S1-18

N91-22308

SPILLOVER, NONLINEARITY, & FLEXIBLE STRUCTURES

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ABSTRACT

Many systems whose evolution in time is governed by Partial Differential Equations (PDEs) are linearized around a known equilibrium before Computer Aided Control Engineering (CACE) is considered. In this case there are infinitely many *independent* vibrational modes, and it is intuitively evident on physical grounds that infinitely many actuators would be needed in order to control *all* modes.

A more precise, general formulation of this grave difficulty (the "spillover" problem) is due to A.V. Balakrishnan [*Applied Functional Analysis*, Springer, 1981, p. 233]. Let the system's state vector x be an element of a separable Hilbert space $\mathcal{H}$ whose dimension is not finite; let A be a closed linear operator with domain dense in $\mathcal{H}$ which is the infinitesimal generator of a strongly continuous semi-group of transition operators $T(t)$ for non-negative times t; and let B denote a bounded linear operator acting on another separable Hilbert space $\mathcal{U}$ (the control space) with range in $\mathcal{H}$. Now consider the control problem $dx/dt = Ax + Bu$, with $x(0)$ given. Then according to Balakrishnan this system is not *exactly controllable* if B is compact.

A possible route to circumvention of this difficulty lies in leaving the PDE in its original *nonlinear* form, and adding the essentially finite-dimensional control action Bu prior to linearization. In many cases it can be shown that the nonlinearity couples the system's modes in such a manner that only a finite-dimensional subset of the modes is functionally independent, with the remaining higher-order modes nonlinearly dependent upon them. Hence control of *all* modes can be achieved by controlling only finitely many modes.

One possibly applicable technique is the Liapunov-Schmidt rigorous reduction of singular infinite-dimensional implicit function problems to finite-dimensional implicit function problems. Such a procedure was employed by Leon Lichtenstein in the 1930's to prove the existence of a solution of the Navier-Stokes equations for a sufficiently small time-interval $0 \le t < \varepsilon$.

Omitting details of Banach-space rigor, the formalities of this approach are as follows. Let $\mathcal{L}$ be a Fredholm operator with pseudo-inverse $\mathcal{K}$; then there exist idempotent projection operators $\mathcal{P} = \mathcal{I} - \mathcal{K}\mathcal{L}$, $\mathcal{Q} = \mathcal{I} - \mathcal{L}\mathcal{K}$ whose ranges are finite-dimensional and such that a NASC for (*) $\mathcal{L}x - \mathcal{F}(x) = 0$ is that (**) $x = \mathcal{P}x + \mathcal{K}\mathcal{F}(x)$ & $\mathcal{Q}\mathcal{F}(x) = 0$. Thus one may set $x = u + v$ where v satisfies the *auxiliary equation* $v = \mathcal{K}\mathcal{F}(u + v)$ and u the *bifurcation equations* [Verzweigungsgleichungen] $\mathcal{Q}\mathcal{F}(u + v) = 0$. Typically one may solve the auxiliary equation by (contraction) iterations to find $v = \mathcal{G}(u)$ where now v is infinite-dimensional but u is finite-dimensional and then insert the result into the (finitely many) bifurcation equations to define a finite-dimensional vector function $f(u) = \mathcal{Q}\mathcal{F}(u + \mathcal{G}(u))$ such that (*) is equivalent to $f(u) = 0$. In summary, a NASC for (*) is

$$(***)\ x = u + \mathcal{G}(u),\quad \mathcal{L}u = 0,\quad \mathcal{P}\mathcal{G}(u) = 0,\quad \mathcal{P}u = u,\quad f(u) = 0.$$

As an illustration the auxiliary equation and bifurcation equations for the problem of *deflection of an in tension* ($a_0 > 0$) *EXTENSIBLE beam* ($a_1 > 0$) is considered, including *viscous damping* ($a_3 > 0$) and *Balakrishnan-Taylor damping* ($a_4 > 0$). Here

$$u_{tt} + a_2 \cdot u_{xxxx} = a_3 \cdot u_{xxt} + \left(a_0 + a_1 \cdot \int_0^L (u_x)^2 \cdot dx + a_4 \cdot \left[\int_0^L (u_x u_{xt}) \cdot dx \right]^{2(n+\beta)+1} \right) \cdot u_{xx},$$

$\forall\ t \ge 0,\ 0 \le x \le L$. As the dimension N of the bifurcation equations increases, the result approaches an N-dimensional *truncated eigenexpansion* (provided that the initial deflections and their initial spatial and temporal rates of change are not too large).

Preface

The basic idea behind the present paper is simply:

SUGGESTION
Don't linearize a PDE until *after* its reduction to a finite-dimensional ODE.

This idea can be implemented by means of the following analytical procedure:

LIAPUNOV-SCHMIDT BIFURCATION EQUATIONS:
A *rigorous* reduction of a singular *infinite*-dimensional implicit equation to the problem of an equivalent, merely *finite*-dimensional implicit equation.

This suggestion is presented as a possible technique for circumvention of the famous "Spillover Problem."

Introduction

If the problem of control of a flexible structure is linearized before one considers the control aspects, then frequently it leads to an abstract problem in functional analysis of the type of the following system of ordinary differential equations:

$$\mathcal{S}: \qquad \frac{dx}{dt} = Ax + Bu, \qquad x(0) = x^0. \tag{1}$$

Here $x \in \mathcal{H}^\infty$ is an element of an infinite-dimensional state space taken to be a separable Hilbert space. Also $u \in \mathcal{U}$ is an element of the control space, taken to be another separable Hilbert space. We take A: $\mathcal{D} \to \mathcal{H}^\infty$ to denote a closed linear mapping of the dense linear-subspace domain $\mathcal{D}$ into $\mathcal{H}^\infty = \overline{\mathcal{D}}$ which is the infinitesimal generator of a strongly continuous semi-group of transition operators $T(t)$ for $t \geq 0$. Finally we require that B: $\mathcal{U} \to \mathcal{H}^\infty$ be a bounded linear operator.

The celebrated "Spillover Problem" now has an exact formulation by means of:

<u>THEOREM</u>. (Balakrishnan, [5], p. 233.) If B is compact, then $\mathcal{S}$ is *NOT exactly controllable.*

A compact operator is one which can be approximated arbitrarily closely by an operator whose range is finite-dimensional. Therefore the practical import of the preceding theorem can be phrased as: *if a linear system has infinitely many independent modes of motion, it cannot be controlled completely with a finite-dimensional actuator suite.*

This suggests that complete control of a flexible structure by a finite actuator suite is foredoomed to impossibility. However, there may be a way to circumvent this difficulty. Note that the preceding theorem has been proved only in the case that the dynamical system $\mathcal{S}$ is linear. The purpose of the present paper is to demonstrate that for many flexible structure problems, such nonlinear mechanisms as Balakrishnan-Taylor damping will couple the higher order modes of motion to the lower order modes in such a way that only a finite number of the lower order modes is functionally independent. This suggests that a finite actuator suite could control such a system. However, we defer consideration of the control problem and deal here only with the free motion of an uncontrolled, but intrinsically nonlinear system. Our purpose is to stimulate further research into this approach rather than to present a finished theory.

