An Experimental Study of Nonlinear Dynamic System Identification

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Abstract

A technique based on the Minimum Model Error optimal estimation approach is employed for robust identification of a nonlinear dynamic system. A simple harmonic oscillator with quadratic position feedback was simulated on an analog computer. With the aid of analog measurements and an assumed linear model, the Minimum Model Error Algorithm accurately identifies the quadratic nonlinearity. The tests demonstrate that the method is robust with respect to prior ignorance of the nonlinear system model, and with respect to measurement record length and regardless of initial conditions.

Introduction

The widespread existence of nonlinear behavior in many dynamic systems is well-documented, e.g, Thompson and Stewart [1]; Nayfeh and Mook [2]. In particular, virtually every problem associated with orbit estimation, flight trajectory estimation, spacecraft dynamics, etc., is known to exhibit nonlinear behavior. Many excellent methods for analyzing nonlinear system models have been developed. However, a key practical link is often overlooked, namely: How does one obtain an accurate mathematical model for the dynamics of a particular complicated nonlinear system? Identification, the process of developing an accurate system model from system output measurements, may provide the answer.

Nonlinear systems are commonly described using linear models. Many efficient algorithms for the identification of linear systems exist and their accuracy and ease of application encourages their use. However, linearization does not work in every application, and even when it does provide a reasonable approximation, the approximation is normally limited to a small region about the operating point of linearization. In the case

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of systems with severe nonlinear behavior using a linear model to describe such a system leads to inconsistencies ranging from inaccurate numerical results to misrepresentation of the system's qualitative behavior. Since nonlinearities are seldomly easily characterized, nonlinear identification techniques may prove beneficial in developing accurate mathematical representations of nonlinear systems.

Numerous methods for the identification of nonlinear systems have been developed in the past two decades (Natke, Juang and Gawronski [3]). Among the most widely used nonlinear identification methods are describing the nonlinear system using a linear model, or representing the nonlinear system in a series expansion and obtaining the respective coefficients either by using a regression estimation technique, by minimizing a cost functional, by using correlation techniques, or by some other approach. Some algorithms use the direct equation approach, while others obtain a graphical representation of the nonlinear term(s) and then find an analytical model for the nonlinearity. The interested reader can find more information on these nonlinear identification techniques in Stry and Mook [4].

The diversity of nonlinear identification techniques prompts the choice of an algorithm based on the needs of the particular application. Typical criteria to look for are: iterations required, robustness in the presence of measurement noise, number of measurements needed, robustness with respect to knowledge of the inital conditions, and robustness with respect to initial assumptions regarding the form of the nonlinearity. The results presented in this paper confirm that the Minimum Model Error algorithm excels in the above mentioned requirements.

In previous papers, the Minimum Model Error algorithm (MME) was explained in detail (Mook and Junkins [5]), modified for nonlinear identification (Mook [6]), and shown to accurately identify exotic nonlinearities in higher order systems (Stry and Mook [4]). In this paper, it is shown how the MME algorithm successfully identifies nonlinearities using experimental data. An analytical model representing a harmonic oscillator with quadratic position feedback is studied. Output data is obtained from an analog computer simulation of the nonlinear system and the quadratic term is accurately identified. It is shown that the Minimum Model Error algorithm is capable of identifying a nonlinear model which accurately reproduces the analog output regardless of knowledge an initially assumed model, initial conditions or record length.

MME Algorithm

In this section, we briefly review the MME algorithm and how it is used to identify nonlinear dynamic systems. A more detailed explanation may be found in Mook and Junkins [5], Mook [6], and Stry and Mook [4].

The MME may be summarized as follows. Suppose there is a nonlinear system whose exact analytical representation is unknown, but for which output measurements are available. Using "normal" means (analysis, finite elements, etc.), a system model, denoted assumed model, is constructed. The MME combines the assumed model with the measurements to determine the correct form of the nonlinear system. The approach consists of adding the to-be-determined correction term to the assumed model. A cost functional composed of the weighted sum of the correction term plus measurements minus estimate residuals is minimized. The minimization yields optimal state trajectories in addition to the correction term. A least squares fit of the correction term is performed to find the form of the dynamic model error.

