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N91-22349

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SUMMARY

In this paper, maximum likelihood estimation for distributed parameter models of large flexible structures has been formulated. Distributed parameter models involve far fewer unknown parameters than independent modal characteristics or finite element models. The closed-form solutions for the partial differential equations with corresponding boundary conditions have been derived. The closed-form expressions of the sensitivity functions lead to highly efficient algorithms for analyzing ground or on-orbit test results. For illustration of this approach, experimental data of the NASA Mini-MAST trust have been used. The estimations of modal properties involve its lateral bending modes and torsional modes. The results show that distributed parameter models are promising in the parameter estimation of large flexible structures.

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INTRODUCTION

Large spacecraft structures, such as, Solar Array Flight Experiment, Mini-MAST (CSI) employ large complex trusses in their constructions. Because of their large size and because of gravitational loads, it is not possible to determine with suitable accuracy their structural dynamic characteristics from ground-based testing of a full-scale prototype. Analysis of on-orbit response data will be necessary. In recent years, numerous variations of system identification methods have been developed. Unfortunately, current approaches to parameter estimation cannot handle the complex models foreseen to be necessary. As the number of modes increases, the accuracy will be decreased and the complexity significantly increases.

Two distinct approaches to the solution of large space structural parameter identification problems have emerged. One is the lumped parameter approach, the other is the distributed parameter approach [1–5]. The obvious fact is that by far the most effort put into the study of the identification problem has been based on the lumped parameter model. With the increase of the number of modes, the number of parameters increases rapidly for the lumped parameter approach. However, for the distributed parameter approach, the total number of parameters needed to be identified almost keeps the same.

Distributed parameter model is based on the partial differential equation (PDE). In this approach, instead of identifying the modal frequency, damping and mode shape deflection of each mode, only the coefficients of the partial differential equation and initial conditions need to be estimated.

This paper intends to create a very simple distributed parameter model, combining the use of maximum likelihood estimation technique (MLE), to identify the modal characteristics of NASA Mini-MAST truss which is treated as a cantilevered beam with two concentrated masses. Wave equation and Bernoulli-Euler beam equation are introduced to describe the torsional and bending behavior of the Mini-MAST truss. Proportional damping will be taken into account simply by adding a damping term in the PDE's, which is proportional to velocity [6].

A series of closed-form solutions of the PDE's will be used to match the measurements of the vibrations. Based on the optimal fitness between the measurement response and the theoretical response in the sense of maximization of likelihood function, the coefficients in the PDE's and the parameters dependent on the initial conditions are estimated. The closed-form expressions of the sensitivity functions are derived, which carry out the innovations of the unknown parameters in the iteration process. The comparison of the results with other methods shows that the proposed approach is promising in the parameter estimation of large flexible structures.

NASA MINI-MAST (CSI) TESTBED

NASA Mini-MAST (see Fig. 1) is a generic space truss built primarily for research in the areas of structural analysis and vibrational control [7]. Mini-MAST is deployed vertically inside a high-bay tower, cantilevered from its base on a rigid foundation. The total height of the truss is 20.16 meters, containing 18 bays (1.12 meters each) in a single-laced configuration with every other bay repeating. The deployable/retractable truss beam has a three-longeron construction forming a triangular cross-section with points inscribed by a circle of diameter 1.4 meters. The beam has three member types: longerons, battens, and diagonals. Longerons are parallel to the beam axis and provide beam stiffness and strength in bending. Battens are in the beam face planes and provide beam stability. Finally, diagonals, also in the beam face planes, provide beam stiffness and strength in torsion and shear. Two instrumentation platforms, holding actuators and sensors for CSI control experiment, have been installed at Bay 10 and Bay 18 (beam tip). These additional components have a significant effect on the structural dynamic characteristics.

In this paper, the real Mini-MAST truss is treated as a cantilevered beam with two concentrated masses at Bay 10 and Bay 18 respectively (Fig. 2). The continuum model is equivalent to the real truss in the sense that both have the same dynamic properties, say, natural frequency, damping ratio and mode shape. To keep the equivalency, the structural parameters of the equivalent must be set up properly. All these parameters will be divided into two types. Some physical quantities of the real structure, such as length, weight, etc., are called unadjustable parameters which are assumed to be known a priori. Another type of parameters, such as the composed parameters appearing in the PDE's, will be adjusted in the estimation process. This type of parameters is called adjustable parameters, which will be the elements of the unknown parameter vector in the maximum likelihood estimator. The parameters of the equivalent of the Mini-MAST truss is listed in Table 1.

The control inputs of the system are three orthogonal torque wheel actuators located at the top platform (Bay 18). Output pulse responses were obtained by applying single pulse signals at each input channel. Twenty seconds of output pulse response were collected for each input channel. A sampling frequency of 50 Hz is used. Two sets of data [8] are selected for our analysis here. The acquisitions of the selected data are recounted as follows. The first set of data was obtained from the measurement of the rotation rate about the x-axis, which was measured by one rate gyro mounted at the tip platform. The second one was obtained from the measurement of y-direction displacement, which was measured by one displacement sensor installed at Bay 18, mounted parallel to the flat face on the corner joints of the structure and positioned to measure deflections normal to the face. The locations of the actuators and sensors concerned are shown in Fig. 3.

MODELLING

In this paper, the analysis of modal characteristics of NASA Mini-MAST truss involves torsional and bending motions. The damped torsional vibration is described by the wave equation

$$J_b \frac{\partial^2 \theta}{\partial t^2} + c \frac{\partial \theta}{\partial t} - k \frac{\partial^2 \theta}{\partial x^2} = 0 \tag{1}$$

where $\theta(x,t)$ angle of twist, $J_b = \rho I_b$ moment of inertia of the beam, $k = GI_b$ torsional rigidity, and C damping constant of proportionality. Two parameters a and b, which relate the coefficients of the PDE, are defined by

$$a^2 = \frac{k}{J_b}, \quad 2b = \frac{c}{J_b} \tag{2}$$

The lateral bending vibration is described by the Bernoulli-Euler beam equation with proportional damping term

$$m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} + k\frac{\partial^4 y}{\partial x^4} = 0 \tag{3}$$

where $y(x,t)$ lateral displacement, $m = \rho A$ mass per length of the beam, and $k = EI$ lateral bending rigidity. Two parameters a and b are also defined by

$$a^2 = \frac{k}{m}, \quad 2b = \frac{c}{m} \tag{4}$$

The solutions of Eqs. (1) and (3) can be written in a generic form (see Appendix)