Liapunov-Schmidt Bifurcation Theory

For the reader's convenience we recall the salient features of this theory from a purely formal point of view. The details of Banach space rigor can be found in pages 173-177 of Deimling [2] and other texts on nonlinear functional analysis [3], [4].

Let $\mathcal{L}$ denote a Fredholm operator, which may be singular, i.e. there may exist elements $u \neq 0$ such that $\mathcal{L}u = 0$. Let $\mathcal{K}$ denote a pseudo-inverse of $\mathcal{L}$, i.e. a linear operator such that

$$\mathcal{K}\mathcal{L}\mathcal{K} = \mathcal{K}\ , \qquad \mathcal{L}\mathcal{K}\mathcal{L} = \mathcal{L}. \tag{2}$$

Now define projection operators

$$\mathcal{P} = \mathcal{I} - \mathcal{K}\mathcal{L}, \qquad \mathcal{Q} = \mathcal{I} - \mathcal{L}\mathcal{K}; \tag{3}$$

it is readily verified that $\mathcal{P}^2 = \mathcal{P}$, $\mathcal{Q}^2 = \mathcal{Q}$, i.e. these operators are idempotent, which justifies referring to them as projection operators. Note that $\mathcal{P}$ is a right zero of $\mathcal{L}$, and $\mathcal{Q}$ is a left zero of $\mathcal{L}$.

Let $\mathcal{F} = \mathcal{F}\{\cdot\}$ denote a nonlinear operator. Then it is easy to verify the equivalence of the implicit equation problem

$$(*) \qquad \mathcal{L}x = \mathcal{F}\{x\}, \quad \mathcal{L}\mathcal{P} = 0, \quad \mathcal{Q}\mathcal{L} = 0, \tag{4}$$

and the problem

$$(**) \qquad x = \mathcal{P}x + \mathcal{K}\mathcal{F}\{x\}, \quad \mathcal{Q}\mathcal{F}\{x\} = 0. \tag{5}$$

Now define $u = \mathcal{P}x$ and verify that $\mathcal{P}u = u$; then we can replace (**) by

$$\text{(definition)} \qquad x = u + v, \tag{6}$$

$$\textbf{(AUXILIARY EQUATION)} \qquad v = \mathcal{K}\mathcal{F}\{u + v\}, \tag{7}$$

$$\textbf{(BIFURCATION EQUATION)} \qquad \mathcal{Q}\mathcal{F}\{u + v\} = 0. \tag{8}$$

Another name for the Bifurcation Equations is Branching Equations.

Suppose that the right-hand side of (7), regarded as a function of v, has a global Lipschitz constant less than unity. Then by the well-known principle of geometric convergence of Contraction Mappings we may, for each fixed u, define a nonlinear mapping $v = \mathcal{G}\{u\}$ as

$$v = \lim_{k\to\infty} v^k, \qquad v^{k+1} = \mathcal{K}\mathcal{F}\{u + v^k\}, \qquad v^0 = 0, \qquad (\ k = 0, 1, 2, 3, \cdots\ ). \tag{9}$$

Here $\mathcal{G}$ is the resolvent of the auxiliary equation in the sense that

$$\mathcal{G} \equiv \mathcal{K}\mathcal{F}\{u + \mathcal{G}\}. \tag{10}$$

Hence we may eliminate the auxiliary equation and replace v in the bifurcation equation by $\mathcal{G}$ to obtain a new finite-dimensional equation

$$f(u) \equiv \mathcal{Q}\mathcal{F}\{u + \mathcal{G}\{u\}\} = 0, \tag{11}$$

which is equivalent to the original infinite-dimensional implicit equation. Thus

$$(*) \iff (**) \iff (***) \quad x = u + \mathcal{G}\{u\}, \quad \mathcal{L}u = 0, \quad \mathcal{P}\mathcal{G}\{u\} = 0, \quad \mathcal{P}u = u, \quad f(u) = 0. \tag{12}$$

If the original functional equation was analytic, then the final finite-dimensional equation $f(u) = 0$ will also be analytic.

If $\mathcal{L}$ was non-singular, then $\mathcal{K} = \mathcal{L}^{-1}$, whence $\mathcal{P} = 0$, $\mathcal{Q} = 0$, $u = 0$, the bifurcation equation does not arise, and the resolvability of the auxiliary equation is equivalent to the resolvability of the original equation:

$$\mathcal{L}x = \mathcal{F}\{x\} \iff x = \mathcal{L}^{-1}\mathcal{F}\{x\} = \mathcal{G}\{0\}. \tag{13}$$

Finally, if $\mathcal{L}$ was singular, then the linear part $F = f_x(0) = (\partial f_i/\partial x_j)$ of $f(x)$ is necessarily also singular. Typically then the solutions of $f(x) = 0$ will not be unique and one studies the branching of these solutions by such methods as Newton's polygon.

Deflection of an Extensible, Nonlinearly Damped Beam

Let u denote the normal deflection from equilibrium. Then the vibrations of the beam can be described by $u = u(t,x)$, $0 \le x \le L$, $0 \le t < +\infty$, which satisfies the PDE

$$\rho \cdot u_{tt} + EI \cdot u_{xxxx} = C \cdot u_{xxt} + \left(H + \frac{EA_c}{L} \cdot \Phi + \Gamma \cdot \Psi \right) u_{xx}, \tag{14a}$$

$$\Phi \equiv (1/2) \cdot \int_0^L (u_x)^2 \cdot dx, \tag{14b}$$

$$\Psi \equiv \left[\int_0^L (u_x u_{xt}) \cdot dx \right]^{2(n+\beta)+1}, \quad (0 \le \beta < 1/2), \quad (n = 0, 1, 2, 3, \cdots), \tag{14c}$$

where

$t \in \mathbb{R}_+ \equiv [0,+\infty) \equiv \{ \tau \mid 0 \le \tau < +\infty \}$,

$x \in J \equiv [0,1] \equiv \{ \xi \mid 0 \le \xi \le 1 \}$,

ρ = density,
E = Young's modulus of elasticity,
I = cross-sectional moment of inertia,
C = coefficient of viscous damping,
H = axial force (tension or compression)
A_c = cross-sectional area,
L = length,
Γ = Balakrishnan-Taylor damping coefficient.

