COUPLED RICCATI EQUATIONS FOR
COMPLEX PLANE CONSTRAINT

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ABSTRACT

A new LQG design method is presented which provides prescribed imaginary-axis pole placement for optimal control and estimation systems. This procedure contributes another degree of design freedom to flexible spacecraft control: Current design methods which interject modal damping into the system tend to have little affect on modal frequencies, i.e. they predictably shift open-loop plant poles horizontally in the complex plane to form the closed-loop controller or estimator pole constellation, but make little provision for vertical (imaginary-axis) pole shifts. Imaginary-axis shifts which reduce the closed-loop modal frequencies (the bandwidth) are desirable since they reduce the sensitivity of the system to noise disturbances. The new method drives the closed-loop modal frequencies to predictable (specified) levels--frequencies as low as zero rad/sec (real-axis pole placement) can be achieved. The design procedure works through rotational and translational destabilizations of the plant, and a coupling of two *independently*-solved algebraic Riccati equations through a structured state-weighting matrix. Two new concepts, *gain transference* and *Q-equivalency*, are introduced and employed in the design process.

1. INTRODUCTION

Multi-input, multi-output systems, such as those encountered in flexible spacecraft control, are often approached with modern optimal control techniques which conveniently generate closed-loop system gain matrices for simultaneous multi-loop closures. However, modern optimal control, as presented in most textbooks, is not a complete control system design methodology. The major problems of translating control system performance requirements, bandwidth constraints, and compensator robustness constraints into the performance index have not been fully developed [1]. The result is a control system design methodology that is iterative and empirical. An approach to solving these problems and de-empiricizing the design process is to use structured performance index (SPI) constraints [2]. SPI constraints may be defined as structured performance index weighting matrices which constrain the weighted variables to approach desired predefined directions and values in the state space as the weighting matrix entries approach infinity. This is in contrast to generalized constraints for which the weighted variables approach zero as the weighting matrix entries approach infinity. To employ structured constraints, and avoid the application of generalized constraints, the weighting matrices for the SPI must be less than full rank. The potential usefulness of the SPI approach is apparent: An appropriately-structured performance index can

drive state variables in predictable directions thereby achieving a desired performance and bandwidth objective. SPI's can provide a non-empirical means of constraining the controller, estimator, and compensator dynamics--the latter is critical for closed-loop system robustness.
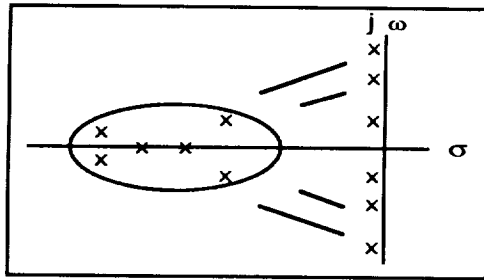

Figure 1. Design Objective: Complex Plane Constraint

The overall design objective for the controller/estimator utilizing the SPI approach is illustrated in Fig. 1: The gain matrices generated through the SPI translate the open-loop poles to some prescribed closed-loop boundary in the complex plane. Similarly the compensation dynamics are constrained to reduce closed-loop system sensitivity. Note that to achieve this objective, two degree-of-freedom control is required over each controlled mode, i.e. poles require movement in two dimensions in the complex plane, both horizontally (along the real-axis) and vertically (along the imaginary-axis).

In the next section we review SPI design methods for prescribed real-axis constraint in the optimal control, estimation, and compensation systems, and introduce a coupled Riccati equation design technique for prescribed imaginary-axis constraint.

2. SPI DESIGN METHODS

Currently a well-known performance index exists for prescribed *real-axis* pole translations in the optimal controller and estimator systems [1]: In the "alpha-shift" technique shown in Fig. 2, the standard Linear Quadratic Gaussian (LQG) performance index is augmented with an exponential weighting. This exponential weighting guarantees that the quadratic terms in the performance index decay with at least a rate of 2α so that the performance index remains finite over the infinite interval. The result is a guaranteed stability margin--all closed-loop poles lie to the left of the -2α-line in the complex plane.

The design procedure with the alpha-shift technique is straight-forward: $[+\alpha I]$ is appended to the nominal plant dynamics, A. This tends to destabilize the plant. Optimal control theory is applied
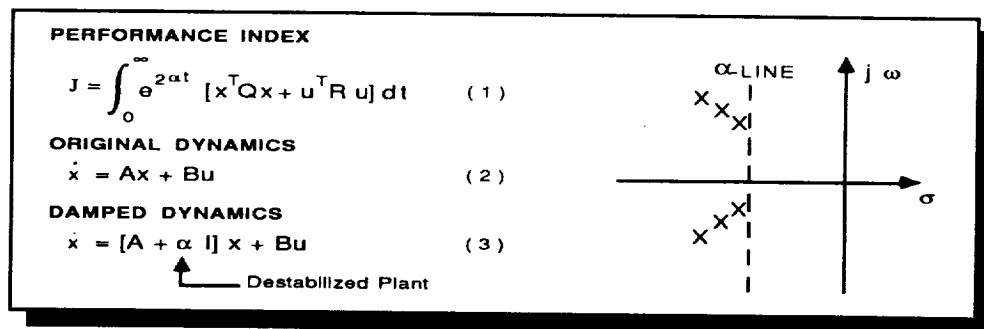

