n+1/2}$ and $\hat{x}_2^{n+1/2}$ are augmented

to the left-hand-side (LHS) of the equations. However, this augmentation into the solution matrix $(M + \delta D + \delta^2 K)$ would destroy the computational advantages of the matrix sparcity and symmetry. Thus, a partitioned solution procedure has been developed for continuous time simulation as described in (Park and Belvin, 1991). The procedure, briefly outlined herein, maintains the control force and state correction on the RHS of the equations as follows.

First, $\hat{x}_1^{n+1/2}$ and $\hat{x}_2^{n+1/2}$ are predicted by

$$\hat{x}_{1p}^{n+1/2} = \hat{x}_1^n, \qquad \hat{x}_{2p}^{n+1/2} = \hat{x}_2^n \tag{35}$$

However, instead of direct substitution of the above predicted quantity to obtain $u_p^{n+1/2}$ and $\bar{z}_p^{n+1/2}$ based on (31) and (32), equation augmentations are introduced to improve the accuracy of $u_p^{n+1/2}$ and $\bar{z}_p^{n+1/2}$. Of several augmentation procedures that are applicable to construct discrete filters for the computations of $u^{n+1/2}$ and $\bar{z}^{n+1/2}$, we substitute (26) into (31) and (32) to obtain

$$\begin{cases} u^{n+1/2} = -\bar{Z}_1 \hat{x}_1^{n+1/2} - \bar{Z}_2(\hat{x}_2^n - \delta K \hat{x}_1^{n+1/2} + \\ \qquad \delta B u^{n+1/2} + \delta \bar{L}_2 \bar{z}^{n+1/2}) \\ \bar{z}^{n+1/2} = z^{n+1/2} - \bar{H}_1 \hat{x}_1^{n+1/2} - \\ \qquad \bar{H}_2(\hat{x}_2^n - \delta K \hat{x}_1^{n+1/2} + \delta B u^{n+1/2} + \delta \bar{L}_2 \bar{z}^{n+1/2}) \end{cases} \tag{36}$$

Rearranging the above coupled equations, one obtains

$$\begin{bmatrix} (I + \delta \bar{Z}_2 B) & \delta \bar{Z}_2 \bar{L}_2 \\ \delta \bar{H}_2 B & (I + \delta \bar{H}_2 \bar{L}_2) \end{bmatrix} \begin{Bmatrix} u^{n+1/2} \\ \bar{z}^{n+1/2} \end{Bmatrix} =$$

$$\begin{Bmatrix} -\bar{Z}_2 \hat{x}_2^n - (\bar{Z}_1 - \delta \bar{Z}_2 K) \hat{x}_1^{n+1/2} \\ z^{n+1/2} - \bar{H}_2 \hat{x}_2^n - (\bar{H}_1 - \delta \bar{H}_2 K) \hat{x}_1^{n+1/2} \end{Bmatrix} \tag{37}$$

which corresponds to a first order filter to reduce the errors in computing $\hat{x}_2 = M\dot{\hat{x}} + D\hat{x}$. A second-order discrete filter for computing u and $\bar{z}$ can be obtained by differentiating u and $\bar{z}$ to obtain

$$\begin{cases} \dot{u} = -\bar{Z}_1 \dot{\hat{x}}_1 - \bar{Z}_2 \dot{\hat{x}}_2 \\ \dot{\bar{z}} = \dot{z} - \bar{H}_1 \dot{\hat{x}}_1 - \bar{H}_2 \dot{\hat{x}}_2 \end{cases} \tag{38}$$

and then substituting $\dot{\hat{x}}_1$ and $\dot{\hat{x}}_2$ from (17). Subsequently, (19) is applied to integrate the equations for u and $\bar{z}$ which yields

$$\begin{bmatrix} I + \delta \bar{Z}_2 B + \delta^2 \bar{Z}_1 M^{-1} B & \delta(\bar{Z}_2 \bar{L}_2 + \bar{Z}_1 \bar{L}_1 + \delta \bar{Z}_1 M^{-1} \bar{L}_2) \\ \delta(\bar{H}_2 B + \delta \bar{H}_1 M^{-1} B) & I + \delta \bar{H}_1(\bar{L}_1 + \delta M^{-1} \bar{L}_2) + \delta \bar{H}_2 \bar{L}_2 \end{bmatrix} \begin{Bmatrix} u^{n+1/2} \\ \bar{z}^{n+1/2} \end{Bmatrix} =$$

$$\begin{Bmatrix} u^n \\ \bar{z}^n \end{Bmatrix} - \delta \begin{Bmatrix} \bar{Z}_1 M^{-1}(\hat{x}_2^n - \delta K \hat{x}_1^{n+1/2} - D \hat{x}_1^{n+1/2}) + \bar{Z}_2 K \hat{x}_1^{n+1/2} \\ \bar{H}_1 M^{-1}(\hat{x}_2^n - \delta K \hat{x}_1^{n+1/2} D \hat{x}_1^{n+1/2}) + \bar{H}_2 K \hat{x}_1^{n+1/2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ z^{n+1/2} - z^n \end{Bmatrix} \tag{39}$$