Now define the constants

$$a_0 = H/\rho, \quad a_1 = EA_c/(2\rho L), \quad a_2 = EI/\rho, \quad a_3 = C/\rho, \quad a_4 = \Gamma/\rho. \tag{15}$$

Then the PDE (14) can be expressed as

$$u_{tt} + a_2 \cdot u_{xxxx} = a_3 \cdot u_{xxt} + \left(a_0 + a_1 \cdot \int_0^L (u_x)^2 \cdot dx + a_4 \cdot \left[\int_0^L (u_x u_{xt}) \cdot dx \right]^{2(n+\beta)+1} \right) \cdot u_{xx}. \tag{16}$$

Boundary Conditions

As usual, we require that

$$u(t,0) = 0, \qquad u(t,L) = 0, \tag{17a}$$

$$u_{xx}(t,0) = 0, \qquad u_{xx}(t,L) = 0. \tag{17b}$$

Initial Conditions

Let ϕ and ψ be functions of x defined on J with the following smoothness requirements. The function ϕ should be continuously once differentiable on J and its second derivative ϕ'' should exist almost everywhere on J and be [Lebesgue] square-integrable on J. The function ψ should be continuous on J, which of course implies that it is [Riemann] square-integrable on J. These smoothness requirements may be summarized as:

$$\phi \in C^{(1)}(J), \qquad \phi'' \in L_2(J), \tag{17c}$$

$$\psi \in C^{(0)}(J), \quad \Rightarrow \quad \psi \in L_2(J). \tag{17d}$$

Now ϕ and ψ are used to define the initial conditions on u as follows:

$$u(0,x) \equiv \phi(x), \qquad u_t(0,x) \equiv \psi(x), \tag{17e}$$

$$\phi(0) = \phi(L) = 0, \qquad \psi(0) = \psi(L) = 0. \tag{17f}$$

Normalized Constants

For future convenience, we define

$$b_0 = (\pi/L)^2 \cdot a_0, \quad b_1 = (\pi/L)^4 \cdot (L/2) \cdot a_1, \quad b_2 = (\pi/L)^4 \cdot a_2, \tag{18a}$$

$$b_3 = (\pi/L)^2 \cdot a_3, \quad b_4 = (\pi/L)^{4[1+n+\beta]} \cdot (L/2)^{2(n+\beta)+1} \cdot a_4. \tag{18b}$$

Function Space Coordinatization

Define the complete orthonormal set $\{e_k\}$ on $L_2(J)$ by

$$e_k = e_k(x) \equiv sin(k\pi[x/L]), \qquad (k = 1, 2, 3, \cdots). \tag{19}$$

Assumptions (17c,d) imply that there exist (as l.i.m.) sequences $\{\alpha_k\}$, $\{\beta_k\}$ such that

$$\phi = \phi(x) = \sum_{k=1}^{\infty} \alpha_k \cdot e_k(x), \qquad \sum_{k=1}^{\infty} k^4 \cdot \alpha_k^2 < +\infty, \tag{20a}$$

$$\psi = \psi(x) = \sum_{k=1}^{\infty} \beta_k \cdot e_k(x), \qquad \sum_{k=1}^{\infty} \beta_k^2 < +\infty. \tag{20b}$$

Now we can seek to find a sequence of time-varying functions $\{u_k(t)\}$ such that

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t) \cdot e_k(x), \tag{21}$$

where, by (17f), the initial values and initial rates of the $\{u_k(t)\}$ must satisfy

$$u_k(0) = \alpha_k, \qquad \dot{u}_k(0) = \beta_k, \qquad (k = 1, 2, 3, \cdots). \tag{22}$$

Infinite System of ODEs

Insert the series expansion (21) into the PDE (16) and use the orthonormality property of the complete basis $\{e_k\}$ to derive an equivalent infinite system of ODEs:

$$\ddot{u}_k + b_3 k^2 \cdot \dot{u}_k + (b_0 + b_1 \hat{\Phi} + b_4 \hat{\Psi} + b_2 k^2) \cdot k^2 \cdot u_k = 0, \qquad (k = 1, 2, 3, \cdots), \tag{23a}$$

$$\hat{\Phi} \equiv \sum_{j=1}^{\infty} j^2 \cdot u_j^2 , \tag{23b}$$

$$\hat{\Psi} \equiv \left[\sum_{j=1}^{\infty} j^2 \cdot u_j \cdot \dot{u}_j \right]^{2(n+\beta)+1} , \tag{23c}$$

where the solutions of (23) are required to satisfy the initial conditions (22). As an alternative to (23a), in which the linear and nonlinear terms are displayed separately, we may write

$$\ddot{u}_k + b_3 k^2 \cdot \dot{u}_k + (b_0 + b_2 k^2) \cdot k^2 \cdot u_k = -\tilde{\Phi} \cdot k^2 \cdot u_k, \qquad (k = 1, 2, 3, \cdots), \tag{23d}$$

$$\tilde{\Phi} \equiv b_1 \hat{\Phi} + b_4 \hat{\Psi} \equiv b_1 \cdot \sum_{j=1}^{\infty} j^2 \cdot u_j^2 + b_4 \cdot \left[\sum_{j=1}^{\infty} j^2 \cdot u_j \cdot \dot{u}_j \right]^{2(n+\beta)+1} . \tag{23e}$$

Energy Integral

Multiply each equation of (23a) by $2\dot{u}_k$ and sum over all k, using the identities $2\ddot{u}_k \cdot \dot{u}_k \equiv (\dot{u}_k^2)^{\cdot}$ and $2u_k \cdot \dot{u}_k \equiv (u_k^2)^{\cdot}$, to obtain

$$\sum_{k=1}^{\infty} \dot{u}_k^2 + b_0 \cdot \sum_{k=1}^{\infty} k^2 \cdot u_k^2 + b_2 \cdot \sum_{k=1}^{\infty} k^4 \cdot u_k^2 + (b_1/2) \cdot \left[\sum_{k=1}^{\infty} k^2 \cdot u_k^2 \right]^2 +$$

$$+ 2b_3 \cdot \int_0^t \left[\sum_{k=1}^{\infty} k^2 \cdot \dot{u}_k^2 \right] dt + (b_4/2) \cdot \int_0^t \left[\sum_{k=1}^{\infty} k^2 \cdot u_k \cdot \dot{u}_k \right]^{2(n+\beta)+1} dt \equiv \tag{24a}$$

$$\equiv \mathcal{E}_0 \equiv constant \equiv \tag{24b}$$

$$\equiv \sum_{k=1}^{\infty} \beta_k^2 + \sum_{k=1}^{\infty} (b_0 + b_2 k^2) \cdot k^2 \cdot \alpha_k^2 + (b_1/2) \left[\sum_{k=1}^{\infty} k^2 \cdot \alpha_k^2 \right]^2 \equiv \tag{24c}$$

$$\equiv \frac{2}{L} \int_0^L \psi^2 \, dx + b_0 \cdot \frac{2}{L} \int_0^L (\phi')^2 \, dx + b_2 \cdot \frac{2}{L} \int_0^L (\phi'')^2 \, dx + (b_1/2) \cdot \left[\frac{2}{L} \int_0^L (\phi')^2 \, dx \right]^2 \equiv \tag{24d}$$

$$\equiv \bar{\psi}^2 + b_0 \cdot (\bar{\phi}')^2 + b_2 \cdot (\bar{\phi}'')^2 + (b_1/2) \cdot \left[(\bar{\phi}')^2 \right]^2 , \tag{24e}$$

where in the last expression we have used an obvious notation for the mean-square values of ψ, ϕ', and ϕ'' on $J = [0,L]$.