$$y(x,t) = \sum_i Y_i(x)\, e^{-\xi_i \omega_{n_i} t}\left(A_i \cos\omega_{d_i} t + B_i \sin\omega_{d_i} t\right) \tag{5}$$

where, $y(x,t)$ represents either angle of twist (θ) or lateral displacement y, A_i and B_i the coefficients dependent on the initial conditions, ω_{n_i} and $\omega_{d_i} = \omega_{n_i}\sqrt{1-\xi_i^2}$ the natural frequency and damped natural frequency, respectively, and ξ_i is the damping ratio. All these modal properties are related to the parameters a and b by

$$\omega_{n_i} = ak_i^q, \quad \omega_{d_i} = \sqrt{\left(ak_i^q\right)^2 - b^2}\ , \quad \xi_i = \frac{b}{ak_i^q} \tag{6}$$

where, q represents the order of power, with $q = 1$ for torsion and $q = 2$ for bending. From Eq. (6), the solution Eq. (5) can be expressed in terms of the parameters a and b,

$$y(x,t) = \sum_i Y_i(x)\, e^{-bt}\left[A_i \cos t\sqrt{\left(ak_i^q\right)^2 - b^2} + B_i \sin t\sqrt{\left(ak_i^q\right)^2 - b^2}\right] \tag{7}$$

$Y_i(x)$ in the solutions (5) or (7) are the eigenfunctions given as follows. For torsional equation:

$$Y_i(x) = \begin{cases} \left[\frac{\cot k_i x_{10}}{g(k_i l)} + 1\right]\sin k_i x & 0 \le x \le x_{10} \\ \frac{\cos k_i x}{g(k_i l)} + \sin k_i x & x_{10} \le x \le l \end{cases} \tag{8}$$

where,

$$g\,(k_i l) = \frac{\sin k_i l + \frac{J_l}{J_b} k_i l \cos k_i l}{\cos k_i l - \frac{J_l}{J_b} k_i l \sin k_i l}$$

For Bernoulli-Euler equation

$$Y_i\,(x) = \begin{cases} r_i\,(\mathrm{sh} k_i x - \sin k_i x) + (\mathrm{ch} k_i x - \cos\ k_i x) & 0 \leq x \leq x_{10} \\ \alpha_i\,[h_1\,(k_i l) \sin k_i x + h_4(k_i l) \cos k_i x + shk_i x] + \\ \beta_i\,[h_3\,(k_i l) \sin k_i x + h_2\,(k_i l) \cos k_i x + chk_i x] & x_{10} \leq x \leq l \end{cases} \tag{9}$$

where,

$$h_1\,(k_i l) = g_1\,(k_i l) + 2\tfrac{W_l}{W_b} k_i l \cos k_i l\ shk_i l; \quad h_2\,(k_i l) = g_2(k_i l) - 2\tfrac{W_l}{W_b} k_i l \sin k_i l\ chk_i l$$

$$h_3\,(k_i l) = g_3(k_i l) + 2\tfrac{W_l}{W_b} k_i l \cos k_i l\ chk_i l; \quad h_4\,(k_i l) = g_4\,(k_i l) - 2\tfrac{W_l}{W_b} k_i l \sin k_i l\ shk_i l$$

and

$$g_1\,(k_i l) = \cos k_i l\ chk_i l + \sin k_i l\ shk_i l, \quad g_2\,(k_i l) = \cos k_i l\ chk_i l - \sin k_i l\ shk_i l$$

$$g_3\,(k_i l) = \cos k_i l\ shk_i l + \sin k_i l\ chk_i l, \quad g_4\,(k_i l) = \cos k_i l\ shk_i l - \sin k_i l\ chk_i l$$

and α_i, β_i and r_i are the modal participant coefficients.

Finally, the most important quantities are the eigenvalues K_i's of the system. They are the roots of the corresponding characteristic equations given as follows,

For torsional equation:

$$\frac{J_{10}}{2J_b} kl \sin 2kx_{10} + \frac{J_{10}}{J_b} kl\ g\,(kl) \sin^2 kx_{10} = 1 \tag{10}$$

For Bernoulli-Euler equation:

$$\mathrm{DET} \begin{vmatrix} SHX & CHX & -SHX & -CHX \\ h_1 SNX + h_4 CSX & h_3 SNX + h_2 CSX & SNX & CSX \\ h_1 CSX - h_4 SNX + CHX & h_3 CSX - h_2 SNX + SHX & CSX - CHX & -SNX - SHX \\ 2CHX & 2SHX & -2CHX + \frac{W_{10}}{W_b} kl(SNX - SHX) & -2SHX + \frac{W_{10}}{W_b} kl(CSX - CHX) \end{vmatrix} = 0 \tag{11}$$

where,

$$SNX = \sin kx_{10}, CSX = \cos kx_{10}, SHX = \mathrm{sh}\ kx_{10}, \text{ and } CHX = \mathrm{ch}\ kx_{10}$$

Maximum Likelihood Estimator

Assume that the outcome Y of an experiment depends on an unknown parameter vector θ. We want to estimate the best value of θ according to the observation Y. One of the advanced technique is based on the Maximum Likelihood Estimate (MLE) principle. The essence of the MLE method is to maximize the conditional probability density function, i.e., so-called likelihood function, $P(Y|\theta)$.

Suppose we have the measurement response sequences $y(1)$, $y(2), ..., y(m)$. The matrix Y_m consisting of all measured outputs is introduced, $Y_m = [y(1), y(2), ..., y(m)]$. If the probability distribution of Y_m has a density $P[Y_m|\theta]$, it then follows from the definition of conditional probability that

$$P[Y_m|\theta] = P[y(1), y(2), ..., y(m)|\theta] = \prod_{i=1}^{m} P[y(i)|Y_{i-1}, \theta] \tag{12}$$

If the assumption of Gaussian distribution is taken, the likelihood function has the form of

$$\begin{aligned} L(\theta) = P[Y_m|\theta] = \prod_{i=1}^{m} \frac{1}{(2\pi)^{mp/2} R^{1/2}} \exp\Big\{ -\frac{1}{2}[y(i|i-1,\theta) - \bar{y}(i|i-1,\theta)]^T \\ \cdot R^{-1}[y(i|i-1,\theta) - \bar{y}(i|i-1,\theta)]\Big\} \end{aligned} \tag{13}$$

where $\bar{y}_o(i|i-1,\theta)$ is the norminal response calculated by using θ_o. If the constants are ignored, we have the log-likelihood function

$$\begin{aligned} J(\theta) &= -\ln\ L(\theta) \\ &= \frac{1}{2}\sum_{i=1}^{m}\Big\{[y(i|i-1,\theta) - \bar{y}(i|i-1,\theta)]^T R^{-1}[y(i|i-1,\theta) - \bar{y}(i|i-1,\theta)] + \ln R\Big\} \end{aligned} \tag{14}$$