Figure 2. The Alpha-Shift Technique Provides a Prescribed Real-Axis Pole Translations [Anderson & Moore]

and state feedback gains are generated for the destabilized plant, characterized by [A+αI], which are guaranteed to stabilize it. When these gains are applied to the *nominal* plant, A, the closed-loop poles have real parts of -2α. This technique provides horizontal (real-axis) translation of the plant poles from their open to closed-loop positions.

For the compensator, predictable *real-axis* pole translations are also possible through *indirect* SPI design techniques which structure control and observation constraints [3]. These constraints tend to normalize the control and observation effort thereby providing indirect control over compensator poles, bandwidth, and closed-loop singular values.

Prescribed *imaginary-axis* pole translations in the optimal control and estimation systems are the focus of this paper: SPI design techniques are presented which drive the modal frequencies of the closed-loop system to desired levels. *Conceptually*, prescribed *imaginary-axis* pole placement may be considered to be composed of a 90 degree rotation, a vertical translation, and a stabilization of the open-loop plant poles as shown in Fig. 3. Stabilization is achieved by generating a stabilization matrix for the plant in rotated space and applying it to a standard alpha-shift

design through a SPI. Using the stabilization matrix from one optimal design process and applying it to another couples two algebraic Riccati equations (ARE's) together.

The next section introduces two key concepts, gain transference and Q-equivalency, that are critical to the development of the SPI for Riccati equation coupling. This is followed by an outline of the actual design steps required for prescribed imaginary-axis pole placement.

3. DESIGN PROCEDURE: PRESCRIBED IMAGINARY-AXIS POLE PLACEMENT

The design procedure for prescribed imaginary-axis pole placement employs a SPI that couples two ARE's together. Gain transference and Q-equivalency are important to understanding the development of this SPI.

Gain transference involves designing optimal gains for one plant and applying them to another, indirectly-related, plant. As shown in Fig. 4, optimal regulator theory is applied to system 1, generating optimal gains $-R^{-1}B^TP_1$. A closed-loop state feedback system is formed for system 2 with these gains. (Note that system
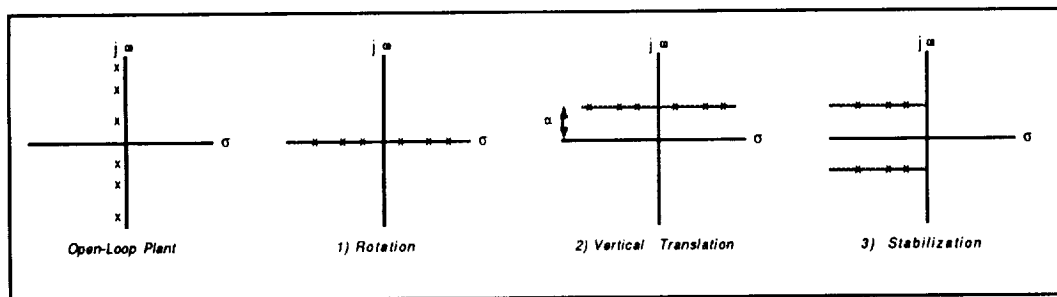

| Open-Loop Plant | 1) Rotation | 2) Vertical Translation | 3) Stabilization |

Figure 3. Conceptual Development: Prescribed Imaginary-Axis Pole Placement

System 1

$\dot{x} = A_1 x + B_1 u$ →(Optimal Regulator Theory)

Optimal
Regulator
Theory

Algebraic Riccati Equation

$A^T P_1 + P_1 A - P_1 B_1 R^{-1} B_1^T P_1 + Q = 0$ →

State Feedback

$u = -R^{-1} B_1 P_1 x$

System 2

$\dot{x} = A_2 x + B_1 u$ →

Gain Transference from System 1
to form Closed-Loop for System 2 →

Closed-Loop System

$\dot{x} = A_2 x - B_1 R^{-1} B_1^T P_1 x$
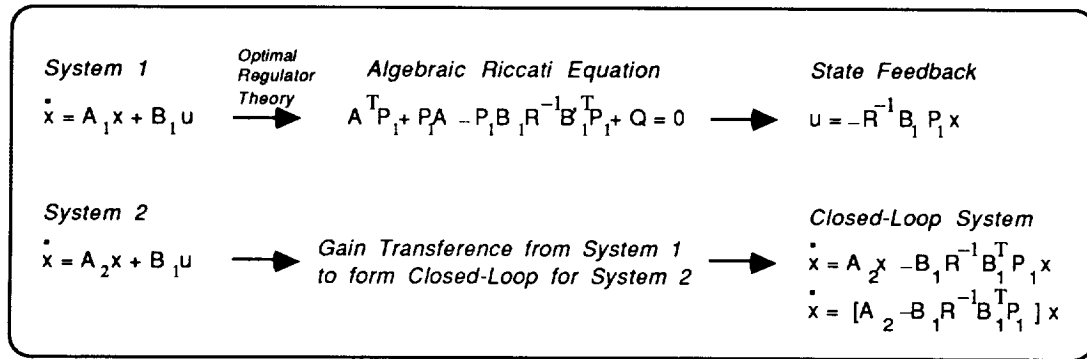
$\dot{x} = [A_2 - B_1 R^{-1} B_1^T P_1] x$