A key technique in what follows is the use of (24) to obtain *a priori* bounds on the solutions of (23).

Vector Notation

For convenience we shall denote the infinite column whose rows are ku_k by $\mathbf{x}$, and we shall partition $\mathbf{x}$ into a finite-dimensional component $\mathbf{u}$ and an infinite-dimensional component $\mathbf{v}$ as follows:

$$\mathbf{x} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \equiv (ku_k), \quad \mathbf{u} \in \mathbb{R}^N, \quad \mathbf{v} \in \mathbb{R}^{\infty}, \tag{25a}$$

$$\mathbf{u}_k = ku_k, \quad \mathbf{v}_k = kv_k, \quad (v_k \equiv u_k , \ k \geq N). \tag{25b}$$

A Priori Bounds

Obviously

$$\|\mathbf{x}\|^2 \equiv \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \equiv \sum_{k=1}^{\infty} k^2 \cdot u_k^2 , \tag{26}$$

while from (24a) it is easy to infer that

$$(b_0 + b_2) \cdot \|\mathbf{x}\|^2 + (b_1/2) \cdot \|\mathbf{x}\|^4 \leq \mathcal{E}_0 , \tag{27}$$

which implies that

$$\|\mathbf{x}\| \leq R, \tag{28}$$

where (choosing the numerically stable quadratic root formula)

$$R \equiv (2\mathcal{E}_0/\{(b_0 + b_2) + [(b_0 + b_2)^2 + 2b_1\mathcal{E}_0]^{1/2}\})^{1/2} \leq (\mathcal{E}_0/[b_0 + b_2])^{1/2}. \tag{29}$$

Similarly,

$$(N+1)^2 \cdot \|\mathbf{v}\|^2 \equiv (N+1)^2 \cdot \sum_{k=N+1}^{\infty} k^2 \cdot v_k^2 \leq \sum_{k=N+1}^{\infty} k^4 \cdot v_k^2 \equiv \sum_{k=N+1}^{\infty} k^4 \cdot u_k^2 \leq \sum_{k=1}^{\infty} k^4 \cdot u_k^2 \leq \mathcal{E}_0/b_2, \tag{30}$$

and

$$\sum_{k=1}^{\infty} \dot{u}_k^2 \leq \mathcal{E}_0, \tag{31}$$

which together with (30) yields, via the Cauchy-Schwarz inequality,

$$\sum_{j=1}^{\infty} j^2 \cdot u_j \cdot \dot{u}_j \leq \left\{ \left(\sum_{k=1}^{\infty} \dot{u}_k^2 \right) \cdot \left(\sum_{k=1}^{\infty} k^4 \cdot u_k^2 \right) \right\}^{1/2} \leq (\mathcal{E}_0[\mathcal{E}_0/b_2])^{1/2} = \mathcal{E}_0/(b_2)^{1/2}. \tag{32}$$

Finally, using (29) and (32),

$$\tilde{\Phi} \equiv b_1 \cdot \|\mathbf{x}\|^2 + b_4 \cdot (\mathbf{x} \cdot \dot{\mathbf{x}})^{2(n+\beta)+1} \leq b_1 \cdot R^2 + b_4 \cdot [\mathcal{E}_0/(b_2)^{1/2}]^{2(n+\beta)+1} \leq b_5 \cdot \mathcal{E}_0, \tag{33}$$

$$b_5 \equiv [b_1/(b_0 + b_2)] + b_4/(b_2)^{n+\beta+(1/2)}, \tag{34}$$

where we have used the assumption that

$$\mathcal{E}_0 < 1. \tag{35}$$

Now let $\mathbb{B}^N(\rho)$ and $\mathbb{B}^\infty(\rho)$ denote balls of radius ρ in $\mathbb{R}^N$ and $\mathbb{R}^\infty$ where $\mathbf{x} = \mathbf{u} \otimes \mathbf{v}$ is considered to be an element of the Cartesian product $\mathbb{R}^N \otimes \mathbb{R}^\infty$; we shall show that for all sufficiently large values of N, $\mathbb{B}^N(R_\varepsilon) \otimes \mathbb{B}^\infty(R_N)$ is a subset of the ball defined in $\mathbb{R}^N \otimes \mathbb{R}^\infty$ by (28), where $0 < \varepsilon < 1$ is an arbitrarily small positive constant, where

$$R_\varepsilon \equiv R \cdot (1 - \varepsilon)^{1/2}, \tag{36}$$

$$R_N \equiv (\mathcal{E}_0/b_2)^{1/2}/(N + 1), \tag{37}$$

and where

$$N + 1 \geq (\mathcal{E}_0/b_2)^{1/2}/(R\varepsilon^{1/2}). \tag{38}$$

In fact,

$$\|\mathbf{x}\|^2 \equiv \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \leq (R_\varepsilon)^2 + (R_N)^2 \leq R^2 \cdot (1 - \varepsilon) + R^2 \cdot \varepsilon \equiv R^2. \tag{39}$$

Initial Conditions

Define

$$\mathbf{a} \equiv (\, k \cdot \alpha_k) \in \mathbb{R}^N, \qquad \mathbf{a}^{(N)} \equiv (\, [N + k] \cdot \alpha_{N+k}) \in \mathbb{R}^\infty, \tag{40a}$$

and note that

$$\mathbf{a} = \mathbf{u}(0), \qquad \mathbf{a}^{(N)} = \mathbf{v}(0). \tag{40b}$$

Next, define

$$\mathbf{b} \equiv (\, \beta_k) \in \mathbb{R}^N, \qquad \mathbf{b}^{(N)} \equiv (\, \beta_{N+k}) \in \mathbb{R}^\infty, \tag{40c}$$

and note that

$$\mathbf{b} \;=\; \mathcal{D}^{-1}\cdot\dot{\mathbf{u}}(0), \qquad \mathbf{b}^{(N)} \;=\; \mathcal{D}_N^{-1}\cdot\dot{\mathbf{v}}(0). \tag{40d}$$

where the bounded linear operators $\mathcal{D}^{-1}$ and $\mathcal{D}_N^{-1}$ are represented by matrices

$$\mathcal{D}^{-1} \;=\; diag\ (\ 1/k\ ), \qquad \mathcal{D}_N^{-1} \;=\; diag\ (\ 1/[k + N]\ ). \tag{40e}$$