The net effects of this augmentation are to filter out the errors committed in estimating both $\hat{x}_1$ and $\hat{x}_2$. Solution of (39) for $u^{n+1/2}$ and $\bar{z}^{n+1/2}$ permits (25) and (26) to be solved in continuous time for $\hat{x}_1^{n+1/2}$ and $\hat{x}_2^{n+1/2}$. Subsequently, (29b) is used for $\hat{x}_1^{n+1}$ and $\hat{x}_2^{n+1}$.

The preceding augmentation (39) leads to an accurate estimate of the control force and observer error correction at the (n+1/2) time step. Although (39) involves the solution of an additional algebraic equation, the equation size is relatively small (size = number of actuators (m) plus the number of measurements (r)). Thus, (39) is an efficient method for continuous time simulation of the Kalman filter equations provided the size of (39) is significantly lower than the first order form of (4). The next section discusses the relative efficiency of the present method and the conventional first order solution. More details on the equation augmentation procedure (39) may be found in Park and Belvin (1991).

Finally, it is noted that by following a similar time discretization procedure adopted for computing $\hat{x}_1^{n+1/2}$ and $\hat{x}_2^{n+1/2}$, the structural dynamics equation (1) can be solved by

$$\begin{cases} (M + \delta D + \delta^2 K)x_1^{n+1/2} = Mx_1^n + \delta x_2^n + \delta^2 Bu^{n+1/2} \\ x_2^{n+1/2} = x_2^n + \delta Bu^{n+1/2} - \delta Kx_1^{n+1/2} \end{cases} \tag{40}$$

Thus, numerical solutions of the structural dynamics equation (1) and the observer equation (20) can be carried out within the second-order solution context, thus realizing substantial computational simplicity compared with the solution of first-order systems of equations (4) and the corresponding first-order observer equations (6).

It is emphasized that the solutions of both the structural displacement x and the reconstructed displacement $\hat{x}$ employ the same solution matrix, $(M + \delta D + \delta^2 K)$. The computational stability of the present procedure can be examined as investigated in Park (1980) and Park and Felippa (1983, 1984). The result, when applied to the present case, can be stated as

$$\delta^2 \lambda_{\max} \leq 1 \tag{41}$$

where $\lambda_{\max}$ is the maximum eigenvalue of

$$(\lambda^2 I + \lambda \bar{Z}_2 B + \bar{Z}_1 M^{-1} B)y = 0 \tag{42}$$

Experience has shown that $|\lambda_{\max}|$ is several orders of magnitude smaller than $\mu_{\max}$ of the structural dynamics eigenvalue problem:

$$\mu My = Ky \tag{43}$$

Considering that a typical explicit algorithm has its stability limit $\mu_{\max} \cdot h \leq 2$, the maximum step size allowed by (42) is in fact several orders of magnitude larger than allowed by any explicit algorithm.

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Computational Efficiency

Solution of the Kalman filtering equations in second-order form is prompted by the potential gain in computational efficiency due to the beneficial nature of matrix sparcity and symmetry in the solution matrix of the second-order observer equations. There is an overhead to be paid for the present second-order procedure, that is, the additional computations introduced to minimize the control force and observer error terms on the right-hand-side of the resulting discrete equations. The following paragraphs show the second-order solution is most advantageous for observer models with sparse coefficient matrices M, D and K.

Solution of the first order Kalman filter equation (6) or the second-order form (25-26, 39) may be performed using a time discretization as given by (19). For linear time invariant (LTI) systems, the solution matrix is decomposed once and subsequently upper and lower triangular system solutions are performed to compute the observer state at each time step. Thus, the computations required at each time step result from calculation of the RHS and subsequent triangular system solutions. For the results that follow, the number of floating point operations per second (flops) are estimated for LTI systems of order $O(N)$. In addition, it is assumed that the mass, damping and stiffness matrices $(M, D$ and $K)$ are symmetric and banded with bandwidth αN, where $0 \le \alpha \le (0.5 - \frac{1}{2N})$.

The first-order Kalman filter equation (6) requires $(4N^2 + 2Nr + O(N))$ flops at each time step. The discrete time second-order Kalman filter solution (25-26, 33-34) require $(8\alpha^2 N^2 + 2\alpha N^2 + 3Nm + 4Nr + O(N))$ flops and the continuous time second-order Kalman filter (25-26, 39) require $(8\alpha N^2 + 2\alpha N^2 + 5Nm + 6Nr + (r + m)^2 + O(N))$ flops at each time step. To examine the relative efficiency of the first-order and second-order forms, several cases are presented as follows.