Figure 4. Gain Transference Theory

2 has identical input vectors as system 1.) The optimal gains generated for system 1 have been transferred to system 2 to form the closed-loop system.

The utility of gain transference lies in its harmonic-restructuring capability. The harmonic structure of the closed-loop system $[A_2 - B_1 R^{-1} B_1^T P_1]$ can be strongly influenced by the harmonic structure of A_1. In the design procedure for prescribed imaginary-axis pole placement, optimal gains, P_1, are generated for the plant in a rotated space, A_1. When these gains are transferred to the nominal plant, A_2, for state feedback, the closed-loop system takes on the harmonic characteristics of the plant in a rotated space. The state feedback transforms the nominal plant to rotational space--a key step in achieving prescribed imaginary-axis pole placement.

Q-equivalency, the other concept central to the design procedure, involves expanding and collecting terms in an ARE to indirectly generate a state-weighting matrix. An example of the concept is shown in Eq. 1 for the ARE employed in the alpha-shift technique.

$(A+\alpha I)^T P + P(A+\alpha I) - PBR^{-1} B^T P + Q = 0$

$A^T P + PA - PBR^{-1} B^T P + 2\alpha IP = 0$

$Q_{eq} = 2\alpha IP$

$\quad = - (A^T P + PA - PBR^{-1} B^T P) \qquad (1)$

The alpha terms are expanded and collected to form a Q-equivalent matrix equal to $2\alpha IP$. In a SPI, Q_{eq} is a state-weighting matrix that will generate the same optimal gains for the *nominal* plant, as those generated through the ARE for the *alpha-shifted* plant. This concept is used in the design procedure to couple two ARE's together: *A Q_{eq} matrix for the ARE in rotational space is used as the state-weighting matrix for an ARE in translational (alpha-shifted) space.*

An overview of the actual design steps that employ the concepts of gain transference and Q-equivalency are illustrated in Fig. 5 and described below:

1) Rotational Plant Destabilization. A simple matrix transformation of the plant rotates poles circularly from their open-loop positions to the real-axis. This removes all harmonic components from the rotated plant dynamics. Half of the rotated
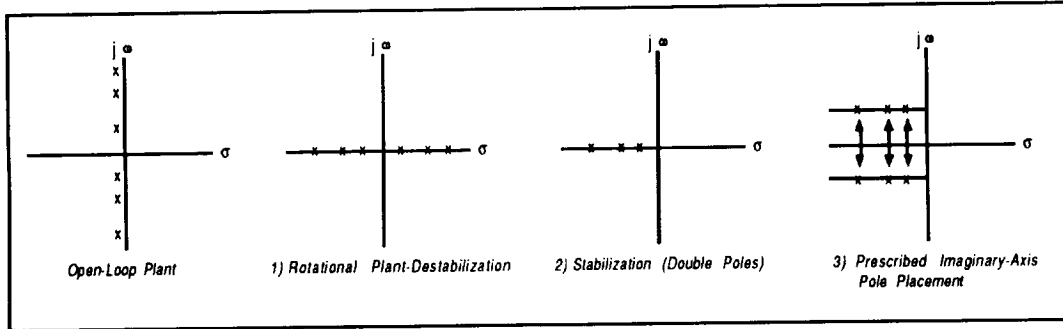
Figure 5. The Three Design Steps for Imaginary-Axis Pole Placement

plant poles in this space are unstable.

2) *Rotational Plant Stabilization.* A stabilization matrix is generated for the rotationally-destabilized plant through a SPI using standard optimal regulator design methods. Unstable right-hand-plane (RHP) poles at $\{+\omega_1, +\omega_2,....,+\omega_n\}$ are moved to the left-hand-plane (LHP) to positions of $\{-\omega_1, -\omega_2,....,-\omega_n\}$, respectively. The stabilization matrix does not affect the stable LHP poles at $\{-\omega_1, -\omega_2,....,-\omega_n\}$. The resulting closed-loop system has *double poles* at each modal frequency in the LHP.

3) *Prescribed Imaginary-Axis Pole Placement.* The stabilization matrix generated for the rotationally-destabilized plant is used in a SPI to transform an alpha-shift design to rotational space. The value of α determines the closed-loop modal frequencies, i.e. α prescribes the amount of imaginary-axis pole translation from the *real-axis*.