Linearizing $\bar{y}(i|i-1,\theta)$ with respect to the unknown parameter vector θ, we have

$$\bar{y}(i|i-1,\theta) = \bar{y}_o(i|i-1,\theta) + (\nabla_\theta y_i)(\theta - \theta_o) \tag{15}$$

where $\nabla_\theta y_i$ gradient of y with respect to θ, and θ_o is nominal θ vector. Substituting Eq. (15) into Eq. (14) for $J(\theta)$ and setting $\frac{\partial J}{\partial \theta}|_{\theta=\hat{\theta}} = 0$, we may obtain.

$$\hat{\theta} = \theta_o + \left[\sum_{i=1}^{m} (\nabla_\theta y_i)^T R^{-1} (\nabla_\theta y_i)\right]^{-1} \left[\sum_{i=1}^{m} (\nabla_\theta y_i)^T R^{-1} (y_i - \bar{y}_{io})\right] \tag{16}$$

In this paper, the unknown parameter vector will be

$$\theta = [a, b, A_1, A_2, ..., A_n, B_1, B_2, ..., B_n]^T$$

The relationship between the modal properties and the unknowns is given in Eq. (6). Thus we can obtain the modal properties through the solution of the PDE as long as these unknown parameters are determined. The gradients of $y(x,t)$ with respect to the unknowns in Eq. (16) can be derived from Eq. (7) simply by taking derivatives.

$$\frac{\partial y}{\partial a}(x,t) = \sum_i Y_i(x) e^{-bt} \frac{k_i^{2q} a t}{\sqrt{(a k_i^q)^2 - b^2}} \left[-A_i \sin t\sqrt{(a k_i^q)^2 - b^2} + B_i \cos t\sqrt{(a k_i^q)^2 - b^2}\right] \tag{17}$$

$$\begin{aligned}\frac{\partial y}{\partial b}(x,t) = \sum_i Y_i(x) t e^{-bt} \Bigg\{ & -A_i \cos t\sqrt{(a k_i^q)^2 - b^2} - B_i \sin t\sqrt{(a k_i^q)^2 - b^2} \\ & + \frac{b}{\sqrt{(a k_i^q)^2 - b^2}} \left[A_i \sin t\sqrt{(a k_i^q)^2 - b^2} - B_i \cos t\sqrt{(a k_i^q)^2 - b^2}\right]\Bigg\}\end{aligned} \tag{18}$$

$$\frac{\partial y}{\partial A_i}(x,t) = Y_i(x) e^{-bt} \cos t\sqrt{(a k_i^q)^2 - b^2} \tag{19}$$

$$\frac{\partial y}{\partial B_i}(x,t) = Y_i(x) e^{-bt} \sin t\sqrt{(a k_i^q)^2 - b^2} \tag{20}$$

Analysis of Modal Properties

As mentioned before, the procedure by using distributed parameter approach to analyze the modal properties is quite different from that of the lumped parameter approach. First, based on the unadjustable structural parameters, the eigenvalues k_i and the eigenfunctions $Y_i(x)$ (mode shapes) of the system can be determined through the solutions of the corresponding characteristic equations before the estimation iteration starts. Second, only the coefficients of the PDE's and the parameters relevant to the initial conditions need to be estimated rather than the modal frequency, damping and mode shape deflection of each mode. These two characteristics greatly decrease the number of unknown parameters and speed up the iterative process.

Solution of the PDE is in the form of infinite series mathematically, so no modal truncation problem is involved theoretically. However the contributions to the response are always so small for the higher frequency modes that only the first several modes (five in this paper) are used in the analysis.

It is noted that the modal coupling must be considered. Because of the eccentric properties of both the tip-mass and the tip-actuating pulses, the lateral bending vibration will be excited while torsional vibration exists and vice versa. However, the experimental data show that bending modes are hardly recognized in the torsional measurement, and the first torsional mode appears clearly in the bending measurement. So the first torsional mode is included in the analysis of the bending vibration.

From the experiment data it is hard to get any a priori information about the initial conditions of the response, that is, the initial values of A_i and B_i for iteration. Fortunately the initial values of A_i and B_i do not affect the convergency significantly, so they are chosen arbitrarily. However, the initial values of parameters a and b are very important to the convergency. From the results of finite element analysis, we can determine the initial values of equivalent stiffness first, then proceed to reckon the initial values of the parameters a and b.

Iterative accuracy is controlled by the innovation residual of the unknown parameter vector θ, which is defined by

$$e_\theta = \left\{ \frac{1}{P-1} \sum_{i=1}^{p} \left[\theta_n(i) - \theta_{n-1}(i)\right]^2 \right\}^{1/2}$$

where, p the number of the unknowns, $\theta(i)$ the ith element of the unknown parameter vector θ, and n is the number of successive iteration. In the algorithm, $e_\theta < 10^{-7}$ is referred to as the criterion which controls the iteration.

Table 2 shows the comparison of the estimated frequencies obtained from Finite Element Analysis (FEA), Eigensystem Realization Algorithm (ERA) [9] and Distributed Parameter Algorithm (DPA). Most modes are comparable to each other. The fifth bending mode from DPA is extremely higher than that of the other approaches. This is due to inadequacy of the Bernoulli-Euler beam model used. Because the rotary inertia and shear deformation of the beam are neglected, the Bernoulli-Euler beam model produces much higher frequencies in the high frequency range. In order to improve the accuracy for high frequency, Timoshenko beam model is proposed for further investigation.

Figure 4 shows that the reconstructed responses obtained from the estimated parameters and the measured responses have a reasonably good fitness.

CONCLUDING REMARKS

This paper proposes a distributed parameter model for the analysis of the modal characteristics of NASA Mini-MAST truss. Wave equation and Bernoulli-Euler equation have been used to describe the torsional and bending vibrations respectively. Closed-form solutions of the PDE's are derived. By using the Maximum Likelihood Estimation method to provide the optimal match between the experimental data and estimated responses, the coefficients in the PDE's and the parameters dependent on the initial conditions are estimated and the modal properties can be further determined. The results are comparable to those from other approaches. The estimates of bending modes in the higher frequency range is expected to be improved by using

Timoshenko beam model. Because the number of unknown parameters is greatly reduced in the distributed parameter model and the maximum likelihood estimation is feasible based on the derived closed form solutions of the PDE's, the proposed approach is particularly attractive for its less computational burden for the large flexible structures.