Resolution of Linear Part

In component form, the infinite ODE can be written, for ($k = 1, 2, 3, \cdots$), as

$$\ddot{u}_k + b_3 k^2\cdot\dot{u}_k + (b_0 + b_2 k^2)\cdot k^2\cdot u_k = -\left\{b_1 \sum_{j=1}^{\infty} j^2\cdot u_j^2 + b_4\left[\sum_{j=1}^{\infty} j^2\cdot u_j\cdot\dot{u}_j\right]^{2(n+\beta)+1}\right\} k^2\cdot u_k. \tag{41}$$

Alternatively, in vector notation

$$\mathcal{D}_0^{-1}\ddot{\mathbf{x}} + b_3\cdot\mathcal{D}_0\dot{\mathbf{x}} + (b_0 + b_2\cdot\mathcal{D}_0^2)\cdot\mathcal{D}_0\mathbf{x} \;=\; -\ \{\ b_1\cdot\|\mathbf{x}\|^2 + b_4\cdot(\mathbf{x}\cdot\dot{\mathbf{x}})^{2(n+\beta)+1}\ \}\cdot\mathcal{D}_0\mathbf{x}, \tag{42}$$

$$b_0 \;=\; \text{axial force coefficient,} \tag{43a}$$
$$b_1 \;=\; \text{extensibility coefficient,} \tag{43b}$$
$$b_2 \;=\; \text{elasticity coefficient,} \tag{43c}$$
$$b_3 \;=\; \text{viscous damping coefficient,} \tag{43d}$$
$$b_4 \;=\; \text{Balakrishnan-Taylor nonlinear damping coefficient,} \tag{43e}$$

where the unbounded linear operator $\mathcal{D}_0$ is defined by the infinite matrix obtained by taking $N = 0$ in the reciprocal of (40e). Upon multiplying (42) through by $\mathcal{D}_0^{-1}\dot{\mathbf{x}}$ and integrating, we obtain the energy integral (24) in the form

$$\|\mathcal{D}_0^{-1}\dot{\mathbf{x}}\|^2 + b_0\cdot\|\mathbf{x}\|^2 + (b_1/2)\cdot\|\mathbf{x}\|^4 + b_2\cdot\|\mathcal{D}_0\mathbf{x}\|^2 + 2\int_0^t \{b_3\cdot\|\dot{\mathbf{x}}\|^2 + b_4\cdot(\mathbf{x}\cdot\dot{\mathbf{x}})^{2(n+\beta+1)}\}\ dt \;\equiv$$

$$\equiv\; \mathcal{E}_0 \;\equiv\; \|\mathbf{b}^{\infty}\|^2 + b_0\cdot\|\mathbf{a}^{\infty}\|^2 + (b_1/2)\cdot\|\mathbf{a}^{\infty}\|^4 + b_2\cdot\|\mathcal{D}_0\mathbf{a}^{\infty}\|^2, \tag{44a}$$

where as before

$$\mathbf{a}^{\infty} \;=\; \mathbf{a}\otimes\mathbf{a}^N, \qquad \mathbf{b}^{\infty} \;=\; \mathbf{b}\otimes\mathbf{b}^N. \tag{44b}$$

For future use define ε_N and δ_N by

$$(N + 1)^2\cdot\|\mathbf{a}^N\|^2 \;\le\; \sum_{k=N+1}^{\infty} k^4\cdot\alpha_k^2 \;\equiv\; \varepsilon_N^2 \;\to\; 0, \qquad N \;\to\; +\infty, \tag{45a}$$

$$\|\mathbf{b}^N\|^2 \;=\; \sum_{k=N+1}^{\infty} \beta_k^2 \;\equiv\; \delta_N^2 \;\to\; 0, \qquad N \;\to\; +\infty, \tag{45b}$$

Homogeneous Linear Part

Henceforth we shall assume that the system is underdamped, i.e. that

$$b_3 \;<\; 2(b_2)^{1/2}. \tag{46}$$

The characteristic polynomial of the homogeneous part of (41) is

$$\lambda_k^2 + b_3 k^2 \cdot \lambda_k + (b_0 + b_2 k^2) \cdot k^2 = 0, \tag{47a}$$

which has roots

$$\lambda_k = -\mu_k \pm i \cdot \nu_k, \quad (i^2 = -1), \tag{47b}$$

$$\mu_k = k^2 \cdot \mu_0, \quad \nu_k = k^2 \cdot \omega_k, \tag{47c}$$

$$\mu_0 \equiv b_3/2, \quad \omega_k \equiv (1/2) \cdot \{4b_2 - (b_3)^2 + 4(b_0/k^2)\}^{1/2}, \tag{47d}$$

where, obviously,

$$\mu_0/\omega_k \leq b_3/\{4b_2 - (b_3)^2\}^{1/2}. \tag{47e}$$

From (47a) the general solution of the homogeneous linear part of (41) is of the form

$$u_k = exp(-\mu_k t) \cdot \{A_k \cdot cos(\nu_k t) + B_k \cdot sin(\nu_k t)\}, \tag{47f}$$

where A_k and B_k are arbitrary constants. Specializing these constants in order to use Lagrange's variation of constants formula to re-express (41) in terms of an impulse response convolution with the right-hand side we get

$$u_k(t) = d_k^{(1)}(t) \cdot \alpha_k + (d_k^{(2)}(t)/k) \cdot \beta_k - \int_0^t d_k^{(3)}(t - \tau) \, \Phi \, u_k(\tau) \, d\tau, \tag{47g}$$

where

$$d_k^{(1)}(t) \equiv exp(-\mu_k t) \cdot \{cos(\nu_k t) + (\mu_0/\omega_k) \cdot sin(\nu_k t)\}, \tag{48a}$$

$$d_k^{(2)}(t) \equiv exp(-\mu_k t) \cdot \{[1/(k \cdot \omega_k)] \cdot sin(\nu_k t)\}, \tag{48b}$$

$$d_k^{(3)}(t) \equiv k \cdot d_k^{(2)}(t). \tag{48c}$$

Next, define three finite and three infinite diagonal matrices as

$$D_j \equiv diag(d_k^{(j)}) : \mathbb{R}^N \to \mathbb{R}^N, \quad (j = 1, 2, 3), \quad (k = 1, 2, 3, \cdots, N), \tag{49a}$$

$$D_{j,N} \equiv diag(d_{k+N}^{(j)}) : \mathbb{R}^\infty \to \mathbb{R}^\infty, \quad (j = 1, 2, 3), \quad (k = 1, 2, 3, \cdots). \tag{49b}$$