We now present details of the prescribed imaginary-axis pole placement design procedure for optimal control and estimation systems. The three steps outlined above are expanded and applied to a low-order system to illustrate their

effects. The complete design procedure is then developed and its application to flexible spacecraft control is illustrated in a numerical example.

DESIGN STEP 1: Rotational Plant-Destabilization

To introduce rotational plant-destabilization we compare it graphically to the alpha-shift technique. As shown in Fig. 6, alpha-shifted plant-destabilization is accomplished via a horizontal translation of the poles into the right-half of the complex plane. Rotational plant destabilization occurs with circular rotations of the open-loop poles to the real-axis.
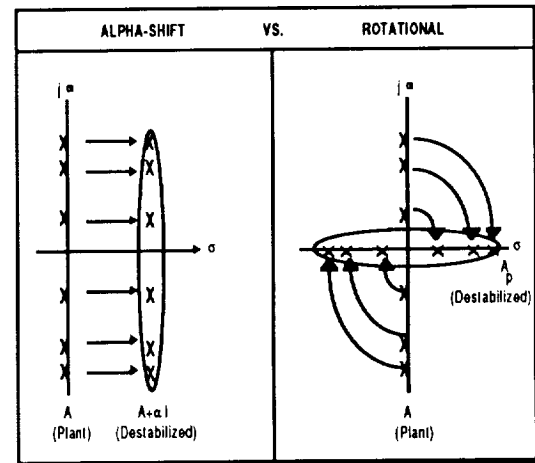

Figure 6. Plant Destabilization Techniques

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The transformation matrix that accomplishes this destabilization of the plant is described below for an n-mode system. If the plant matrix, A, in block-diagonal form is:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & 0 \\ & & 0 & 1 \\ & & -\omega_2^2 & 0 \\ & & & & \ddots \\ & & & & & 0 & 1 \\ & & & & & -\omega_n^2 & 0 \end{bmatrix}_{2n \times 2n}$$

then the rotationally-destabilized plant matrix, A_r, is defined to be:

$$A_r \overset{\Delta}{=} A\, T$$

where

$$T = - \begin{bmatrix} 0 & 1/\omega_1 \\ \omega_1 & 0 \\ & & 0 & 1/\omega_2 \\ & & \omega_2 & 0 \\ & & & & \ddots \\ & & & & & 0 & 1/\omega_n \\ & & & & & \omega_n & 0 \end{bmatrix}$$

and

$$A_r = \begin{bmatrix} -\omega_1 \\ & \omega_1 \\ & & -\omega_2 \\ & & & \omega_2 \\ & & & & \ddots \\ & & & & & -\omega_n \\ & & & & & & \omega_n \end{bmatrix}$$

with eigenvalues $\lambda(\pm\omega_1, \pm\omega_2, ..., \pm\omega_n)$. We note that this set of eigenvalues is not unique to A_r. Other transformations of the plant will produce *block-diagonal* or *off-diagonal* matrices with equivalent eigenvalue sets. These matrices may be used in lieu of A_r in designing the controllers and estimators, but the closed-loop system bandwidths tend to be larger than those designed with A_r. A_r's diagonal structure provides the smaller bandwidth controller for this design procedure, and we use this structure in the following two-mode example

illustrating rotational destabilization, and in the flexible spacecraft example at end of the paper.

Given System: $\omega_1 = 10$ rad/sec
                $\omega_2 = 15$ rad/sec

One actuator and collocated sensor corresponding to torque actuation and velocity sensing are employed.

$$A = \begin{bmatrix} 0 & 1 \\ -100 & 0 \\ & & 0 & 1 \\ & & -225 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ .3 \\ 0 \\ .4 \end{bmatrix}$$

Transformation for A:

$$T = - \begin{bmatrix} 0 & 1/10 & 0 & 0 \\ 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/15 \\ 0 & 0 & 15 & 0 \end{bmatrix}$$

Rotationally-Destabilized Plant:

$$A_r = AT = \begin{bmatrix} -10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & -15 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

Note that half the eigenvalues of A_r are unstable. Optimal regulator design theory may now be used to stabilize the rotationally-destabilized plant.

DESIGN STEP 2: Stabilization of Rotational Plant

In this intermediate step, optimal regulator theory is applied to the plant in rotational space. The optimal gains that are generated will stabilize the rotationally-destabilized plant, but their prac-

tical function in this design algorithm is to structure a performance index that will rotate and stabilize an alpha-shift design for the *nominal* plant. The motivation for using the gains in this way came from applying them to the *nominal* plant and observing their significant effect: They eliminate harmonic components from the closed-loop system poles. State feedback with this set of optimal gains rotates the open-loop plant poles to the real-axis. This suggests that a SPI employing a feedback structure with these optimal gains as its state-weighting matrix can produce rotation and stabilization of a prescribed damping design.