REFERENCES

1. J. Y. SHEN, L. W. TAYLOR, JR., and J. K. HUANG, 1990, "An algorithm for maximum likelihood estimation for distributed parameter models of large beam-like structures," 2nd USAF/NASA Workshop On System Identification and Health Monitoring of Precision Space Structures.
2. L. W. TAYLOR, JR., 1989, "Nonlinear and distributed parameter models of the Mini-MAST Truss," 3rd NASA/DoD Controls-Structures Interaction Technology Conference.
3. C. T. SUN and J. N. JUANG, 1986, "Modeling global structural damping in trusses using simple continuum models," *AIAA Journal* 24, No. 1.
4. A. K. NOOR, M. S. ANDERSON and W. H. GREENE, 1978, "Continuum models for beam- and plate-like lattice structures" AIAA/ASME 19th Structures, Structural Dynamics, and Materials Conference.
5. L. W. TAYLOR, JR., 1990, "PDEMOD — computer software for distributed parameter Estimation for flexible spacecraft applied to NASA Mini-MAST Truss experiment," 2nd USAF/NASA Workshop On System Identification and Health Monitoring of Precision Space Structures.
6. H. T. BANKS and D. J. INMAN, 1989, "On damping mechanisms in beams," NASA CR 181904.
7. L. R. ADAMS, 1987, "Design, development and fabrication of a deployable/retractable truss beam model for large space structures application," NASA CR 178287, Astro Aerospace Corporation.
8. K. LIU and R. SKELTON, 1990, "The Q-markov cover and its application to flexible structure identification," School of Aeronautics And Astronautics, Purdue University.
9. R. S. PAPPA, A. SCHENK and C. NOLL, 1990, "ERA modal identification experiences with mini-MAST," 2nd USAF/NASA Workshop on System Identification and Health Monitoring of Precision Space Structures.
10. L. LJUNG, 1987, "System identification: theory for the user," Prentice-Hall.
11. S. TIMOSHENKO, D. M. YOUNG and W. WEAVER, JR., 1974, "Vibration problems in engineering," the 4th Ed. John Wiley & Sons, Inc.
12. W. T. THOMSON, 1972, "Theory of vibration with application," Prentice-Hall, Inc.

FIGURE LEGEND

Table 1 Structural Parameters of Equivalent Continuum of NASA Mini-MAST Truss

A. Unadjustable Parameters

Length of the Beam L = 66.24 ft.

X-Coordinate of Bay 10 x_{10} = 36.80 ft.

Radius of Gyration of the Section r = 1.6237 ft.

Mass per Length ρA = 0.1076 slug/ft.

Ratios of Weights:

Bay 10-body/Beam: $\frac{W_{10}}{W_b} = 0.4760$

Tip-body/Beam: $\frac{W_t}{W_b} = 1.4547$

Ratios of Moments of Inertia:

Bay 10-body/Beam: $\frac{J_{10}}{J_b} = 0.6206$

Tip-body/Beam: $\frac{J_t}{J_b} = 0.6206$

B. Adjustable Parameters: Initial Values for Iteration

Longitudinal Stiffness EA = 10,530,000 Lb

Bending Rigidity EI = 27,760,000 Lb.ft.^{2}

Torsional Rigidity GI_p = 1,970,000 Lb.ft.^{2}

Table 2 Comparison of Estimated Frequencies (Hz)

A. Bending Modes

No.	F.E.A.	E.R.A.	D.P.A.
1	0.80	0.86	0.768
2	6.16	6.18	6.637
3	32.06	32.39	29.773
4	44.86	43.23	50.923
5	70.18	67.27	102.973

B. Torsional Modes

No.	F.E.A.	E.R.A.	D.P.A.
1	4.37	4.19	4.527
2	21.57	22.89	21.671
3	39.01	38.06	42.521
4	54.27	51.55	56.509
5	72.87	67.27	70.559