Finally, multiply (47g) through by k in order to convert to vector notation:

$$\mathbf{x}(t) = D_{1,0}(t) \cdot \mathbf{a}^\infty + D_{2,0}(t) \cdot \mathbf{b}^\infty - \int_0^t D_{3,0}(t - \tau) \cdot \tilde{\Phi} \cdot \mathbf{x}(\tau) \, d\tau, \tag{50a}$$

$$\dot{\mathbf{x}}(t) = \dot{D}_{1,0}(t) \cdot \mathbf{a}^\infty + \dot{D}_{2,0}(t) \cdot \mathbf{b}^\infty - \int_0^t \dot{D}_{3,0}(t - \tau) \cdot \tilde{\Phi} \cdot \mathbf{x}(\tau) \, d\tau, \tag{50b}$$

$$\tilde{\Phi} \equiv b_1 \cdot \|\mathbf{x}\|^2 + b_4 \cdot (\mathbf{x} \cdot \dot{\mathbf{x}})^{2(n+\beta)+1}. \tag{50c}$$

(In deriving (50b) we used the fact that $D_{3,0}(0) \equiv 0$.) The fact that we have been able to reformulate the original PDE boundary-value and initial-value problem in the form (50) is the equivalent of (13), i.e the non-singular case, wherein there is no requirement for a Bifurcation Equation since the entire system now has the form of the Auxiliary Equation. If now we can prove that the iteration, for $m = 0, 1, 2, 3, \cdots$,

$$\mathbf{x}^{m+1}(t) = D_{1,0}(t)\cdot\mathbf{a}^{\infty} + D_{2,0}(t)\cdot\mathbf{b}^{\infty} - \int_0^t D_{3,0}(t-\tau)\cdot\tilde{\Phi}(\mathbf{x}^m,\dot{\mathbf{x}}^m)\cdot\mathbf{x}^m(\tau)\,d\tau, \qquad (51a)$$

$$\mathbf{x}^0(t) \equiv 0, \qquad \mathbf{x}^1(t) \equiv D_{1,0}(t)\cdot\mathbf{a}^{\infty} + D_{2,0}(t)\cdot\mathbf{b}^{\infty}, \qquad (51b)$$

converges, then we have constructed a solution of the original problem.

To avoid certain difficulties, we shall consider this iterative solution only for the problem

$$b_4 = 0, \quad \tilde{\Phi} = \tilde{\Phi}(\mathbf{x}) \equiv b_1\cdot\|\mathbf{x}\|^2; \qquad (52)$$

the more general problem will be approached by a non-constructive homotopy method. Consider now the fixed-point problem

$$\mathbf{x} = \mathcal{F}\{\mathbf{x}\}, \qquad \|\mathbf{x}\|_{\infty} \equiv \sup_{(t>0)} \|\mathbf{x}(t)\|, \qquad (53a)$$

$$\mathcal{F} \equiv \mathcal{F}\{\mathbf{x}(\cdot)\}(t) \equiv D_{1,0}(t)\cdot\mathbf{a}^{\infty} + D_{2,0}(t)\cdot\mathbf{b}^{\infty} - \int_0^t D_{3,0}(t-\tau)\cdot\tilde{\Phi}(\mathbf{x}(\tau))\cdot\mathbf{x}(\tau)\,d\tau. \qquad (53b)$$

We want to find a Lipschitz constant for $\mathcal{F}$, i.e. a constant κ such that

$$\|\mathcal{F}\{\mathbf{x}^{j+1}\} - \mathcal{F}\{\mathbf{x}^j\}\|_{\infty} \le \kappa\cdot\|\mathbf{x}^{j+1} - \mathbf{x}^j\|_{\infty} \quad \forall\ \mathbf{x}^j \in \mathbb{B}^{\infty}(R), \quad (j = 0, 1, 2, \cdots). \qquad (54)$$

Later we shall prove that

$$\int_0^{+\infty} \|D_{3,0}(\tau)\|\,d\tau \le b_6 \equiv 4/\{b_3[4b_2 - (b_3)^2]^{1/2}\}. \qquad (55)$$

In the Hilbert space norm of $\mathbb{R}^{\infty}$ it is clear that

$$\|\,\{\|\mathbf{x}^2\|^2\}\,\mathbf{x}^2 - \{\|\mathbf{x}^1\|^2\}\,\mathbf{x}^1\,\| \equiv \|\,\{\|\mathbf{x}^2\|^2\}\,(\mathbf{x}^2 - \mathbf{x}^1) + \{(\mathbf{x}^2 + \mathbf{x}^1)\cdot(\mathbf{x}^2 - \mathbf{x}^1)\}\,\mathbf{x}^1\,\| \le$$

$$\le \{\|\mathbf{x}^1\|^2 + \|\mathbf{x}^2\|^2 + \|\mathbf{x}^1\|\cdot\|\mathbf{x}^2\|\}\cdot\|\mathbf{x}^2 - \mathbf{x}^1\| \le 3R^2\cdot\|\mathbf{x}^2 - \mathbf{x}^1\|. \qquad (56)$$

Hence we may take

$$\kappa = 3R^2 b_1\cdot b_6 \le 3\{b_1\cdot b_6/[b_0 + b_2]\}\mathcal{E}_0 \le \vartheta < 1, \qquad (57a)$$

if $\mathcal{E}_0$ is taken to be sufficiently small that

$$\mathcal{E}_0 < (1/12)\cdot\vartheta\cdot\{(b_0 + b_2)\cdot b_3\cdot[4b_2 - (b_3)^2]^{1/2}\}/b_1. \qquad (57b)$$

The *a priori* bounds (28)-(29) apply to the first iterate (51b) and so the first iterate is inside the ball $\mathbb{B}^{\infty}(R)$; now, using (54) with $j = 0, 1, 2,$ etc. it is clear that $\mathbf{x}^2$, $\mathbf{x}^3$, all remain in the ball provided that $\|\mathbf{x}^1\|_{\infty} + \vartheta\cdot\|\mathbf{x}^1\|_{\infty} + \vartheta^2\cdot\|\mathbf{x}^1\|_{\infty} + \cdots < R$, i.e. summing the geometric series, provided that

$$\{\|\mathbf{x}^1\|_{\infty}/(1 - \vartheta)\} < R, \qquad (58a)$$

which is readily obtainable simply by taking ϑ so small that

$$\vartheta < 1 - (\|\mathbf{x}^1\|_{\infty}/R). \qquad (58b)$$

Now we can apply the well known principle of contraction mappings (also called the Banach fixed point theorem and Caccioppoli's fixed point theorem) to prove that the map (53) has a unique fixed point in the ball $\mathbb{B}^{\infty}(R)$ and that this fixed point can be computed constructively by the iterative procedure just described.

Next, replace the coefficient b_4 by μb_4, where $0 \le \mu \le 1$, and note that the *a priori* bounds previously derived remain valid for all values of $\mu \in [0,1]$. Hence we may infer the existence of a solution for all $\mu \in [0,1]$ by a homotopy method. For sufficiently small values of μ, this is Poincaré's method of analytic continuation of solutions of functional equations. Here the fact that the problem was non-singular at $\mu = 0$ implies that the Leray-Schauder Index [2], [3] or topological degree of the map is of magnitude unity at $\mu = 0$. Consequently the existence of the *a priori* bound for any solution on $0 \le \mu \le 1$ implies that there is a continuum of solutions connecting the solution at $\mu = 0$ with one at $\mu = 1$. Thus in summary we have proved the following result.