An example of rotational stabilization is now presented for the low-order system used previously. The rotationally-destabilized plant matrix, A_r, developed in step 1, becomes a parametric design matrix in the algebraic Ricatti equation (ARE), i.e. the nominal plant, A, is replaced with A_r in the ARE. We note also that the parametric design matrix Q is set equal to zero in the example. This results in *double* poles in the closed-loop systems. Double poles are not mandatory. Alternative selections of Q may include an identity matrix which will split the closed-loop system poles, but still maintain them in the LHP. Positive scaling of the identity matrix will provide as much separation of the poles as desired. Negative scaling of the identity matrix Q adds harmonics to the closed-loop system and can be used, if desired, to obtain an additional increase in the modal frequencies in the final design step, or to decrease the optimal gains. Other structurings of the Q matrix are currently being evaluated for their closed-loop system effects. All examples in this paper employ a zero matrix Q which

produces the double-pole structuring in the intermediate closed-loop systems. We now perform the rotational stabilization:

Algebraic Riccati Equation:
$$A^T P_1 + P_1 A - P_1 B R^{-1} B^T P_1 + Q = 0$$

Parametric Design Matrices:
$$A = A_r$$
$$R = 1$$
$$Q = [0]_{4 \times 4}$$

Intermediate Riccati Solution for A_r:

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5.5556E3 & 0 & -5.0000E3 \\ 0 & 0 & 0 & 0 \\ 0 & -5.0000E3 & 0 & 4.6875E3 \end{bmatrix}$$

Eigenvalues of Intermediate Closed-Loop Systems:
$$\lambda(A_r - BR^{-1}B^T P_1) = \{-10, -10, -15, -15\}$$
$$\lambda(A - BR^{-1}B^T P_1) = \{+5, +5, -30, -30\}$$

Note that P_1 is sparse and singular. Also, as indicated earlier, all harmonic components are completely eliminated from the closed-loop design model when the optimal gains are applied to the *nominal* plant, i.e. all poles have imaginary parts equal to zero. The real parts are positive or negative values which typically have values given by one-half or two times the modal frequencies. This intermediate closed-loop system must now be stabilized in a final design step with an additional algebraic Ricatti equa-tion which will also add a prescribed degree of harmonics to the closed-loop system.

DESIGN STEP 3: Prescribed Imaginary-Axis Pole Placement

In this section we develop the SPI that is employed to design optimal controllers and estimators

with prescribed (closed-loop) modal frequencies. The exponentially-weighted performance index of the alpha-shift technique is modified with the optimal gains of the rotational plant stabilization step. The modification of one SPI with the optimal gains from another results in a coupling of two *independently*-solved ARE's. As developed below, the coupling occurs through the ARE parametric matrix, Q.

The optimal gains generated in the rotational plant stabilization step are used to structure the parametric matrix, Q. "Q-equivalency", the expansion and collection of terms in an ARE to indirectly generate a state-weighting matrix, is used to structure Q. A Q_{cq} equation, parallel to that shown in Eq. 1, is developed for the rotationally-destabilized ARE in Eq. 2. (The unity subscripts indicate that this is the first ARE that is solved in the design algorithm.)

It is Q_{eq1} that is used to modify the exponentially-weighted performance index used in alpha-shift designs. The modified performance index and its accompanying ARE are shown in Eqs. 3a, 3b, and 4a

respectively. Q_{eq1} transforms the alpha-shift design to rotational space. After rotation, the alpha parameter prescribes the amount of imaginary-axis pole translation that is desired from the *real-axis*.

A Q_{eq} may be developed for the modified ARE: Terms in Eq. 4a are expanded and collected as shown in Eq. 4b. Eq. 4c is formed by substitution of Q_{eq1} and defining $Q_{eq2} = 2\alpha I P_2$. Q_{eq} then is the sum of two terms--Q_{eq1} from the rotationally-destabilized ARE and Q_{eq2} from the alpha-shift design as shown in Eq. 5. If $Q_{eq2} \gg Q_{eq1}$, i.e. if α is large relative to the modal frequencies, then the alpha-shift term will dominate, and the imaginary-parts of the poles will *asymptotically* approach the desired alpha value.

We now demonstrate pre-scribed imaginary-axis pole placement for the low-order system used previously. The optimal gains, P_1, designed under step 2 are used to form Q_{eq1} and modify the performance index for three alpha-shift designs: $\{\alpha=0, \alpha=1, \text{ and } \alpha=2\}$.