First, a worst case condition is examined whereby M, D and K are fully populated ($\alpha = 0.5 - \frac{1}{2N}$) and $r = m = N$. For this condition, the number of flops are:

$$\begin{cases} First\ Order & 6N^2 + O(N) \\ Second\ Order\ Discrete & 10N^2 + O(N) \\ Second\ Order\ Continuous & 18N^2 + O(N) \end{cases}$$

Thus, for non-sparse systems with large numbers of sensors and actuators relative to the system order, the first order Kalman filter is 300 percent more efficient than the second-order continuous Kalman filter solution presented herein.

For structural systems, M, and K are almost always banded. In addition, the number of sensors and actuators is usually small compared to the system order N. Hence, the value of α for which the second-order form becomes more computationally attractive than the first order form must be determined. If the assumption is made that the number of

actuators (m) and the number of measurements (r) is proportional to the bandwidth ($r = m = \alpha N$), the value of α which renders the second-order solution more efficient is readily obtained. For the second-order discrete Kalman filter, when $\alpha \leq 0.394$ the second-order form is more efficient. Similarly, the second-order continuous Kalman filter form is more efficient when $\alpha \leq 0.279$. Since α obtains values approaching 0 when a modal based structural representation is used with few sensors and actuators, the second-order form can be substantially more efficient than the classical first-order form. A more detailed discussion can be found in Belvin (1989).

Implementation and Numerical Evaluations

The second-order discrete Kalman filtering equation derived in (25) and (26) have been implemented along with the stabilized form of the controller u and the filtered measurements z̄ in such a way the observer computational module can be interfaced with the partitioned control-structure interaction simulation package developed previously (Belvin, 1989; Belvin Park, 1991; Alvin and Park, 1991). Table 1 contrasts the present CSI simulation procedure to conventional procedures. It is emphasized that the solution procedure of the present second-order discrete Kalman filtering equations (25) and (26) follows exactly the same steps as required in the solution of symmetric, sparse structural systems (or the plant dynamics in the jargon of control). It is this attribute that makes the present discrete observer attractive from the simulation viewpoint.

The first example is a truss beam shown in Fig. 1, consisting of 8 bays with nodes 1 and 2 fixed for cantilevered motions. The locations of actuator and sensor applications as well as their directions are given in Table 2. Figures 2, 3 and 4 are the vertical displacement histories at node 9 for open-loop, direct output feedback, and dynamically compensated feedback cases, respectively. Note the effectiveness of the dynamically compensated feedback case by the present second-order discrete Kalman filtering equations as compared with the direct output feedback cases. Figure 5 illustrates a testbed evolutionary model of an Earth-pointing satellite. Eighteen actuators and 18 sensors are applied to the system for vibration control and their locations are provided in Tables 3 and 4. Figures 6, 7, and 8 are a representative of the responses for open-loop, direct output feedback, and dynamically compensated cases, respectively. Note that u_x response by the dynamically compensated case does drift away initially even though the settling time is about the same as that by the direct output feedback case. However, the sensor output are assumed to be noise-free in these two numerical experiemnts. Although the objective of the present paper is to establish the computational effectiveness of the second-order discrete Kalman filtering equations, we conjecture that for noise-contaminated sensor output for which one would apply dynamic compensated strategies, the relative control performance may turn

out to be the opposite. Further simulations with the present procedure should shed light on the performance of dynamically compensated feedback systems for large-scale systems as they are computationally more feasible than heretofore possible.

Table 5 illustrates the computational overhead associated with the direct output feedback vs. the use of a dynamic compensation scheme by the output present Kalman filtering equations. In the numerical experiments reported herein, we have relied on Matlab software package (Wolfram, 1988) for the synthesis of both the control law gains and the discrete Kalman filter gain matrices. It is seen that the use of the present second-order discrete Kalman filtering equations for constructing dynamically compensated control laws adds computational overhead, only an equivalent of open-loop transient analysis of symmetric sparse systems of order N instead of $2N \times 2N$ dense systems.

Summary

The present paper has addressed the advantageous features of employing the same direct time integration algorithm for solving the structural dynamics equations also to integrate the associated continuous Kalman filtering equations. The time discretization of the resulting Kalman filtering equations is further facilitated by employing a canonical first-order form via a generalized momenta. When used in conjunction with the previously developed stabilized form of control laws (Park and Belvin, 1991), the present procedure offers a substantial computational advantage over the solution methods based on a first-order form when computing with large and sparse observer models.