Consider a forced nonlinear dynamic system which may be modeled in state-space form by the equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{F}(t) + \underline{f}(\underline{x}(t), \dot{\underline{x}}(t)) \tag{1}$$

where $\underline{x}(t)$ is the $n \times 1$ state vector consisting of the system states, A is the $n \times n$ state matrix, $\underline{F}(t)$ is an $n \times 1$ vector of known external excitation, and $\underline{f}(\underline{x}(t), \dot{\underline{x}}(t))$ is an $n \times 1$ vector which includes all of the system nonlinearities. State-observable discrete time domain measurements are available for this system in the form

$$\tilde{\underline{y}}(t_k) = \underline{g}_k(\underline{x}(t_k), t_k) + \underline{v}_k, \qquad t_0 \leq t_k \leq t_f \tag{2}$$

where $\tilde{\underline{y}}(t_k)$ is an $m \times 1$ measurement vector at time t_k, $\underline{g}_k$ is the accurate model of the measurement process, and $\underline{v}_k$ represents measurement noise. $\underline{v}_k$ is assumed to be a zero-mean, gaussian distributed process of known covariance R_k. The measurement vector $\tilde{\underline{y}}(t_k)$ may contain one or more of the system states. To implement MME, assume that a model, which is generally not the true system model because of the difficulties inherent in obtaining the true system model, is constructed in state-vector form as

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{F}(t) \tag{3}$$

Here, we show a linear model because in practice, linearization is the most common approach to modeling nonlinear systems. MME uses the assumed linear model in (3) and the noisy measurements in (2) to find the model error.

The model error, which might include linear terms as well as unknown nonlinear term(s), is represented by the addition of a correction term to the assumed linear model as

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{F}(t) + \underline{d}(t) \tag{4}$$

where $\underline{d}(t)$ is the $n \times 1$ correction term (dynamic model error) to be estimated later.

A cost functional, J, that consists of the weighted integral square of the correction term plus the weighted sum square of the measurement-minus-estimated measurement residuals, is formed:

$$J = \sum_{k=1}^{M} \left\{ [\tilde{\underline{y}}(t_k) - \underline{g}_k(\hat{\underline{x}}(t_k), t_k)]^T R_k^{-1} [\tilde{\underline{y}}(t_k) - \underline{g}_k(\hat{\underline{x}}(t_k), t_k)] \right\}$$

$$+\int_{t_o}^{t_f} \underline{d}(\tau)^T W \underline{d}(\tau) d\tau \tag{5}$$

where M is the number of measurement times, $\hat{\underline{x}}(t_k)$ is the estimated state vector and W is a weight matrix to be determined.

J is minimized with respect to the correction term, $\underline{d}(t)$. The necessary conditions for the minimization lead to the following two point boundary value problem (TPBVP), (see Geering [7]),

$$\dot{\underline{x}}(t) = A\underline{x}(t) + \underline{F}(t) + \underline{d}(t) \tag{5a}$$

$$\dot{\underline{\lambda}}(t) = -A^T \underline{\lambda}(t) \tag{5b}$$

$$\underline{d}(t) = -\frac{1}{2} W \underline{\lambda}(t) \tag{5c}$$

$$\underline{\lambda}(t_k^+) = \underline{\lambda}(t_k^-) + 2H_k R_k^{-1} [\tilde{\underline{y}}(t_k) - \underline{g}_k(\hat{\underline{x}}(t_k), t_k)] \tag{5d}$$

$$H_k = \frac{\delta g}{\delta \underline{x}} |_{\hat{\underline{x}}(t_k), t_k}$$

$$\underline{x}(t_o) = \underline{x}_o \quad or \quad \underline{\lambda}(t_o) = 0 \tag{5e}$$

$$\underline{x}(t_f) = \underline{x}_f \quad or \quad \underline{\lambda}(t_f) = 0 \tag{5f}$$

where $\underline{\lambda}(t)$ is a vector of costates (Lagrange multipliers). Estimates of the states and of the dynamic model error are produced by the solution of this two-point boundary value problem. The estimates depend on the particular value of W. The solution is repeated until a value of W is obtained which produces state estimates which satisfy the "covariance constraint", explained next.