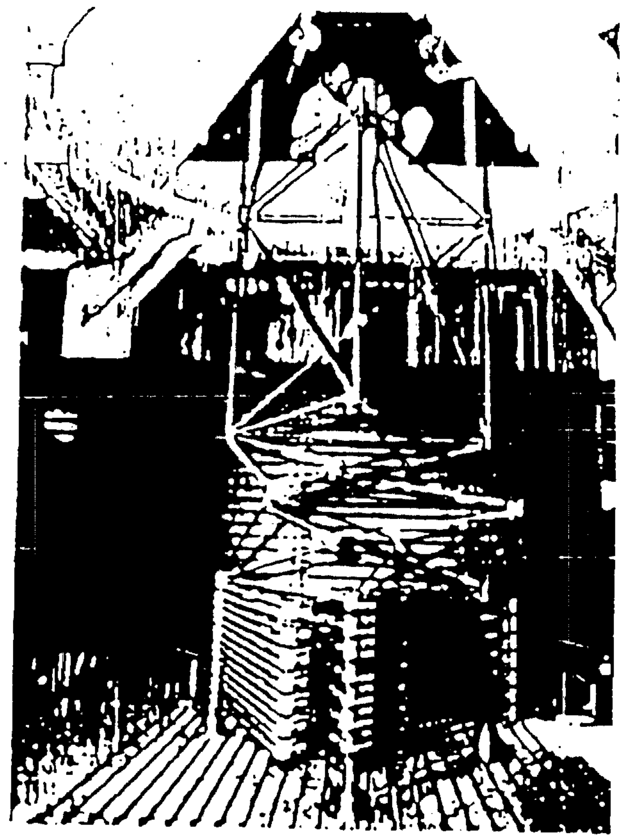


Fig. 1 Sketch of NASA Mini-MAST Truss

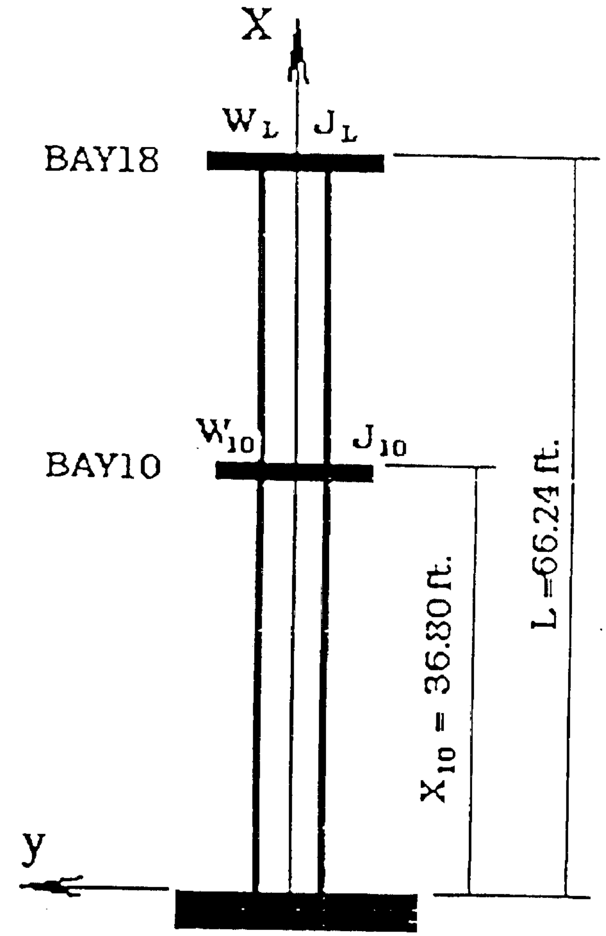
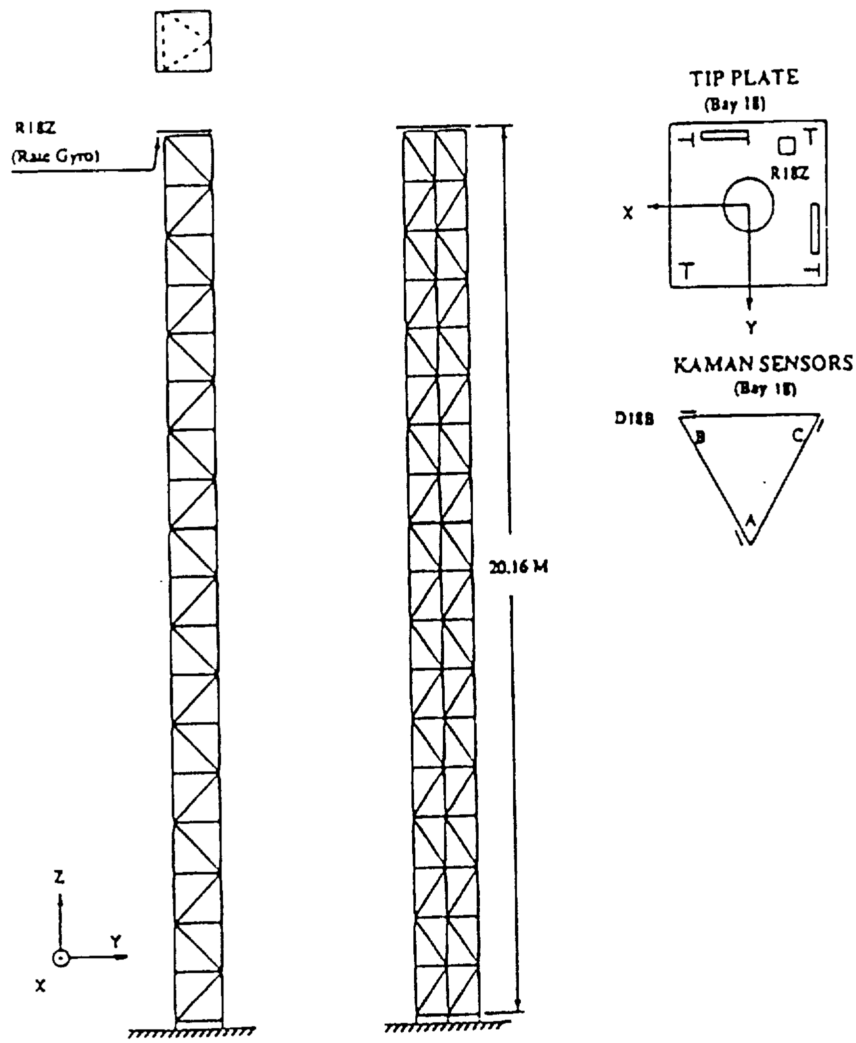


Fig. 2 Equivalent Cantilevered Beam



Output No.	Notation	Sensor	Location
y_7	R18Z	Rate Gyro	Bay 18 z axis
y_9	D18B	Displacement Sensor	Bay 18 Corner B