Computational stability of the present solution method for the observer equation has been assessed based on the stability analysis result of partitioned solution procedures (Park, 1980). To obtain a sharper estimate of the stable step size, a more rigorous computational stability analysis is being carried out and will be reported in the future.

Acknowledgements

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$$\begin{cases}
\text{Structure:} & a) & \mathbf{M\ddot{q}} + \mathbf{D\dot{q}} + \mathbf{Kq} = \mathbf{f} + \mathbf{Bu} + \mathbf{Gw} \\
& & \mathbf{q}(0) = \mathbf{q}_0, \qquad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \\
\text{Sensor Output:} & b) & \mathbf{z} = \mathbf{Hx} + \mathbf{v} \\
\text{Estimator:} & c) & \mathbf{M\ddot{\tilde{q}}} + \mathbf{D\dot{\tilde{q}}} + \mathbf{K\tilde{q}} = \mathbf{f} + \mathbf{Bu} + \mathbf{ML}_2\gamma \\
& & \tilde{\mathbf{q}}(0) = 0, \quad \dot{\tilde{\mathbf{q}}}(0) = 0 \\
\text{Control Force:} & d) & \dot{\mathbf{u}} + \mathbf{F}_2\mathbf{M}^{-1}\mathbf{Bu} = \mathbf{F}_2(\mathbf{M}^{-1}\dot{\mathbf{p}} + \mathbf{L}_2\gamma) + \mathbf{F}_1\dot{\tilde{\mathbf{q}}} \\
\text{Estimation Error:} & e) & \dot{\gamma} + \mathbf{H}_v\mathbf{L}_2\gamma = \dot{\mathbf{z}} - \mathbf{H}_v\mathbf{M}^{-1}(\dot{\mathbf{p}} - \mathbf{Bu}) - \mathbf{H}_d\dot{\tilde{\mathbf{q}}}
\end{cases}$$

Table 1a Partitioned Control-Structure Interaction Equations

$$\begin{cases}
\text{Structure:} & a) & \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Ef} + \bar{\mathbf{B}}\mathbf{u} + \bar{\mathbf{G}}\mathbf{w} \\
& & \mathbf{q}(0) = \mathbf{q}_0, \qquad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \\
\text{Sensor Output:} & b) & \mathbf{z} = \mathbf{Hx} + \mathbf{v} \\
\text{Estimator:} & c) & \dot{\tilde{\mathbf{x}}} = \mathbf{A\tilde{x}} + \mathbf{Ef} + \bar{\mathbf{B}}\mathbf{u} + \mathbf{L}\gamma \\
& & \tilde{\mathbf{x}}(0) = 0 \\
\text{Control Force:} & d) & \mathbf{u} = -\mathbf{F\tilde{x}} \\
\text{Estimation Error:} & e) & \gamma = \mathbf{z} - (\mathbf{H}_d\tilde{\mathbf{q}} + \mathbf{H}_v\dot{\tilde{\mathbf{q}}})
\end{cases}$$

where

$$\mathbf{x} = \left\{ \begin{array}{c} \mathbf{q} \\ \dot{\mathbf{q}} \end{array} \right\}, \qquad \tilde{\mathbf{x}} = \left\{ \begin{array}{c} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \end{array} \right\}$$

and

$$\mathbf{H} = [\mathbf{H}_d \quad \mathbf{H}_v], \qquad \mathbf{L} = \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0 \\ \mathbf{M}^{-1} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \qquad \bar{\mathbf{B}} = \begin{bmatrix} 0 \\ \mathbf{M}^{-1}\mathbf{B} \end{bmatrix}, \qquad \mathbf{F} = [\mathbf{F}_1 \quad \mathbf{F}_2]$$

Table 1b   Conventional Control-Structure Interaction Equatioons

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TABLE 2a:
Actuator Placement for Truss Example Problem

Actuator	Node	Component
1	2	y
2	18	y
3	9	y
4	9	x

TABLE 2b:
Sensor Placement for Truss Example Problem

Sensor	Type	Node	Component
1	Rate	2	y
2	Rate	18	y
3	Rate	9	y
4	Rate	9	x
5	Position	9	y
6	Position	9	x

TABLE 3:
Actuator Placement for EPS Example Problem

Actuator	Node	Component
1	97	x
2	97	z
3	96	x
4	96	z
5	65	y
6	68	y
7	59	y
8	62	y
9	45	y
10	45	z
11	70	y
12	70	z
13	95	x
14	95	y
15	95	z
16	95	ϕ_x
17	95	ϕ_y
18	95	ϕ_z