According to the covariance constraint, the measurement-minus-estimated measurement residual covariance matrix must match the measurement-minus-truth error covariance matrix. This may be written as

$$[\tilde{\underline{y}}(t_k) - \underline{g}_k(\hat{\underline{x}}(t_k), t_k)]^T [\tilde{\underline{y}}(t_k) - \underline{g}_k(\hat{\underline{x}}(t_k), t_k)] \approx R_k \tag{6}$$

During the minimization, the weight W is varied until the state estimates satisfy the covariance constraint, i.e., the left hand side of Eq. (6) is approximately equal to the right hand side. The correction term or model error is, therefore, the minimum adjustment to the model required for the estimated states to predict the measurements with approximately the same covariance as the measurement error.

After W has been determined such that the state estimates satisfy the covariance constraint, the final step in the identification procedure is to use a least squares algorithm to fit the model error $d(t)$ to the unknown dynamic term(s). The error is expanded into some combination of linear and nonlinear terms, for example,

$$d(t) = \alpha x(t) + \beta x^2(t) + \gamma x^3(t) + \dots \tag{7}$$

where α, β, γ, ... are unknown coefficients to be determined by least squares. The least squares approach is explained in detail in Mook[6]

The TPBVP represented by Eqs. (5a) to (5f) contains jumps in the costates and, consequently, in the correction term. As evident from Eq. (5d), the size of the jump is directly proportional to the measurement residual at each measurement time. The noisier the measurements, the larger the jump size. A multiple shooting algorithm, developed by Mook and Lew [8], converts this jump-discontinuous TPBVP into a set of linear algebraic equations which may be solved using any linear equation solver. Multiple shooting also facilitates the analysis of a large number of measurements, by processing the solution at the end of every set of jumps.

The multiple shooting algorithm presented by Mook and Lew [8] was used to obtain the MME solutions used in the tests presented in this paper. It was assumed in the examples that MME obtained the dynamic error term without knowledge of the boundary conditions on $\underline{x}$, so some distortion of the correction term at the initial and final times was expected due to the constraints of Eqs. (5e-5f), i.e., by assuming no state knowledge is available at t_0 or t_f, we constrain $\lambda(t_0) = 0$ and $\lambda(t_f) = 0$. Therefore, in all test cases, the initial and final ten percent of the correction term data was ignored in the least squares fit.

Application Examples

Two nonlinear equations of motion were studied, which represent the motion of an undamped harmonic oscillator with different amounts of quadratic position feedback (identical equations may arise in other physical systems as well). The equations in state space form are

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ -0.526x^2 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ -1.137x^2 \end{pmatrix} \tag{9}$$

where x is position, and the dot indicates differentiation with respect to time. No forcing was applied.

In the following discussion, Eq. (8) is denoted Model A and Eq. (9) is denoted Model B. Different initial conditions were used for each system, for a total of four different tests. These are shown in Table 1.

Table 1. List of conditions used for each test

Test #	x(0)	v(0)	x*x
A1	0.000	0.261	-0.526
A2	0.000	0.523	-0.526
B1	0.000	0.087	-1.137
B2	0.000	0.261	-1.137

To utilize MME, the linear part of Eqs. (8) and (9) was chosen as the assumed model, rendering the model error equivalent to the nonlinear term, $c * x^2$. Measurements for the MME nonlinear identification were generated by simulating test A1 to B2 on an analog computer. Position measurements for all four tests were recorded and nonlinear models identified. The results were compared with the analytical position and analytical error term data, $c * x^2$, which were generated for Models A and B using a digital computer. MME proved capable of accurately identifying the nonlinear quadratic term in spite of ignorance of the assumed model, true initial conditions and record length.