Fig. 3 Mini-MAST Sensor and Actuator Locations

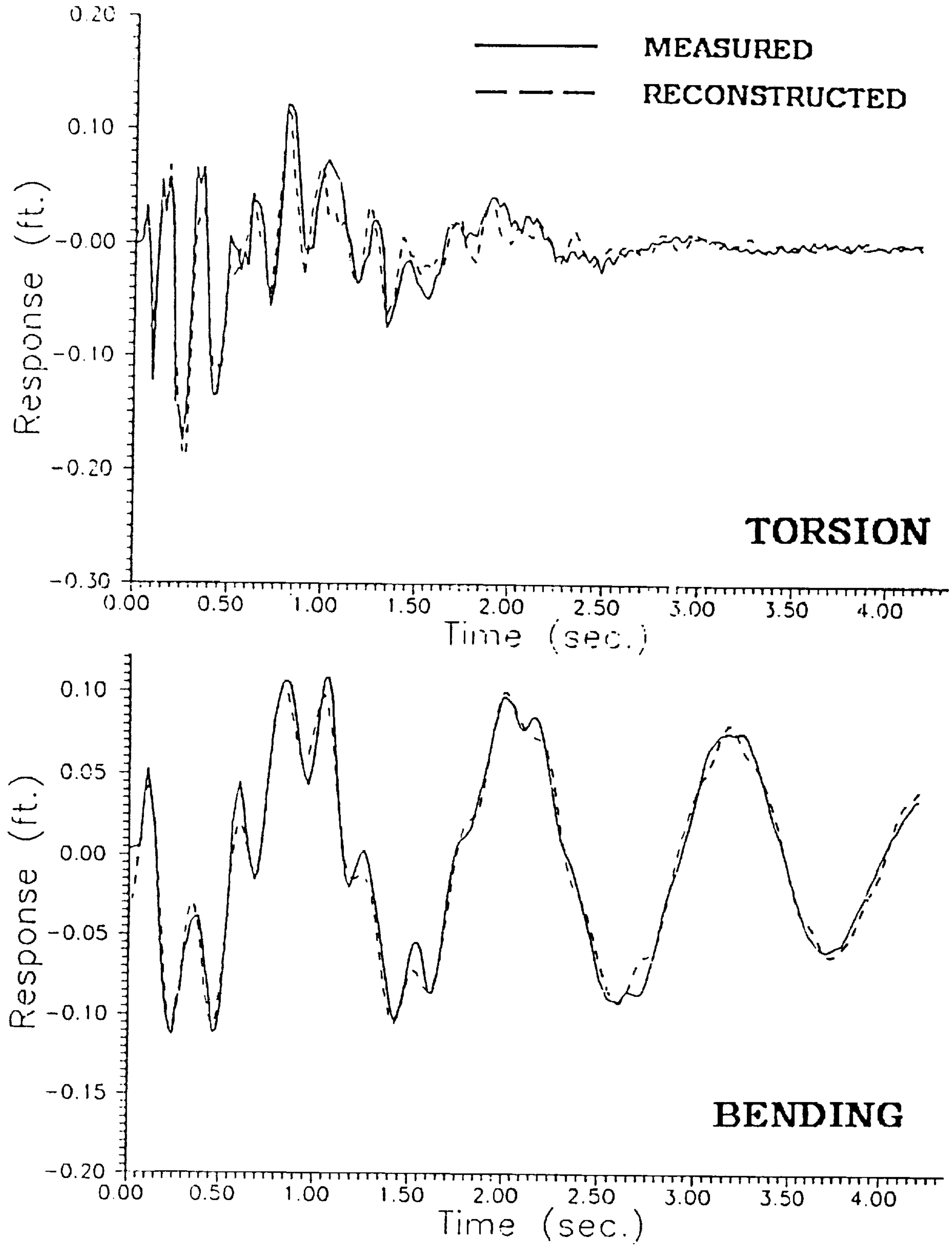


Fig. 4 Comparison of Reconstructed and Measured Responses

APPENDIX A

Solution to the Bernoulli-Euler Beam Equation

In using distributed parameter approach to identify the structural systems, one of the most important procedures is to solve the partial differential equation.

Here, only solution to the Bernoulli-Euler beam equation will be provided. Solution to the torsional vibration equation can be derived by following similar procedures.

The Bernoulli-Euler beam equation describing lateral bending is

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} = 0 \tag{A.1}$$

where, $a^2 = \frac{EI}{\rho A}$. The general solution to Eq. (A.1) may be expressed as

$$y(x,t) = \sum_i Y_i(x)\left[A_i \cos\omega_i t + B_i \sin\omega_i t\right] \tag{A.2}$$

where, $Y_i(x)$ are the eigenfunctions which are of the form

$$Y_i(x) = C_1 \sin k_i x + C_2 \cos k_i x + C_3 sh\ k_i x + C_4 ch\ k_i x \tag{A.3}$$

Now we derive the specific form of the solution for the equivalent Mini-MAST truss — cantilevered beam with two lumped masses. The procedure consists of three steps as follows.

Step 1 For the right segment, i.e. $x = x_{10} \sim l$.

The PDE for the lateral vibration of the right segment is

$$\frac{\partial^4 y_R}{\partial x^4} + \frac{1}{a^2}\frac{\partial^2 y_R}{\partial t^2} = 0 \tag{A.4}$$

The boundary conditions for the free end ($x = l$) are:

$$\begin{cases} \frac{\partial^2 y_R}{\partial x^2}(l,t) = 0 \\ EI\frac{\partial^3 y_R}{\partial x^3}(l,t) = \frac{W_1}{g}\frac{\partial^2 y_R}{\partial t^2}(l,t) \end{cases} \tag{A.5}$$

After the separation of variables

$$y_R(x,t) = Y_R(x)\,T(t)$$

we have a Y_R – ODE,

$$Y_R''''(x) - k^4 Y_R(x) = 0 \tag{A.6}$$

with B.C.'s

$$\begin{cases} Y_R''(l) = 0 \\ Y_R'''(l) = -\frac{W_t}{W_b} l k^4 Y_R(l) \end{cases} \tag{A.7}$$

The general solution to Eq. (A.6) and the corresponding derivatives are as follows,

$$\begin{cases} Y_R(x) = C_1 \sin kx + C_2 \cos kx + C_3 shkx + C_4 chkx \\ Y_R'(x) = K(C_1 \cos kx - C_2 \sin kx + C_3 chkx + C_4 shkx) \\ Y_R''(x) = K^2(-C_1 \sin kx - C_2 \cos kx + C_3 shkx + C_4 chkx) \\ Y_R'''(x) = K^3(-C_1 \cos kx + C_2 \sin kx + C_3 chkx + C_4 shkx) \end{cases} \tag{A.8}$$

By using the B.C., we have

$$\begin{cases} C_1 = C_3\left(g_1(kl) + 2\frac{W_t}{W_b} kl \cos kl shkl\right) + C_4\left(g_3(kl) + 2\frac{W_t}{W_b} kl \cos kl chkl\right) \\ C_2 = C_3\left(g_4(kl) - 2\frac{W_t}{W_b} kl \sin kl shkl\right) + C_4\left(g_2(kl) - 2\frac{W_t}{W_b} kl \sin kl chkl\right) \end{cases} \tag{A.9}$$

where,

$$\begin{cases} g_1(kl) = \cos kl chkl + \sin kl shkl \\ g_2(kl) = \cos kl chkl - \sin kl shkl \\ g_3(kl) = \cos kl shkl + \sin kl chkl \\ g_4(kl) = \cos kl shkl - \sin kl chkl \end{cases} \tag{A.10}$$

Thus, $Y_R(x)$ and its derivatives can be expressed in terms of the coefficients C_3 and C_4.

$$\begin{cases} Y_R(x) = C_3[h_1(kl)\sin kx + h_4(kl)\cos kx + shkx] \\ \qquad + C_4[h_3(kl)\sin kx + h_2(kl)\cos kx + chkx] \\ Y_R'(x) = C_3 K[h_1(kl)\cos kx - h_4(kl)\sin kx + chkx] \\ \qquad + C_4 K[h_3(kl)\cos kx - h_2(kl)\sin kx + shkx] \\ Y_R''(x) = C_3 K^2[-h_1(kl)\sin kx - h_4(kl)\cos kx + shkx] \\ \qquad + C_4 K^2[-h_3(kl)\sin kx - h_2(kl)\cos kx + chkx] \\ Y_R'''(x) = C_3 K^3[-h_1(kl)\cos kx + h_4(kl)\sin kx + chkx] \\ \qquad + C_4 K^3[-h_3(kl)\cos kx + h_2(kl)\sin kx + shkx] \end{cases} \tag{A.11}$$

900

where,

$$\begin{cases} h_1(kl) = g_1(kl) + 2\frac{W_l}{W_b} kl \cos kl shkl \\ h_2(kl) = g_2(kl) - 2\frac{W_l}{W_b} kl \sin kl chkl \\ h_3(kl) = g_3(kl) + 2\frac{W_l}{W_b} kl \cos kl chkl \\ h_4(kl) = g_4(kl) - 2\frac{W_l}{W_b} kl \sin kl shkl \end{cases} \tag{A.12}$$

Step 2 For the left segment, i.e. $x = 0 \sim x_{10}$.

