"A Model for the Three-Dimensional Spacecraft Control Laboratory Experiment"

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A MODEL FOR THE THREE-DIMENSIONAL SPACECRAFT CONTROL LABORATORY EXPERIMENT

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In this paper, a model for the three-dimensional Spacecraft COntrol Laboraory Experiment (SCOLE) is developed. The objective behind this method of modeling is to utilize the basic partial differntial equations of motion for this distributed parameter system and not to use the modal expansion in developing the model. The final model obtained is in terms of a transfer function matrix which relates the flexible mast parameters like displacement, slope, shear stress etc. to external forces and moments.

1. INTRODUCTION

It is widely recognized that the future space exploration would require a wide array of very large and flexible spacecrafts with very stringent pointing and vibration suppression requirements. Some of these spacerafts would also be deployed as an assemblage of a number of flexible members. In order to design control systems to meet these requirements, accurate dynamical models of the flexible spacrafts would have to obtained. Generally, the basic dynamical equations are developed in terms of a system of partial differential equations and one common approach is to formaulate solutions of these equations in terms of an infinite modal expansions and use this approach for developing control systems.

In this paper, an attempt is made to work with the basic partial differential equations and by using Laplace Transforms and incorporating boundary condition relationships an alternate modeling scheme is proposed. This methodolgy is based on extensive details documented in reference [1] and is applied to NASA Langley Research Center's SCOLE problem [2,3].

2. NOMENCLATURE

$u_x(t,z)$ Displacement at point z in roll bending

$\theta_x(t,z)$ Slope of beam at point z in roll bending

$M_x(t,z)$ Bending moment at point z in roll bending

$\sigma_x(t,z)$ Shear stress at point z in roll bending

$f_x(t,z)$ External force per unit length in roll bending

I_x Moment of inertia of area about neutral axis in roll bending

B_x Damping in roll bending

$u_x(t,z)$ Displacement at point z in pitch bending

$\theta_y(t,z)$ Slope of beam at point z in pitch bending

$M_y(t,z)$ Bending at beam at point z in pitch bending

$\sigma_y(t,z)$ Shear stress at point z in pich bending

$f_y(t,z)$ External force per unit length in pitch bending

I_y Moment of inertia of area about neutral axis in pitch bending

B_y Damping in pitch bending

$\psi(t,z)$ The angular displacement at z of an element dz of the beam

I_p The polar moment of inertia of the cross-section

ρ_p The mass per unit volume

I_pG The torsional stiffness of the beam

G The shear modulus of the material

I Modulus of elsticity

ρ Mass per unit length

3. METHODOLOGY

The partial differential equations governing the roll bending motion are

$$\frac{\partial u_x}{\partial z} = \theta_x \tag{1}$$

This is obtained from the definition of slope.

$$\frac{\partial \theta_x}{\partial z} = -\frac{1}{(EI)_x} M_x = -\frac{1}{(EI)} M_x \tag{2}$$

This equation is based on beam theory and here $(EI)_x$ and $(EI)_y$ are considered equal and represented by (EI)

$$\frac{\partial M_x}{\partial z} = \sigma_x \tag{3}$$

This is obtained from the definition of bending moment.

$$\frac{\partial \sigma_x}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2} - f_x(t,z) \tag{4}$$

This equation is based on Newton's law of motion and the term $f_x(t,z) = -B_x \frac{\partial x}{\partial t}$ if there is no external force per unit length on the beam.

The corresponding equations for pitch bending are

$$\frac{\partial u_y}{\partial z} = \theta_y \tag{5}$$

$$\frac{\partial \theta_y}{\partial z} = -\frac{1}{(EI)_y} M_y \tag{6}$$

$$\frac{\partial M_y}{\partial z} = \sigma_y \tag{7}$$

$$\frac{\partial \sigma_y}{\partial z} = \rho \frac{\partial^2 u_y}{\partial t^2} - f_y(t,z) \tag{8}$$

The following equations describe torsional bending

$$\frac{\partial \psi}{\partial z} = \frac{1}{I_p G} T \tag{9}$$

This is based on torsional flexibility.

$$\frac{\partial T}{\partial z} = \rho I_p \frac{\partial \psi}{\partial t^2} - M_\psi(t,z) \tag{10}$$

This equation is based on Newton's Law of motion and the term $M_\psi(t,z) = -B_\psi \frac{\partial \psi}{\partial z}$ if there is no external torque per unit length of the beam.