Analog computer results

One hundred position measurements were generated on an EAI–2000 analog computer for all four test cases. All measurements with a sampling rate of 4 Hertz were used in the analysis. Position, velocity, and position squared were chosen as the basis functions for the least squares fit. It was uncertain if the analog computer would add some damping to the system or if it was able to correctly reproduce the stiffness term. By including position and velocity in the least squares fit, stiffness and damping could be identified if they existed. The identification procedure yielded the numerical values shown in Table 2.

Table 2. Least Square estimates of the nonlinear terms using measurements generated by the analog computer.

Test #	True x	MME x	True v	MME v	True x*x	MME x*x
A1	0.000	0.003	0.000	0.000	-0.526	-0.528
A2	0.000	0.003	0.000	0.000	-0.526	-0.526
B1	0.000	0.008	0.000	0.005	-1.137	-1.141
B2	0.000	0.003	0.000	0.000	-1,.137	-1.135

The numerical results for the least squares fit of the error term matched the analytically predicted coefficients with great accuracy. Figures (1a-4a) show the analytical position, analog measurements and position predicted by the MME analysis for all analog tests. Figures (1b-4b) show the analytical correction term and the error term estimated by MME. In all cases the MME identification produced good state estimates.

The MME algorithm could accurately identify a nonlinear model regardless of the initial conditions. As seen from Figures (1a) and (4a) (test A1 and B2), the measured position and the analytical position differ significantly. The analytical position was digitally recalculated for test A1 and B2 using the initial analog measurements as initial conditions instead of the initial conditions presented in Table 1. The results are shown in Figures (5a) and (6a). In this set of plots the analytical position and the measurements are almost identical. Also, as shown in Figures (5b) and (6b), the analytic correction term is much more similar to the estimated correction term, confirming that MME does not need any knowledge of the initial or final state vector value.[3]

MME could identify the nonlinear term accurately idenpendent of the record length. In test B1 only 40 measurements were employed in the analysis because subsequent measurements were saturated. The nonlinear term is identified very well.

Note that the data appears to be noiseless, as shown in Figures (1a-4a). Successfull analysis of noisy data using the MME algorithm can be found in Mook[6] and Stry and Mook[4].

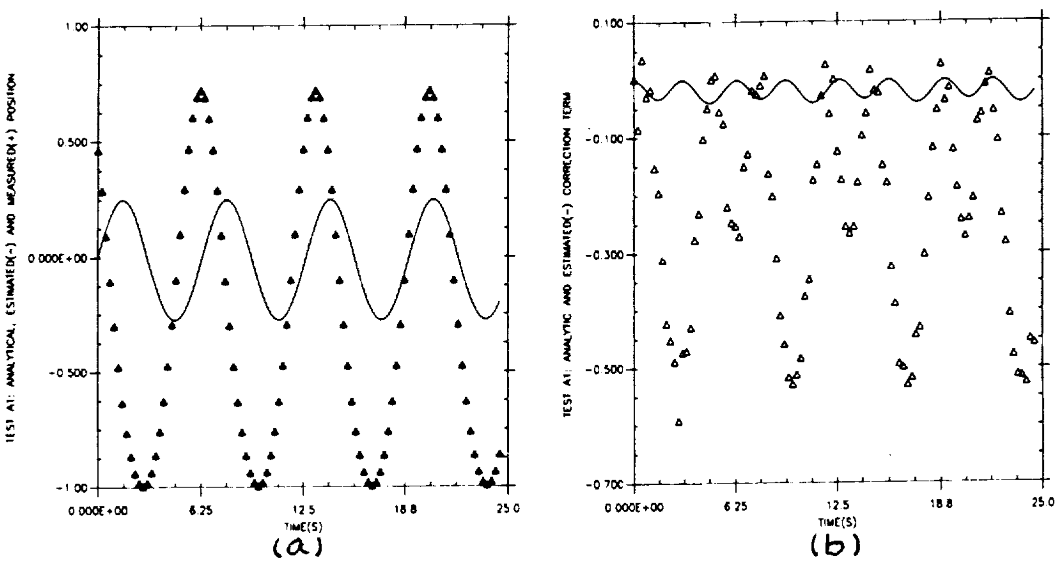


Figure 1. Test A1 a) Analytical, measured (+), and MME estimated (˚) position. b) Analytical and MME estimated (˚) correction term. The MME estimates are essentially identical to the measurements.