The PDE for the lateral vibration of the left segment is

$$\frac{\partial^4 y_L}{\partial x^4} + \frac{1}{a^2}\frac{\partial^2 y_L}{\partial t^2} = 0 \tag{A.13}$$

The boundary conditions for the fixed end ($x = 0$) are:

$$\begin{cases} y_L(0,t) = 0 \\ \frac{\partial y_L}{\partial x}(0,t) = 0 \end{cases} \tag{A.14}$$

After the separation of variables

$$y_L(x,t) = Y_L(x)\, T(t)$$

we have a Y_L – ODE

$$Y_L''''(x) - K^4 Y_L(x) = 0 \tag{A.15}$$

with B.C.'s

$$\begin{cases} Y_L(0) = 0 \\ Y_L'(0) = 0 \end{cases} \tag{A.16}$$

The general solution to Eq. (A.15) has the form of

$$Y_L(x) = d_1 \sin kx + d_2 \cos kx + d_3 shkx + d_4 chkx \tag{A.17}$$

By using the B.C., we can express $Y_L(x)$ and its derivatives in terms of the coefficients d_3 and d_4

$$\begin{cases} Y_L(x) = d_3(shkx - \sin kx) + d_4(chkx - \cos kx) \\ Y_L'(x) = d_3 K(chkx - \cos kx) + d_4 K(shkx + \sin kx) \\ Y_L''(x) = d_3 K^2(shkx + \sin kx) + d_4 K^2(chkx + \cos kx) \\ Y_L'''(x) = d_3 K^3(chkx + \cos kx) + d_4 K^3(shkx - \sin kx) \end{cases} \tag{A.18}$$

Step 3 The compatible conditions at the common point, $x = x_{10}$, between the two segments.

I. The lateral displacement must be the same, i.e., $Y_L(x_{10}) = Y_R(x_{10})$. Thus,

$$\begin{aligned} & d_3\left(shkx_{10} - \sin kx_{10}\right) + d_4\left(chkx_{10} - \cos kx_{10}\right) \\ & = C_3\left[h_1(kl)\sin kx_{10} + h_4(kl)\cos kx_{10} + shkx_{10}\right] \\ & + C_4\left[h_3(kl)\sin kx_{10} + h_2(kl)\cos kx_{10} + chkx_{10}\right] \end{aligned} \tag{A.19}$$

II. The slope of the center line of the beam must be the same, i.e, $Y_L'(x_{10}) = Y_R'(x_{10})$. Thus

$$\begin{aligned} & d_3\left(chkx_{10} - \cos kx_{10}\right) + d_4\left(shkx_{10} + \sin kx_{10}\right) \\ & = C_3\left[h_1(kl)\cos kx_{10} - h_4(kl)\sin kx_{10} + chkx_{10}\right] \\ & + C_4\left[h_3(kl)\cos kx_{10} - h_2(kl)\sin kx_{10} + shkx_{10}\right] \end{aligned} \tag{A.20}$$

III. The bending moment must be the same, i.e, $Y_L''(x_{10}) = Y_R''(x_{10})$. Thus,

$$\begin{aligned} & d_3\left(shkx_{10} + \sin kx_{10}\right) + d_4\left(chkx_{10} + \cos kx_{10}\right) \\ & = C_3\left[-h_1(kl)\sin kx_{10} - h_4(kl)\cos kx_{10} + shkx_{10}\right] \\ & + C_4\left[-h_3(kl)\sin kx_{10} - h_2(kl)\cos kx_{10} + chkx_{10}\right] \end{aligned} \tag{A.21}$$

IV. Because of the inertia of the lumped weight W_{10}, the shear has a jump at $x = x_{10}$, i.e.,

$$EIY_L'''(x_{10})T = EIY_R'''(x_{10})T + \frac{W_{10}}{g}Y_L(x_{10})\ddot{T}$$

which yields

$$Y_L'''(x_{10}) = Y_R'''(x_{10}) - \frac{W_{10}}{W_b}lk^4Y_L(x_{10})$$

902

Thus

$$
\begin{aligned}
& d_3\left(chkx_{10}+\cos kx_{10}\right)+d_4\left(shkx_{10}-\sin kx_{10}\right) \\
& =C_3\left[-h_1(kl)\cos kx_{10}+h_4(kl)\sin kx_{10}+chkx_{10}\right] \\
& +C_4\left[-h_3(kl)\cos kx_{10}+h_2(kl)\sin kx_{10}+shkx_{10}\right] \\
& -\frac{W_{10}}{W_b}kl\left[d_3\left(shkx_{10}-\sin kx_{10}\right)+d_4\left(chkx_{10}-\cos kx_{10}\right)\right]
\end{aligned}
\tag{A.22}
$$

Then a set of equations forms from Eq. (A. 19) to Eq. (A.22)

$$
\left\{
\begin{aligned}
& C_3 shkx_{10}+C_4 chkx_{10}-d_3 shkx_{10}-d_4 chkx_{10}=0 \\
& C_3\left(h_1\sin kx_{10}+h_4\cos kx_{10}\right)+C_4\left(h_3\sin kx_{10}+h_2\cos kx_{10}\right) \\
& \quad +d_3\sin kx_{10}+d_4\cos kx_{10}=0 \\
& C_3\left(h_1\cos kx_{10}-h_4\sin kx_{10}+chkx_{10}\right)+C_4(h_3\cos kx_{10} \\
& \quad -h_2\sin kx_{10}+shkx_{10})+d_3\left(\cos kx_{10}-chkx_{10}\right) \\
& \quad -d_4\left(\sin kx_{10}+shkx_{10}\right)=0 \\
& C_3\left(2chkx_{10}\right)+C_4\left(2shkx_{10}\right)+d_3\left[\left(-2chkx_{10}\right)+\tfrac{W_{10}}{W_b}kl\left(\sin kx_{10}-shkx_{10}\right)\right] \\
& \quad +d_4\left[\left(-2shkx_{10}\right)+\tfrac{W_{10}}{W_b}kl\left(\cos kx_{10}-chkx_{10}\right)\right]=0
\end{aligned}
\right.
\tag{A.23}
$$