$$A_r^T P_1 + P_1 A_r - P_1 B R^{-1} B^T P_1 = 0$$

$$- P_1 B R^{-1} B^T P_1 + (A_r P_1 + P_1 A_r) = 0$$

$$Q_{cq1} = (A_r P_1 + P_1 A_r) = P_1 B R^{-1} B^T P_1 \qquad (2)$$

$$J_2 = \int_0^\infty e^{2\alpha\tau} [x^T Q_{eq1} x + u^T R u] \, dt \qquad (3a)$$

$$J_2 = \int_0^\infty e^{2\alpha\tau} [x^T P_1 B R^{-1} B^T P_1 x + u^T R u] \, dt \qquad (3b)$$

$$(A+\alpha I)^T P_2 + P_2 (A+\alpha I) - P_2 B R^{-1} B^T P_2 + P_1 B R^{-1} B^T P_1 = 0 \qquad (4a)$$

$$A^T P_2 + P_2 A - P_2 B R^{-1} B^T P_2 + P_1 B R^{-1} B^T P_1 + 2\alpha I P_2 = 0 \qquad (4b)$$

$$A^T P_2 + P_2 A - P_2 B R^{-1} B^T P_2 + Q_{eq1} + Q_{eq2} = 0 \qquad (4c)$$

$$Q_{eq} = Q_{eq1} + Q_{eq2} = P_1 B R^{-1} B^T P_1 + 2\alpha I P_2 \qquad (5)$$

Modified ARE: $(A+\alpha I)^T P_2 + P_2(A+\alpha I) - P_2 BR^{-1}B^T P_2 + P_1 BR^{-1}B^T P_1 = 0$

Q_{eq1} Formed From Optimal Gains P_1:

$$Q_{eq1} = P_1 BR^{-1}B^T P_1 = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1.1111E5 & 0 & -1.2500E5 \\ 0 & 0 & 0 & 0 \\ 0 & -1.2500E5 & 0 & 1.4062E5 \end{matrix}$$

Optimal Gains P_2 for Design 1: $\alpha=0$

6.2844E5	-5.1200E4	-1.0080E6	3.0400E4
-5.1200E4	9.2711E3	1.0440E5	-6.7200E3
-1.0080E6	1.0440E5	1.7601E6	-6.4800E4
3.0400E4	-6.7200E3	-6.4800E4	5.3025E3

Optimal Gains P_2 for Design 2: $\alpha=1$

7.1690E5	-5.2195E4	-1.1158E6	3.0532E4
-5.2195E4	1.2205E4	1.2129E5	-8.7124E3
-1.1158E6	1.2129E5	1.9794E6	-7.5562E4
3.0532E4	-8.7124E3	-7.5562E4	6.6674E3

Optimal Gains P_2 for Design 3: $\alpha=2$

8.2904E5	-4.8182E4	-1.2239E6	2.7139E4
-4.8182E4	1.6620E4	1.4013E5	-1.1753E4
-1.2239E6	1.4013E5	2.2091E6	-8.7720E4
2.7139E4	-1.1753E4	-8.7720E4	8.7729E3

Closed-Loop Eigenvalues $\lambda(A-BR^{-1}B^T P_2)$:

$\alpha=0$:   {-5.0, -5.0, -30.0, -30.0}

$\alpha=1$:   {-6.1 ± 1.0 i, -31.1 ± 1.0 i}

$\alpha=2$:   {-7.2 ± 1.8 i, -32.2 ± 2.0 i}

For $\alpha=0$, the imaginary-parts of the closed-loop poles are zero rad/sec--stable, real-axis pole placement is achieved. The intermediate closed-loop system of step 2, characterized by $[A-BR^{-1}B^T P_1]$, has been stabilzed; only the RHP double poles at +5 are affected by the new optimal gain matrix, $R^{-1}B^T P_2]$.

For $\alpha=1$, the imaginary-parts of the closed-loop poles have values of 1 rad/sec. The real-parts of the eigenvalues have been increased from their values for the previous design as Q_{eq2} has begun to have an effect.

For $\alpha=2$, the imaginary-parts of the closed-loop poles have values

of 2 rad/sec., or approaching 2 rad/sec. As α increases, Q_{eq2} will begin to dominate the Q_{eq} term for the modified ARE--some closed-loop modal frequencies may be slightly less than the specified modal frequencies.

Step 3 concludes the prescribed imaginary-axis pole placement procedure for optimal controller design.

The optimal estimator is designed via duality theory using the same 3-step procedure.

We now present the design procedure in algorithmic form, and illustrate its effects on a higher-order example derived from flexible spacecraft control.