[3] It was shown in Eqs. (5e) and (5f), that by setting the initial and final costate values to zero, MME does not need any knowledge of the initial or end conditions.

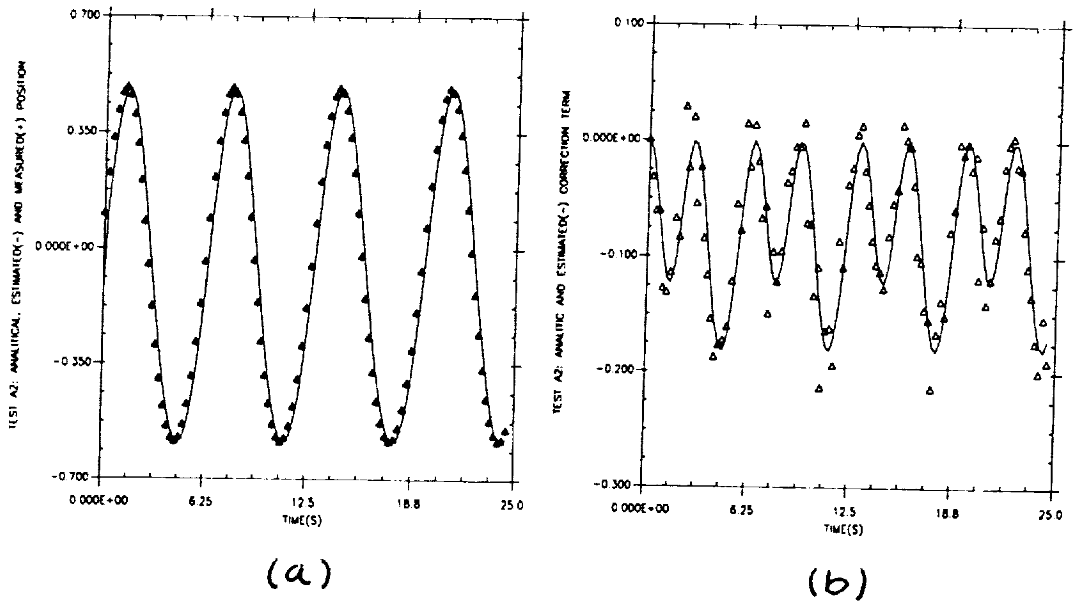


Figure 2. Test A2 a) Analytical, measured (+), and MME estimated (^) position. b) Analytical and MME estimated (^) correction term. The MME estimates are essentially identical to the measurements.