TABLE 4:
Sensor Placement for EPS Example Problem

Sensor	Type	Node	Component
1	Rate	97	x
2	Rate	97	z
3	Rate	96	x
4	Rate	96	z
5	Rate	65	y
6	Rate	68	y
7	Rate	59	y
8	Rate	62	y
9	Rate	45	y
10	Rate	45	z
11	Rate	70	y
12	Rate	70	z
13	Position	95	x
14	Position	95	y
15	Position	95	z
16	Position	95	ϕ_x
17	Position	95	ϕ_y
18	Position	95	ϕ_z

TABLE 5:

CPU Results for ACSIS Sequential and Parallel Versions

Model	Problem Type	Sequential	Parallel
54 DOF	Transient	4.5	5.6
Truss	FSFB	9.4	10.2
	K. Filter	13.0	10.7
582 DOF	Transient	98.6	100.3
EPS7	FSFB	190.2	294.5
	K. Filter	284.2	321.5
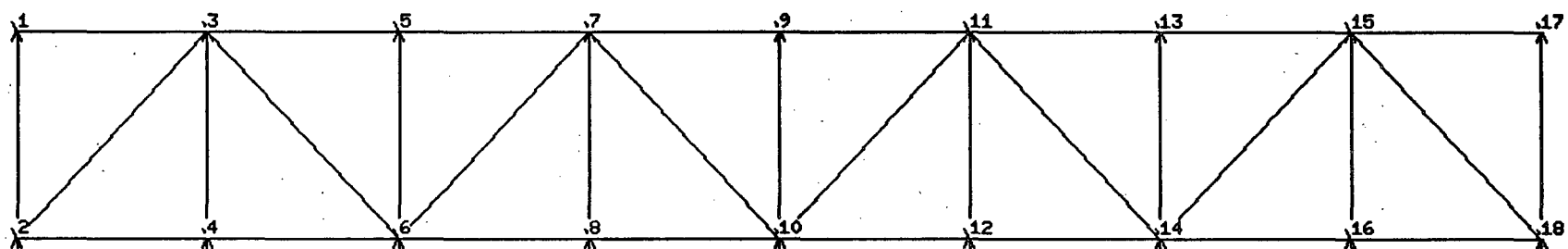
Figure 1: Truss Beam Problem

Truss Model: Open Loop Transient Response

Node 9, uy
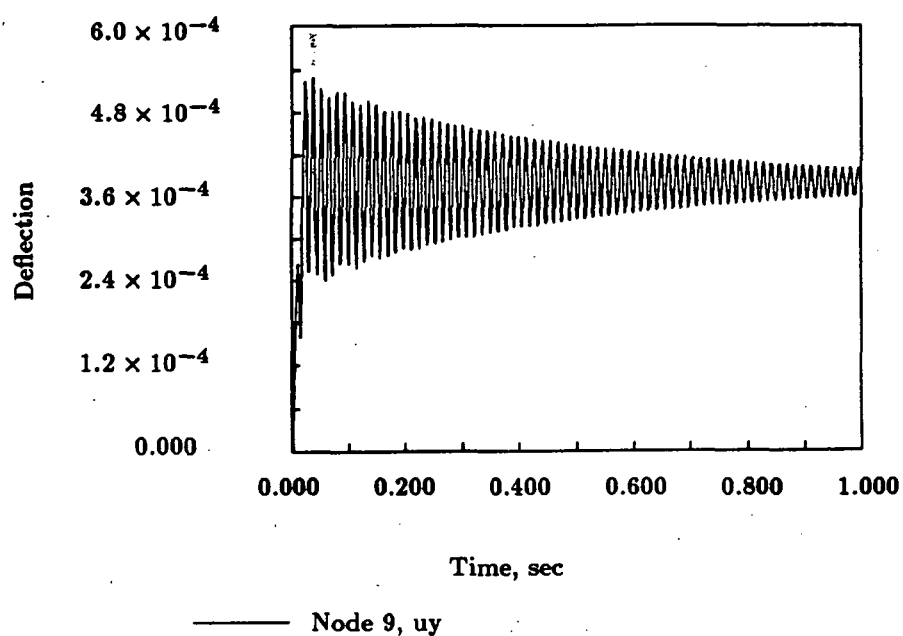
Figure 2: Truss Transient Response