Defining the state variables as

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \end{bmatrix} = \begin{bmatrix} u_x \\ \theta_x \\ M_x \\ \sigma_x \\ u_y \\ \theta_y \\ M_y \\ \sigma_y \\ \psi \\ T \end{bmatrix} , \tag{11}$$

the following equations are obtained

$$\frac{\partial q_1}{\partial z} = q_1$$

$$\frac{\partial q_2}{\partial z} = -\frac{1}{(EI)} q_3$$

$$\frac{\partial q_3}{\partial z} = q_4$$

$$\frac{\partial q_4}{\partial z} = \rho(\frac{\partial^2}{\partial t^2}) q_1 - f_x(t,z)$$

$$\frac{\partial q_5}{\partial z} = q_6$$

$$\frac{\partial q_6}{\partial z} = -\frac{1}{(EI)} q_7 \tag{12}$$

$$\frac{\partial q_7}{\partial z} = q_8$$

$$\frac{\partial q_8}{\partial z} = \rho(\frac{\partial^2}{\partial t^2}) q_5 - f_y(t,z)$$

$$\frac{\partial q_9}{\partial z} = \frac{1}{I_p G} q_{10}$$

$$\frac{\partial q_{10}}{\partial z} = \rho I_p(\frac{\partial^2}{\partial t^2}) q_9 - M_\psi(t,z)$$

These equations can be expressed in the form

$$\frac{\partial \underline{q}}{\partial z} = F_0 \underline{q} + F_1 \frac{\partial \underline{q}}{\partial t} + F_2 \frac{\partial^2 \underline{q}}{\partial t^2} + \underline{u}(t,z) \tag{13}$$

Taking Laplace Transforms of the previous matrix-vector equation, the following equation is obtained.

$$\frac{d\underline{Q}}{dz} = (F_0 + F_1 s + F_2 s^2)\,\underline{Q} - F_1 \underline{q}(0,z) - F_2\,[\,\dot{\underline{q}}(0,z) + s\underline{q}(0,z)\,] + \underline{U}(s,z)$$

$$\frac{d\underline{Q}}{dz} = (F_0 + F_1 s + F_2 s^2)\,\underline{Q} + \overline{\underline{U}}(s,z) \qquad (14)$$

Here F_1 is a 10 x 10 null matrix. The matrix F_0 is a 10 x 10 matrix with zeros except the following nonzero elements.

$$F_0(21) = 1$$

$$F_0(3,2) = \frac{-1}{(EI)}$$

$$F_0(4,3) = 1$$

$$F_0(7,6) = \frac{-1}{(EI)}$$

$$F_0(8,7) = 1$$

$$F_0(9,9) = \frac{1}{I_p G}$$

The matrix F_2 is also a 10 x 10 matrix with zero entries except for the following elements

$$F_2(4,1) = \rho$$
$$F_2(8,5) = \rho$$
$$F_2(10,9) = \rho\, I_p$$

The equation (14) represents a linear system and this can be solved by using the state-transition matrix as

$$\underline{Q}(s,z) = H(s,z - z_0)\,\underline{Q}(s,z_0) + \int_{z_0}^{z} H(s,z-\xi)\overline{\underline{U}}\,(s,\xi)d\xi \quad . \qquad (15)$$

The SCOLE model is of finite length; i.e. $0 \le z \le L$. Then, at $z = L$,

$$\underline{Q}(s,L) = H(s,L)\,\underline{Q}(s,0) + \int_{0}^{L} H(s,L-\xi)\overline{\underline{U}}\,(s,\xi)d\xi \quad . \qquad (16)$$

To determine $Q(s,L)$ and $Q(s,0)$ which in turn would allow determination of $Q(s,z)$ for any z between 0 and L, ten terminal relations must be specified. These terminal relations can be expressed in the form of ten ordinary differential equations in the following vector-matrix form using linear differential operators.