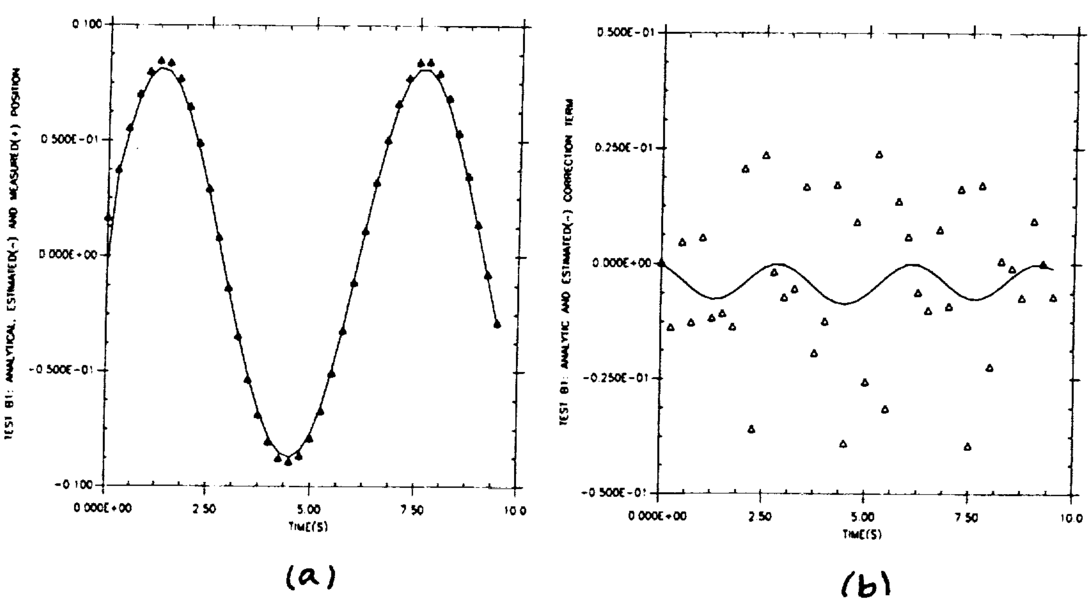
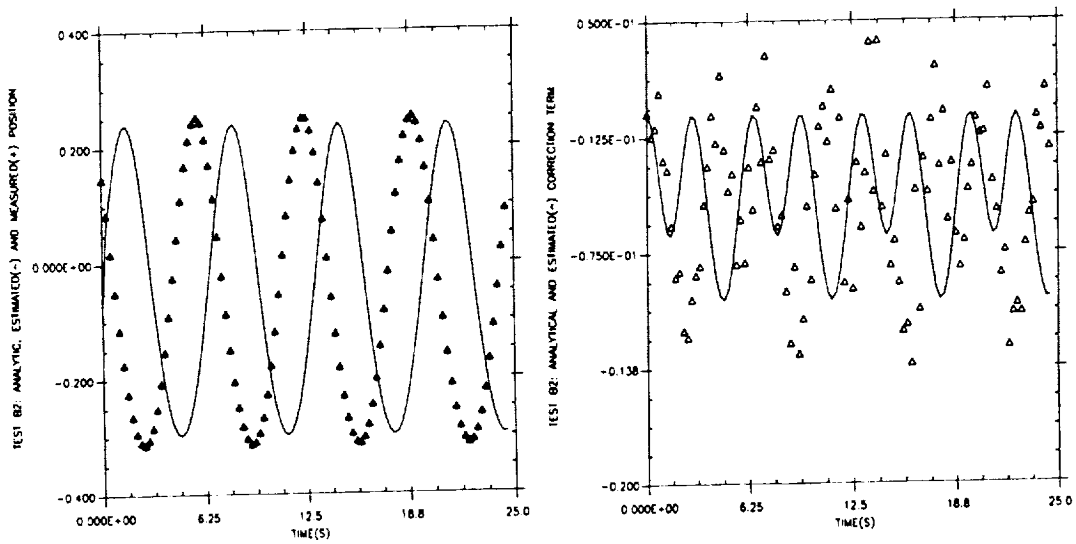
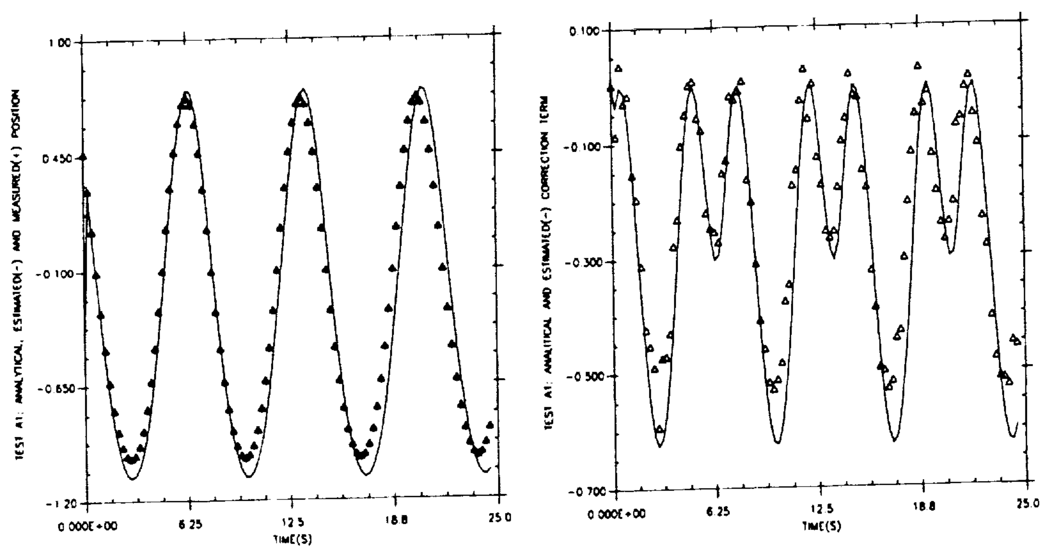