The condition for set (A.23) having non-trivial solution is that the determinant of the set (A.23) equals to zero, that is

$$
\mathrm{DET}\begin{vmatrix}
SHX & CHX & -SHX & -CHX \\
h_1 SNX+h_4 CSX & h_3 SNX+h_2 CSX & SNX & CSX \\
h_1 CSX-h_4 SNX+CHX & h_3 CSX-h_2 SNX+SHX & CSX-CHX & -SNX-SHX \\
2CHX & 2SHX & -2CHX+\frac{W_{10}}{W_b}kl(SNX-SHX) & -2SHX+\frac{W_{10}}{W_b}kl(CSX-CHX)
\end{vmatrix}=0
\tag{A.24}
$$

Actually, Eq. (A.24) is the characteristic equation of the Bernoulli-Euler beam equation used in this paper, from which the eigenvalues K_i will be derived. Solving the characteristic equation by using binary searching technique, we can obtain the eigenvalues K_i,

$$K_i = 0.0172,\ 0.0507,\ 0.1074,\ 0.1405,\ 0.1998,$$
$$0.2258,\ 0.2263,\ 0.2268,\ 0.2278,\ 0.2283, \ldots$$

After determining the eigenvalues K_i's we can solve the set (A.23) for the coefficients C_3, C_4, d_3 and d_4. In fact, there are infinite number of solutions to the set (A.23) because C_3, C_4, $d_3$and d_4 are not totally independent. Assuming the solution to the set (A.23) is

$$\begin{cases} C_3 = \alpha_i C_i \\ C_4 = \beta_i C_i \\ d_3 = r_i C_i \\ d_4 = C_i \end{cases} \tag{A.25}$$

then, from Eqs. (A.11) and (A.18) we have the characteristic functions

$$Y_i(x) = \begin{cases} Y_{L_i}(x) = r_i\,(shk_i x - \sin k_i x) + (chk_i x - \cos k_i x) & 0 \le x \le x_{10} \\ Y_{R_i}(x) = \alpha_i\,[h_1(k_i l)\sin k_i x + h_4(k_i l)\cos k_i l + shk_i x] & \\ \quad +\beta_i\,[h_3(k_i l)\sin k_i x + h_2(k_i l)\cos k_i l + chk_i x] & x_{10} \le x \le l \end{cases} \tag{A.26}$$

By superposition, then, the solution to the PDE should be

$$\begin{aligned} y(x,t) &= \sum_i Y_i(x)\,T_i(t) \\ &= \sum_i Y_i(x)\,[A_i \cos\omega_i t + B_i \sin\omega_i t] \end{aligned} \tag{A.27}$$

where, $\omega_i = K_i^2 a$.

When proportional damping is taken into account, the PDE describing damped lateral vibration will be

$$m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} + k\frac{\partial^4 y}{\partial x^4} = 0 \tag{A.28}$$

Here, two parameters a and b, which relate the coefficients of the PDE, are defined,

$$\begin{cases} a^2 = \frac{k}{m} = \frac{EI}{\rho A} \\ 2b = \frac{c}{m} \end{cases} \tag{A.29}$$

The substitution of $y(x,t) = \sum_i T_i(t) Y_i(x)$ into Eq. (A.28) yields

$$\sum_i \left(m\ddot{T}_i Y_i + C\dot{T}_i Y_i + kT_i Y_i'''' \right) = 0$$

which bears further a set of independent equations under the generalized coordinates $T_i(t)$ by the orthogonality property of the eigenfunctions,

$$m_i \ddot{T}_i + c_i \dot{T}_i + k_i T_i = 0 \tag{A.30}$$

where,

$$m_i = m \int_0^l Y_i^2 dx \qquad \text{generalized mass}$$

$$k_i = k \int_0^l Y_i Y_i'''' dx = m\omega_{n_i}^2 \int_o^l Y_i^2 dx \qquad \text{generalized stiffness}$$

$$c_i = c \int_o^l Y_i^2 dx \qquad \text{generalized damping}$$

Eq. (A.30) can be expressed in modal form

$$\ddot{T}_i + 2\xi_i \omega_{n_i} \dot{T}_i + \omega_{n_i}^2 T_i = 0 \tag{A.31}$$

where,

$$2\xi_i \omega_{n_i} = \frac{c_i}{m_i}$$

$$\omega_{n_i}^2 = \frac{k_i}{m_i}$$

Note that the damping ratio ξ_i is related to the eigenvalue K_i through the parameter a and b. In fact,

$$\xi_i = \frac{1}{2\omega_{n_i}} \frac{c_i}{m_i} = \frac{1}{2\omega_{n_i}} \frac{c}{m} = \frac{b}{ak_i^2} \tag{A.32}$$

Eq. (A.31) turns out to be the equation of motion of a simple mass-spring-dashpot system. Therefore, the ith-component of the response can be expressed as

$$T_i(t) = e^{-\xi_i \omega_{n_i} t}(A_i \cos \omega_{d_i} t + B_i \sin \omega_{d_i} t) \tag{A.33}$$

where,

$$\xi_i \omega_{n_i} = \frac{c_i}{2m_i} = \frac{c}{2m} = b$$

$$\omega_{d_i} = \omega_{n_i}\sqrt{1-\xi_i^2} = (ak_i^2)\sqrt{1-\left(\frac{b}{ak_i^2}\right)^2} = \sqrt{(ak_i^2)^2 - b^2}$$

Thus, $T_i(t)$ can be expressed in terms of parameters a and b,

$$T_i(t) = e^{-bt}\left(A_i \cos t\sqrt{(ak_i^2)^2 - b^2} + B_i \sin t\sqrt{(ak_i^2)^2 - b^2}\right) \tag{A.34}$$

By superposition, finally, the solution to the Eq. (A.28) should be

$$\begin{aligned} y(x,t) &= \sum_i Y_i(x)\, T_i(t) \\ &= \sum_i Y_i(x)\, e^{-bt}\left(A_i \cos \sqrt{(ak_i^2)^2 - b^2} + B_i \sin t\sqrt{(ak_i^2)^2 - b^2}\right) \end{aligned} \tag{A.35}$$

where, $Y_i(x)$ are the eigenfunctions shown in Eq. (A.26).