$$M\,(D)\,\underline{q}(t,0) + N\,(D)\,\underline{q}(t,L) = \underline{F}(t) \tag{17}$$

Here, $F(t)$ represents external control forcing functions (the physical inputs like forces on the flexible beam or shuttle moments etc. or linear combinations thereof). Taking Laplace Transforms on both the sides of equation (17), the following vector-matrix equation is obtained.

$$M\,(s)\,\underline{Q}(s,0) + N(s)\,\underline{Q}(S,L) = \underline{F}(s) \tag{18}$$

The vector $F(s)$ is the sum of the Laplace Transforms of $F(t)$ and any initial condition terms. The termination is said to bo homogeneous if $\underline{F}(s) = \underline{0}$.

It is important to note here that each of the differential equations given in (17) in case of a distributed parameter system may involve quantities at both ends of the distributed system. This represents termination characteristics of a feedback system where quantities at one end are made to depend on quantities at the other. The boundary conditions for SCOLE model without any external forcing functions are given as follows.

At the shuttle end where $z = 0$,

$$(EI)\frac{\partial\, u_x}{\partial z^2}\,(0,t) = 0 \quad \textit{Moment} \tag{19a}$$

$$(EI)\frac{\partial\, u_y}{\partial z^2}\,(0,t) = 0 \quad \textit{Moment} \tag{19b}$$

$$(EI)\frac{\partial\, u_x}{\partial z^3}\,(0,t) = 0 \quad \textit{ShearForce} \tag{19c}$$

$$(EI)\frac{\partial\, u_y}{\partial z_3}\,(o,t) = 0 \quad \textit{ShearForce} \tag{19d}$$

$$(I_pG)\frac{\partial\psi}{\partial z}\,(0,t) = 0 \quad \textit{Torquq} \tag{19e}$$

At the reflector end where $z = L$, the corresponding boundary conditions are

$$(EI)\frac{\partial\, u_x}{\partial z^2}\,(L,t) = 0 \tag{20a}$$

$$(EI)\frac{\partial u_y}{\partial z^2}(L,t) = 0 \tag{20b}$$

$$(EI)\frac{\partial u_x}{\partial z^3}(L,t) = 0 \tag{20c}$$

$$(EI)\frac{\partial u_y}{\partial z^3}(L,t) = 0 \tag{20d}$$

$$(I_pG)\frac{\partial \psi}{\partial z}(L,t) = 0 \quad . \tag{20e}$$

In order to obtain a complete representation of the system governed by equations (16) and (18), we substitute (16) into (18) with $z_0 = 0$ and $z = L$;

$$[\, M(s) + N(s)\, H(s,L)\,]\, \underline{Q}(s,0) = \underline{V}(s) \tag{21}$$

where,

$$\underline{V}(s) = \underline{F}(s) - N(s)\int_0^L H(s,L-\xi)\underline{\overline{U}}(s,\xi)d\xi \tag{22}$$

It can be shown that the relationship given in (17) is independent and as a result $[\, M(s) + N(s)H(s)\,]$ is of rank 10 for SCOLE and so has an inverse. Hence, we can write

$$\underline{Q}(s,0) = [\, M(s) + N(s)H(s,L)\,]^{-1}\underline{V}(s) \tag{23}$$

Thus,

$$\underline{Q}(s,z) = H(s,z)\,[\, M(s) + N(s)H(s,L)\,]^{-1}\underline{V}(s) + \int_0^z H(s,z-\xi)\underline{\overline{U}}(s,\xi)d\xi \tag{24}$$

If there are no distributed forcing terms and initial conditions, equation (24) can be written as

$$\underline{Q}(s,z) = H(s,z)\,[\, M(s) + N(s)H(s,L)\,]^{-1}\underline{F}(s) \tag{25}$$

where $F(s)$ is the Laplace transform of the external forcing terms in (16). Hence the matrix of transfer functions from $\underline{F}(s)$ to $\underline{Q}(s,z)$ is given by

$$G(s,z) = H(s,z)\,[\, M(s) + N(s)H(s,L\,]^{-1} \tag{26}$$