Figure 3. Test B1 a) Analytical, measured (+), and MME estimated (^) position. b) Analytical and MME estimated (^) correction term. The MME estimates are essentially identical to the measurements.



(a) (b)

Figure 4. Test B2 a) Analytical, measured (+), and MME estimated (ˆ) position. b) Analytical and MME estimated (ˆ) correction term. The MME estimates are essentially identical to the measurements.



(a) (b)

Figure 5. Test A1 a) Analytical, measured (+), and MME estimated (ˆ) position. b) Analytical and MME estimated (ˆ) correction term. The analytical position was calculated using as initial conditions the initial position and velocity measurements from the analog computer.

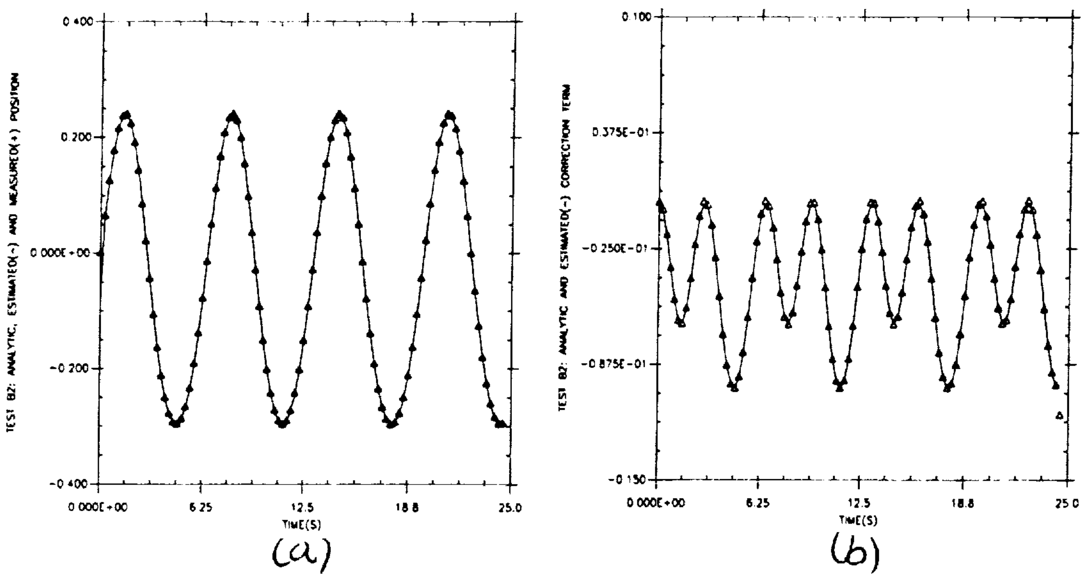


Figure 6. Test B2 a) Analytical, measured (+), and MME estimated (^) position. b) Analytical and MME estimated (^) correction term. The analytical position was calculated using as initial conditions the initial position and velocity measurements from the analog computer.

Conclusion

In this paper, an MME based algorithm was used to accurately identify the quadratic term of a nonlinear harmonic oscillator. Data was obtained from an analog computer simulation of the nonlinear system. It is demonstrated that the method is robust with respect to (lack of) a priori knowledge of the system dynamics. The identification was accurate regardless of initial conditions or data record length.

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