Existence Theorem

THEOREM. If $\bar{\phi}'$, $\bar{\phi}''$, and $\bar{\psi}$ [as defined in (24d-e)] are all sufficiently small, then the nonlinear functional PDE (16) has at least one solution which exists for all $t \ge 0$ and satisfies the boundary conditions (17a-b) and initial conditions (17e) as well as the *a priori* bounds

$$\frac{2}{L}\int_0^L [u_x(t,x)]^2\, dx \;\le\; R^2 \;\le\; \mathcal{E}_0/(b_0 + b_2), \tag{59}$$

$$\int_0^{+\infty}\left\{\frac{2}{L}\int_0^L [u_{xt}(t,x)]^2\, dx\right\} dt \;\le\; \mathcal{E}_0/(2b_3), \tag{60}$$

where

$$\mathcal{E}_0 \;\equiv\; \bar{\psi}^2 + b_0\cdot(\bar{\phi}')^2 + b_2\cdot(\bar{\phi}'')^2 + (b_1/2)\cdot\left[(\bar{\phi}')^2\right]^2. \tag{62}$$

Proof. Existence has already been proved. Equation (59) holds because the left-hand side of (59) is by (26) equal to $\|\mathbf{x}\|^2$ which in (28) is proved smaller than R^2. Equation (60) follows from inspection of (24a) and (24b), which hold for all $t \ge 0$, making it permissible to let $t \to +\infty$.

Rigorous Truncation

In "naive truncation" one simply sets $\mathbf{v} \equiv 0$, i.e. in (41) one takes

$$u_k(t) \;\equiv\; 0, \quad (k = N + 1,\ N + 2,\ N + 3,\ \cdots). \tag{63}$$

Here we shall prove that the solution proved above to exist can be derived as in the Liapunov-Schmidt bifurcation theory, and that, for all $t \ge 0$,

$$\|\mathbf{v}(t)\| \;\le\; R_N\cdot exp(-b_3[N + 1]^2\cdot t/2) \;\le\; R\varepsilon^{1/2}, \tag{64}$$

provided that N is taken larger than the lower bounds in (38) and in (68b)-(69) below.

By inspection of (50), we can express the problem as follows.

Bifurcation Equation

$$\mathbf{u}(t) \;=\; D_1(t)\cdot\mathbf{a} + D_2(t)\cdot\mathbf{b} - \int_0^t D_3(t - \tau)\cdot\tilde{\tilde{\Phi}}\cdot\mathbf{u}(\tau)\, d\tau, \tag{65a}$$

$$\tilde{\tilde{\Phi}} \equiv b_1\cdot\{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\} + b_4\cdot(\mathbf{u}\cdot\dot{\mathbf{u}} + \mathbf{v}\cdot\dot{\mathbf{v}})^{2(n+\beta)+1}. \tag{65b}$$

Auxiliary Equation

$$\mathbf{v}(t) = D_{1,N}(t)\cdot\mathbf{a}^N + D_{2,N}(t)\cdot\mathbf{b}^N - \int_0^t D_{3,N}(t-\tau)\cdot\tilde{\tilde{\Phi}}\cdot\mathbf{v}(\tau)\,d\tau. \tag{65c}$$

More A Priori Bounds

From (48a),

$$|d_k^{(1)}| \le \{1 + (\mu_0/\omega_k)^2\}^{1/2}\cdot exp(-\mu_0 k^2 t) \le \{1 + b_3^2/[4b_2 - (b_3)^2]\}^{1/2}\cdot exp(-b_3 t/2) \equiv$$

$$\equiv [1/\{1 - (b_3^2/4b_2)\}^{1/2}]\cdot exp(-b_3 t/2). \tag{66a}$$

The induced Euclidean norm of any diagonal matrix is equal to the absolute value of its (absolutely) largest element. Hence

$$\|D_1\| \le [1/\{1 - (b_3^2/4b_2)\}^{1/2}]\cdot exp(-b_3 t/2). \tag{66b}$$

Similarly,

$$\|D_2\| \le \max_{1\le k\le N} |d_k^{(2)}| \le \max_{1\le k\le N} \{1/(k\omega_k)\}\cdot exp(-b_3 t/2) \le$$

$$\le [2/\{4b_2 - (b_3^2)\}^{1/2}]\cdot exp(-b_3 t/2), \tag{66c}$$

$$\|D_3\| \le \max_{1\le k\le N} \{1/(k\omega_k)\}\cdot exp(-b_3 t/2) \le [2/\{4b_2 - (b_3^2)\}^{1/2}]\cdot exp(-b_3 t/2), \tag{66d}$$

$$\|D_{1,N}\| \le \{1/[1 - (b_3^2/4b_2)]^{1/2}\}\cdot exp(-b_3[N+1]^2 t/2), \tag{66e}$$

$$\|D_{2,N}\| \le \{[2/\{4b_2 - (b_3^2)\}^{1/2}]/(N+1)\}\cdot exp(-b_3[N+1]^2 t/2). \tag{66f}$$

$$\|D_{3,N}\| \le [2/\{4b_2 - (b_3^2)\}^{1/2}]\cdot exp(-b_3[N+1]^2 t/2). \tag{66g}$$

Integrating (66d), one obtains

$$\int_0^{+\infty} \|D_3(\tau)\|\,d\tau \le b_6 \equiv 4/\{b_3[4b_2 - (b_3)^2]^{1/2}\}. \tag{66h}$$

Similarly,

$$\int_0^{+\infty} \|D_{3,N}(\tau)\|\,d\tau \le b_6/(N+1)^2, \tag{66i}$$

which we have used above in (55) in the case $N = 0$.

Next, from (65c), (66e,f,g), and (45a,b),

$$\|\mathbf{v}(t)\| \le \gamma_N\cdot exp(-b_3[N+1]^2 t/2) + \{b_5 b_6 \mathcal{E}_0/(N+1)^2\}\cdot\|\mathbf{v}(t)\|, \tag{67a}$$

where

$$\gamma_N \equiv \{1/[1 - (b_3^2/4b_2)]^{1/2}\}\|\mathbf{a}^N\| + \{[2/\{4b_2 - (b_3^2)\}^{1/2}]/(N+1)\}\|\mathbf{b}^N\| \le$$

$$\leq \left(\{1/[1 - (b_3^2/4b_2)]^{1/2}\}\cdot\varepsilon_N + [2/\{4b_2 - (b_3^2)\}^{1/2}]\cdot\delta_N \right)/(N + 1) \equiv \sigma_N/(N + 1). \tag{67b}$$

Consequently, if $\mathcal{E}_0$ is so small that

$$b_6 b_5 \mathcal{E}_0 < 1/2, \tag{68a}$$

i.e. so small that

$$\mathcal{E}_0 < 1/\left(\{8/(b_3[4b_2 - (b_3)^2]^{1/2}\}\cdot\{[b_1/(b_0 + b_2)] + b_4/(b_2)^{n+\beta+(1/2)}\} \right), \tag{68b}$$

then we may subtract the second term on the right hand side of (67a) from the left hand side, and then, by (45a,b), making N still larger (if necessary) so that

$$\sigma_N/\{1 - (1/2)]\} \equiv 2\sigma_N \leq (\mathcal{E}_0/b_2)^{1/2} \equiv (N + 1)\cdot R_N \tag{69}$$

we can ensure that, for all $t \geq 0$,

$$\|\mathbf{v}(t)\| \leq R_N\cdot exp(-b_3[N + 1]^2\cdot t/2) \leq R_N \leq R\cdot\varepsilon^{1/2}, \tag{70}$$

as claimed in (64).