Truss Model: Full State Feedback Response

Node 9, uy
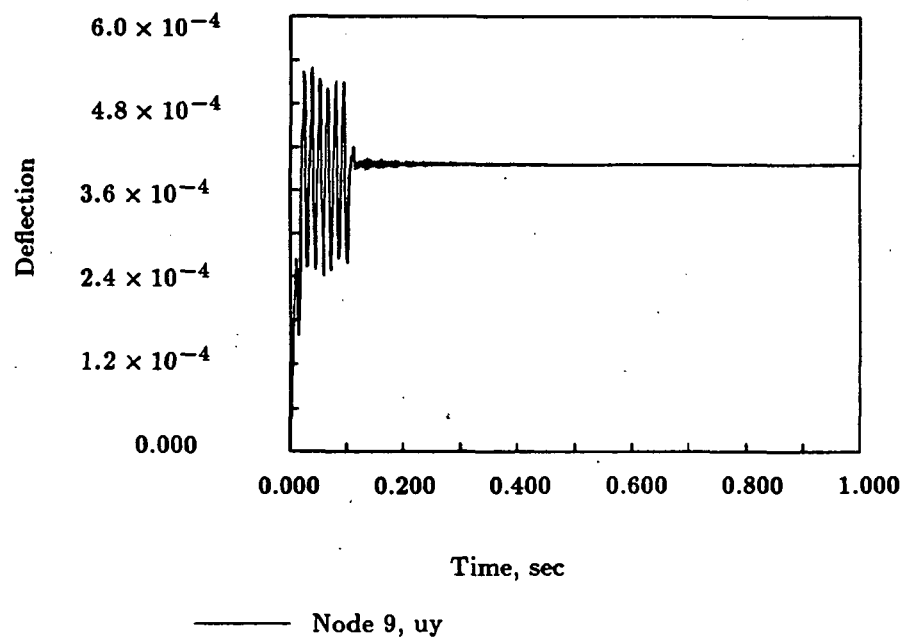
Figure 3: Truss FSFB Response

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