4. DESIGN ALGORITHM

Fig. 7 illustrates the design algorithm for prescribed imaginary-axis pole placement in the optimal controller system. Two independently-solved ARE's are employed: the ARE in rotational space, and the ARE in translational space. The coupling between the
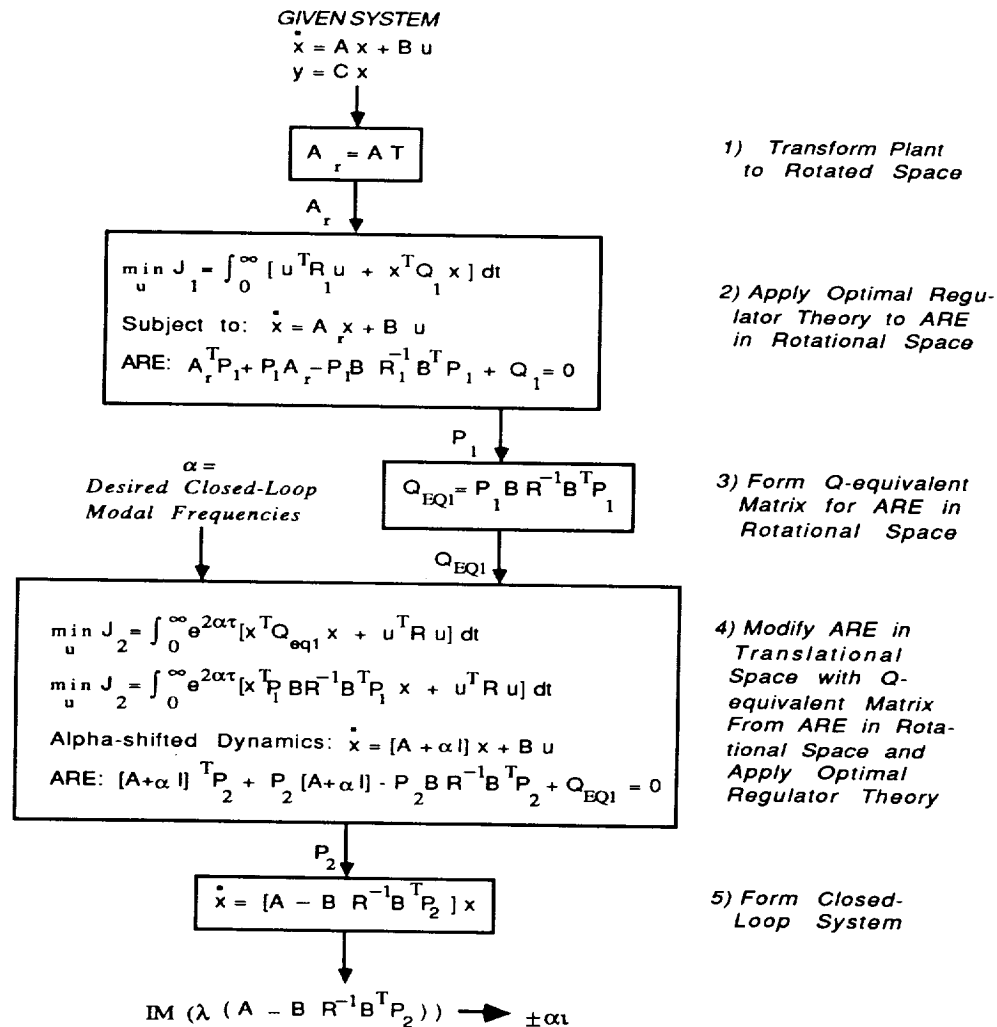
GIVEN SYSTEM
$$\dot{x} = A\,x + B\,u$$
$$y = C\,x$$

$$A_r = A\,T$$

1) Transform Plant to Rotated Space

$$A_r$$

$$\min_u J_1 = \int_0^\infty [\,u^T R_1 u + x^T Q_1 x\,]\,dt$$

Subject to: $\dot{x} = A_r x + B\,u$

ARE: $A_r^T P_1 + P_1 A_r - P_1 B\,R_1^{-1} B^T P_1 + Q_1 = 0$

2) Apply Optimal Regulator Theory to ARE in Rotational Space

$$P_1$$

$\alpha =$
Desired Closed-Loop Modal Frequencies

$$Q_{EQ1} = P_1 B\,R^{-1} B^T P_1$$

3) Form Q-equivalent Matrix for ARE in Rotational Space

$$Q_{EQ1}$$

$$\min_u J_2 = \int_0^\infty e^{2\alpha\tau}[x^T Q_{eq1} x + u^T R\,u]\,dt$$

$$\min_u J_2 = \int_0^\infty e^{2\alpha\tau}[x^T_1 P_1 BR^{-1}B^T P_1 x + u^T R\,u]\,dt$$

Alpha-shifted Dynamics: $\dot{x} = [A + \alpha I] x + B\,u$

ARE: $[A+\alpha I]^T P_2 + P_2 [A+\alpha I] - P_2 B\,R^{-1} B^T P_2 + Q_{EQ1} = 0$

4) Modify ARE in Translational Space with Q-equivalent Matrix From ARE in Rotational Space and Apply Optimal Regulator Theory

$$P_2$$

$$\dot{x} = [A - B\,R^{-1}B^T P_2\,]\,x$$

5) Form Closed-Loop System

$$IM\,(\lambda\,(A - B\,R^{-1}B^T P_2\,)) \longrightarrow \pm\alpha_i$$

Figure 7. Design Algorithm

ARE's occurs with the Q-equivalent matrix for the ARE in rotational space. This Q-equivalent matrix acts as a state-weighting matrix for the ARE in translational space. The design algorithm for the optimal estimator follows a parallel structure: Dual variables are substituted into J_2, and A^T replaces A in the ARE in translational space.

5. FLEXIBLE SPACECRAFT CONTROL EXAMPLE

The design algorithm is applied to a model for the spacecraft boom shown in Fig. 8. The model contains twelve modes with frequencies ranging from 0.67 to 11.4 Hz. Four collocated actuators and sensors are positioned at the tip of the boom and mid-boom. All modes are modeled with zero damping.