Quite similarly, from (65a) and (68a),

$$\|\mathbf{u}\| \leq \|D_1\|\cdot\|\mathbf{a}\| + \|D_2\|\cdot\|\mathbf{b}\| + (b_6 b_5 \mathcal{E}_0)\cdot\|\mathbf{u}\| \leq \|D_1\|\cdot\|\mathbf{a}\| + \|D_2\|\cdot\|\mathbf{b}\| + (1/2)\cdot\|\mathbf{u}\|, \tag{71a}$$

whence, precisely as before,

$$\{1 - (1/2)\}\cdot\|\mathbf{u}\| \leq \|D_1\|\cdot\|\mathbf{a}\| + \|D_2\|\cdot\|\mathbf{b}\|, \tag{71b}$$

i.e.

$$\|\mathbf{u}\| \leq 2\cdot\{\|D_1\|\cdot\|\mathbf{a}\| + \|D_2\|\cdot\|\mathbf{b}\|\} \leq R_\varepsilon, \tag{71c}$$

provided only that, by (66b,c)

$$\|\mathbf{a}\| \leq (1/4)\cdot\{1 - (b_3^2/4b_2)\}^{1/2}\cdot R_\varepsilon, \tag{72a}$$

$$\|\mathbf{b}\| \leq (1/8)\cdot\{4b_2 - (b_3^2)\}^{1/2}\cdot R_\varepsilon, \tag{72a}$$

which can be assured simply by taking $\bar{\phi}'$ and $\bar{\psi}$ small enough.

Recalling that, by (62), $\mathcal{E}_0$ can be made arbitrarily small by making the initial mean-square spatial rates of change $\bar{\phi}'$, $\bar{\phi}''$, and $\bar{\psi}$ of u_x, u_{xx} and u_t arbitrarily small, we may conclude that when those initial rates are sufficiently small then every solution of (65a,b) must, for all $t \geq 0$, satisfy *a priori*

$$\|\mathbf{u}(t)\| \leq R_\varepsilon \equiv R\cdot(1 - \varepsilon)^{1/2}, \tag{73a}$$

and every solution of (65b,c) must, for all $t \geq 0$, satisfy *a priori*

$$\|\mathbf{v}(t)\| \leq R_N \leq R\cdot\varepsilon^{1/2}, \tag{73b}$$

whence finally, as in (39), every solution of the combined systems (65a,b,c) must for all $t \geq 0$, satisfy *a priori*

$$\|\mathbf{x}\| \equiv \{\|\mathbf{u}(t)\|^2 + \|\mathbf{v}(t)\|^2\}^{1/2} \leq R. \tag{73c}$$

Conclusion: Naive vs Rigorous Truncation

Let $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$ be arbitrarily given continuous N-vector functions and insert them into the nonlinear function $\tilde{\tilde{\Phi}}$ defined in (65b), and then insert this functional of $\mathbf{v}$

only into (65c), giving an infinite system of integral equations for $\mathbf{v}(t)$. As in (28) through (35), the nonlinear term can be made arbitrarily small by taking $\mathcal{E}_0$ arbitrarily small. Similarly the nonlinear term can be made to have a Lipschitz constant less than unity by restrictions upon $\mathcal{E}_0$ as in (53)-(58).

This infinite system of Auxiliary Equations can be solved by one of the methods illustrated above (iteration or homotopy), and the result inserted into the Bifurcation Equations (65a) to provide a *finite-dimensional* system of functional integral equations *exactly-equivalent* to the the original infinite-dimensional system.

When the arbitrarily given $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$ in the Auxiliary Equations (65b,c) are taken to be the projection into $\mathbb{R}^N$ of the solution $\mathbf{x}(t) \in \mathbb{R}^N \otimes \mathbb{R}^\infty$ proved to exist in the Theorem concerning (59)-(62), then the Auxiliary Equations have a solution corresponding to the projection $\mathbf{v}(t)$ of $\mathbf{x}(t)$ into $\mathbb{R}^\infty$. The resulting Bifurcation Equations then must be satisfied by the same $\mathbf{u}(t)$ used to define the Auxiliary Equations. However, all of the *a priori* bounds proved above to apply to the solution $\mathbf{x}$ of the complete problem now apply to the projections $\mathbf{u}$, $\mathbf{v}$ of the rigorously truncated problem.

Consequently we can compare the rigorous version of the Bifurcation Equations, namely (65a), with the naively truncated version wherein one sets $\mathbf{v} \equiv 0$, and note that that, as $\mathcal{E}_0$ becomes sufficiently small for the bounds to apply, then as N becomes arbitrarily large the difference between the solutions of the naively truncated version of (65a) and its rigorously truncated version becomes arbitrarily small.

Consequently in attempting to solve the given nonlinear function PDE boundary-value initial-value problem, we may be confident that if we truncate naively for some finite N, the results become arbitrarily accurate as N increases without limit provided that the initial mean-square spatial rates of change $\bar{\phi}'$, $\bar{\phi}''$, and $\bar{\psi}$ of u_x, u_{xx} and u_t are kept sufficiently small.

Further research is needed in order to ascertain whether or not this conclusion would still apply if the homogeneous boundary conditions were replaced by inhomogeneous boundary conditions corresponding to a finite number of actuators.

REFERENCES

Apology: We have mislaid a reprint which treats the above problem in the case wherein $b_4 \equiv 0$ by the present approach in a manner equivalent to (23a) or (41). Although we believe that in this case some of the details above may be novel, this paper should be our **primary reference** and we feel abjectly apologetic to the author and to the journal in question and would appreciate being reminded of this reference by any reader who recognizes it in order to rectify this injustice in a future paper.

Acknowledgements: We are deeply grateful to Dr. A.V. Balakrishnan and to Dr. L. W. Taylor, Jr. for having brought reference [1] to our attention. Also R.W. Bass thanks NASA-Ames Dryden Flight Research Center's support of the UCLA Flight Systems Research Lab (EE Dept.), where an earlier version of this research was done by him.

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