Truss Model: Controlled Response with Kalman Filter
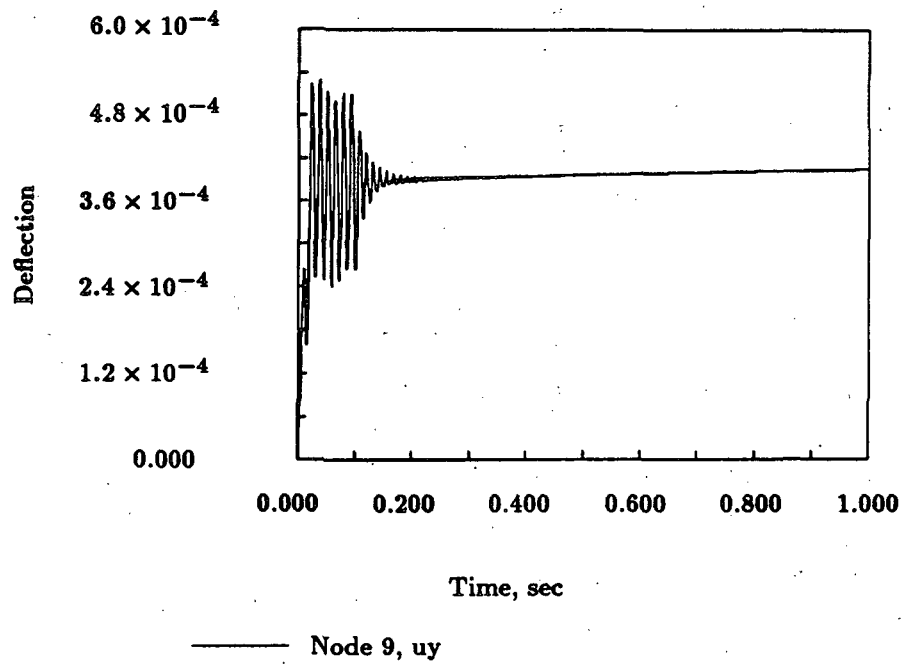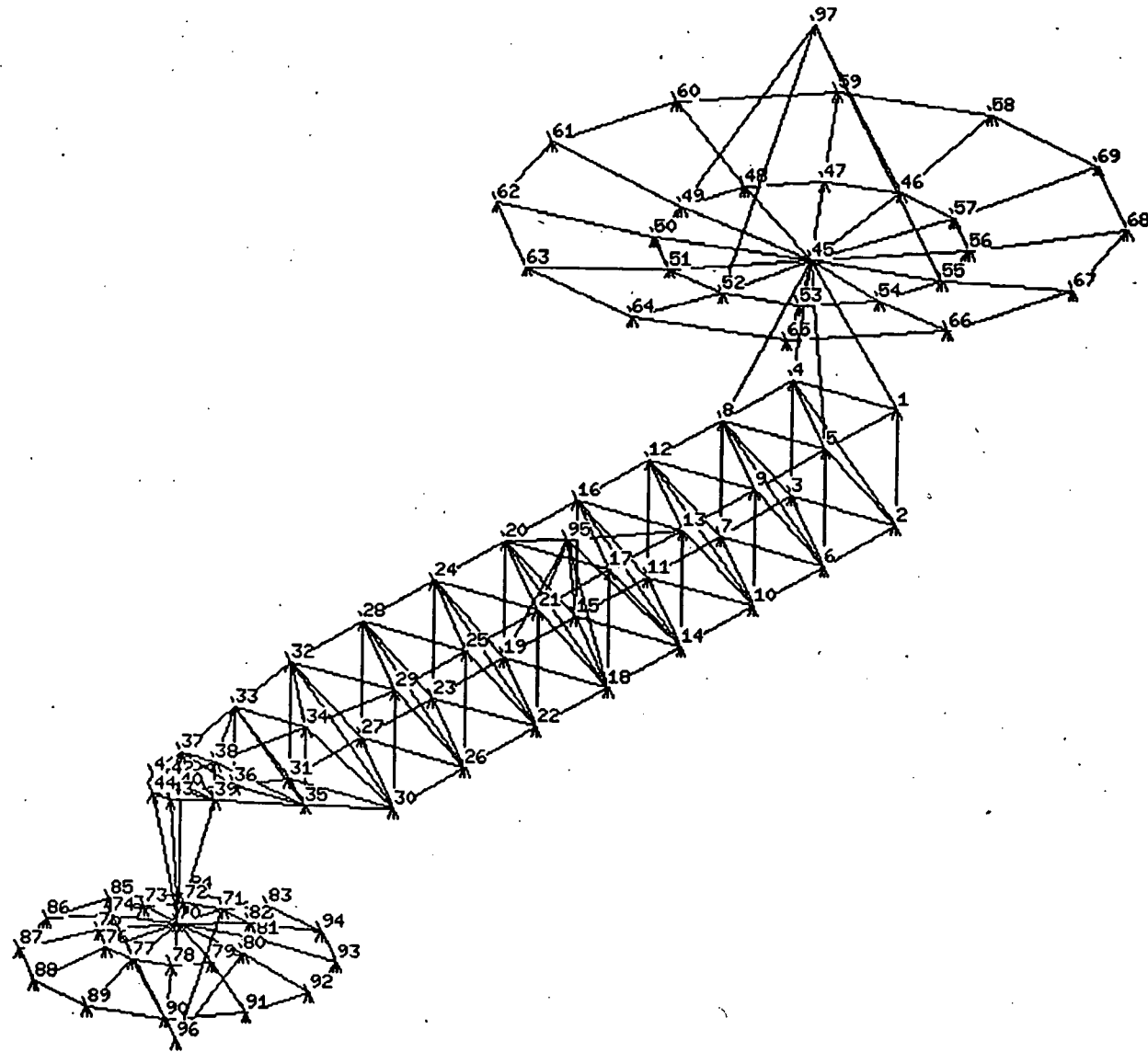