For the example, we design five optimal controllers and compare their pole constellations. The
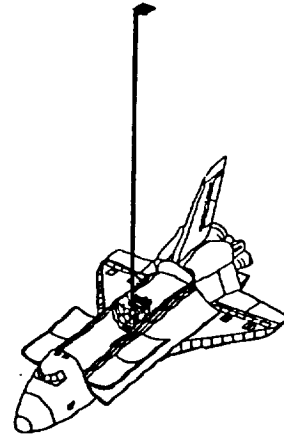

Figure 8. Flexible Spacecraft Boom

alpha values for the five designs, i.e. the prescribed imaginary-axis pole placement that is desired, are as follows: $\{\alpha_1=0, \alpha_2=1, \alpha_3=5, \alpha_5=10, \alpha_5=15\}$.

The design results are shown in Fig. 9 which plots the $\lambda(A-BR^{-1}B^TP_2)$ for the five designs. (Only the upper-half of the complex plane is shown.) Small values of
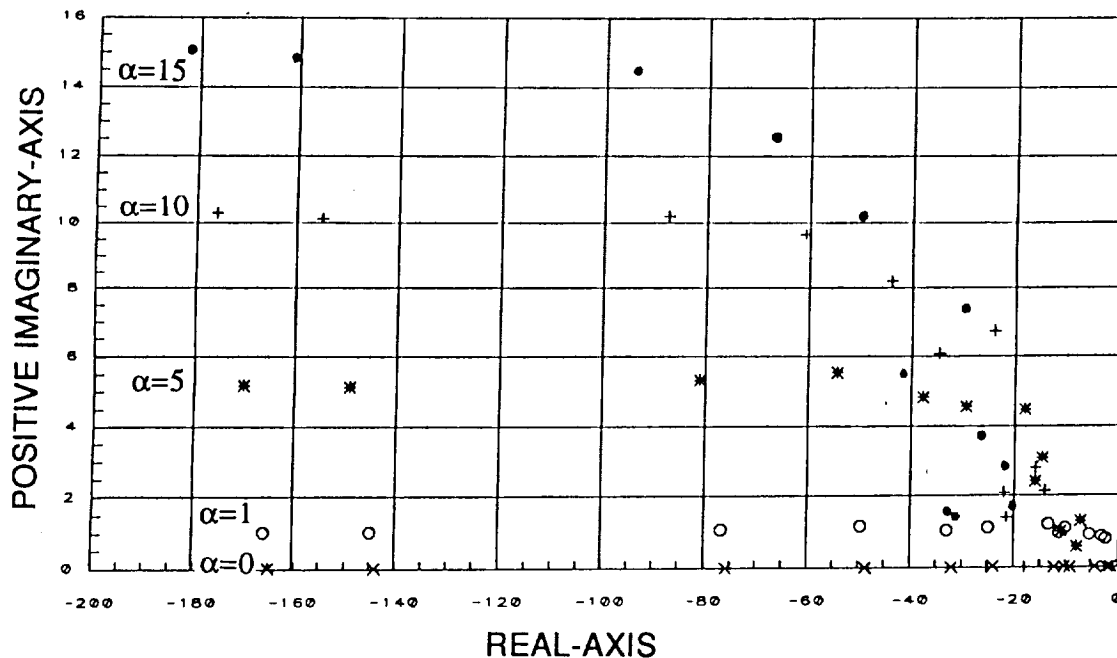

FIGURE 9. Controller Eigenvalues for Five Design Values of α

alpha ($\alpha_1=0$, $\alpha_2=1$) most closely approach the "prescribed" response, i.e. the imaginary-parts of all poles are approximately equal to the prescribed α value. This is due to a small or non-existent contribution of Q_{eq2} to the Q_{eq} matrix for the modified ARE, as explained in the previous section. For large values of α, the Q_{eq2} term begins to contribute to the Q_{eq} matrix, and the imaginary-parts of the pole asymptotically approach the desired α value. Poles corresponding to higher frequency modes have imaginary-parts that are closer to the α-asymptote.

6. SUMMARY/FUTURE WORK

A design procedure has been developed for prescribed-imaginary axis pole constraints for the optimal control and estimation systems: The imaginary-parts of the closed-loop system poles asymptotically approach a prescribed value, α. At this stage in the development, the maximum value that α may assume for a given system is constrained, possibly by a computational problem with solutions for the alpha-shifted ARE. Values of α that are large relative to the lowest modal frequency in the system can produce root migration from the desired α-asymptote. Small or mid-range frequency values of α produce excellent results as shown in the example of Section 5. Further analysis of the computational problem is required.

The design procedure developed empirically as the result of numerical experiments in gain transference and Q-equivalency theory. Future work calls for developing an analytical basis for the procedure. Additional work requires extending the design procedure to cover prescribed imaginary-axis constraints for the optimal compensator system.

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