Figure 4: Truss Response with Filter

Figure 5: Evolutionary Earth-Pointing Satellite

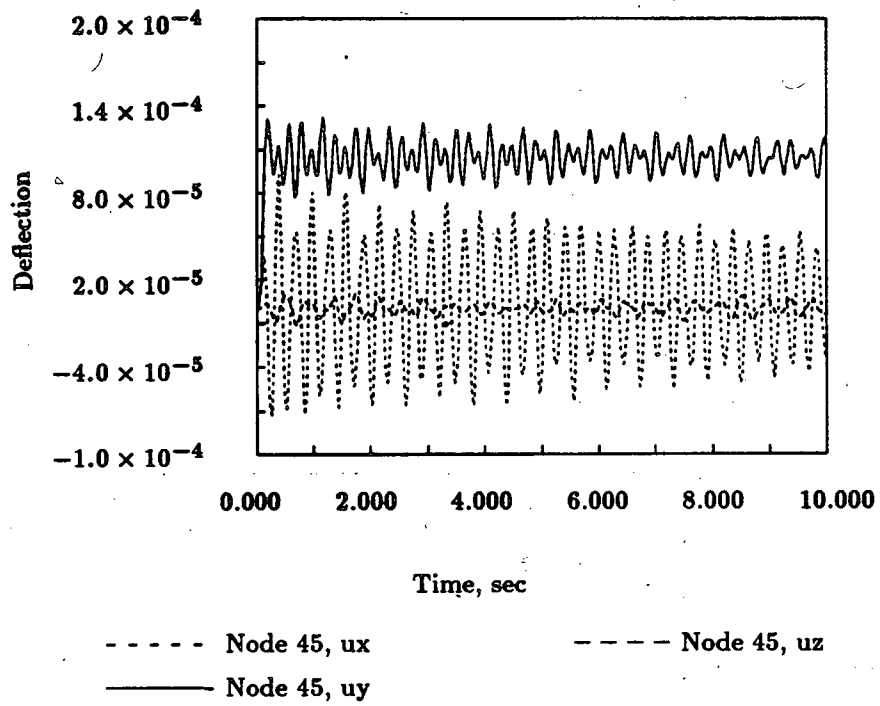## EPS7 Model: Open Loop Transient Response
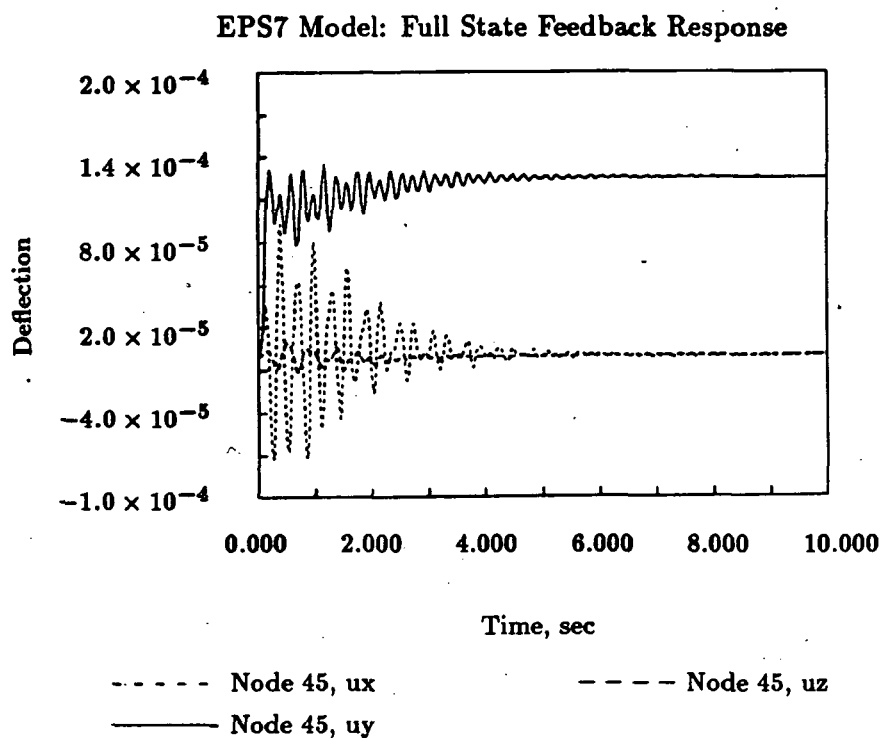

Figure 6: EPS Transient Response

EPS7 Model: Full State Feedback Response
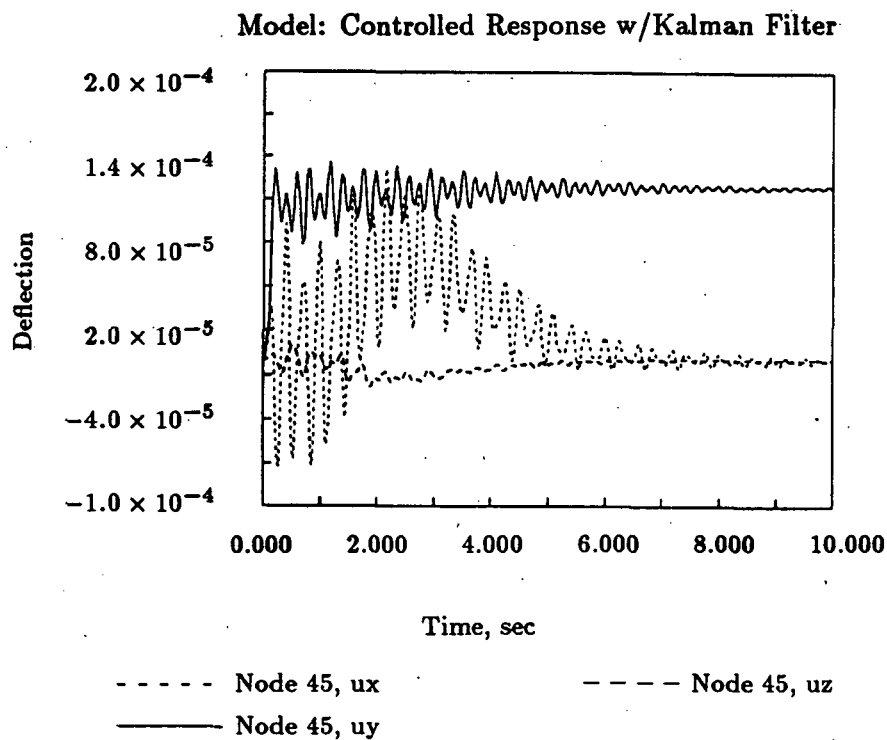

Figure 7: EPS FSFB Response

Model: Controlled Response w/Kalman Filter

Figure 8: EPS Response with Filter