PROBABILISTIC FINITE ELEMENTS FOR FRACTURE AND FATIGUE ANALYSIS

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The fusion of the probabilistic finite element method (PFEM) and reliability analysis for probabilistic fracture mechanics (PFM) is presented. The authors have developed a comprehensive method for determining the probability of fatigue failure for curved crack growth. The criterion for failure or performance function is stated as: the fatigue life of a component must exceed the service life of the component; otherwise failure will occur. An enriched element that has the near crack-tip singular strain field embedded in the element is used to formulate the equilibrium equation and solve for the stress intensity factors at the crack-tip. The crack growth law (e.g., Paris law) and crack direction law (e.g., direction which gives rise to the maximum tangential stress at the crack-tip) are also incorporated into the formulation. The loading is mixed-mode (i.e., opening mode and in-plane shear mode which are designated mode I and mode II, respectively) with randomness in the applied loads, material properties, component geometry, crack geometry, crack growth law, and the crack direction law. Examples of random crack geometry are the initial and final crack lengths, initial crack angle, and initial crack position. The methodology consists of calculating the reliability index, which is used to calculate the first-order probability of failure, by solving a constrained optimization problem. Included in the optimization for reliability is the equilibrium equation for stress intensity factors, the crack growth law, and the crack direction law.

Performance and accuracy of the method is demonstrated on a classical mode I fatigue problem. The initial and final crack lengths, material parameters in the Paris law,

and the applied stress are modeled with random variables. The results imply that the service life of a component must be well below the deterministic fatigue life of the component when many parameters are considered random. The fusion of PFEM and reliability for fatigue crack growth is computationally quite efficient and provides a powerful tool for the design engineer.

SCOPE OF WORK

Probabilistic Finite Element Method (PFEM)

- Uncertainties
    - Random Load
    - Random Material Properties
    - Random Geometry

- Statics and Dynamics

- Elasticity and Nonlinear Elasticity

Brittle Fracture by PFEM

- Probability of the Stress Intensity Factor exceeding the Fracture Toughness of the Material

$$P (\kappa_I > \kappa_{Ic})$$
$$P (G > G_c)$$

Fatigue Crack Growth by PFEM

- Probability of the Desired Service Life exceeding the Fatigue Life

$$P (T_s > T)$$

INTRODUCTION

Probabilistic Fracture Mechanics (PFM)

- Model the crack-tip singularity in stress.

- Determine statistics of stress intensity factors.

- Probability of fracture and fatigue failure.

Uncertainties which need to be considered in PFM are:

- Random load, material properties, and geometry;

- Random crack geometry;
    - length,
    - location,
    - orientation,

- Random crack growth laws;

- Random crack direction laws;

- Random correlation among detectable cracks and micro-cracks.

RELIABILITY OF COMPONENTS
SUBJECT TO
FATIGUE CRACK GROWTH
BY
PROBABILISTIC FINITE ELEMENTS

METHODOLOGY AND OBJECTIVE

- Given :

  - Component with Crack Subject to Cyclic Stress
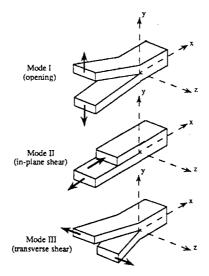 - Method for Determining Stress Intensity Factors
  - Crack Growth Law
  - Crack Direction Law

- With Randomness in :

  - Applied Loads
  - Component Geometry
  - Crack Geometry
  - Crack Direction Parameters
  - Crack Growth Parameters
  - Material Properties

- Determine Probability of Fatigue Failure

  - Probability that the desired service life is greater than the fatigue life.

    $$P(T_s > T)$$

STRESS INTENSITY FACTOR

κ_I = mode I stress intensity factor

κ_{II} = mode II stress intensity factor

κ_{III} = mode III stress intensity factor

Consider only Modes I and II: $\quad \kappa = \begin{bmatrix} \kappa_I \\ \kappa_{II} \end{bmatrix}$

RANDOM VARIABLES

System Uncertainties

- Applied Loads
- Component Geometry
  length, height, width, etc.
- Crack Geometry
  length, orientation, position
- Crack Direction Parameters
- Crack Growth Parameters
- Material Properties
  elastic modulus, poisson's ratio,
  fracture toughness, etc.,

Modeled by a correlated random variable vector

$$\mathbf{b} \ (q \times 1)$$

Transform $\mathbf{b}$ to a Independent Standard Normal Random Variable Vector

$$\mathbf{r} \ (q \times 1)$$

- Eigenvalue Orthogonalization (Normal R.V.'s)
  Covariance Matrix for $\mathbf{b}$

- Rosenblatt Transformation (Non-normal R.V.'s)
  Joint Probability Distribution for $\mathbf{b}$

RELIABILITY ANALYSIS

Performance Function or Margin of Safety

$$g > 0 \qquad g = 0 \qquad g < 0$$

safe state   *limit-state*   *failure state*

Probability of Failure

$$p_f = P(g \leq 0)$$
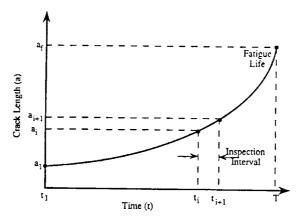
First-Order Estimate of the Probability of Failure

$$p_{f1} \approx \Phi(-\beta)$$

Φ is the standard normal cumulative distribution,
β is the reliability index.

Reliability Index

Minimum Distance from Limit-State Surface to Origin in Independent Standard Normal Space

$$\beta^2 = \sqrt{\mathbf{r}^T \mathbf{r}}$$

CRACK GROWTH LAW

Experimental fatigue crack growth data for different materials.

$$\frac{da}{dt} = f\left(\Delta\kappa_{eq}\right)$$

$\Delta\kappa_{eq}$ is equivalent stress intensity factor range

Region A: A sufficient level of activity at the crack-tip has not been attained resulting in very slow crack growth.

Region C: Corresponds to rapid unstable crack growth.

Region B: Represents the zone where most fatigue crack growth is analyzed.

Paris Law: $\quad \dfrac{da}{dt} = D\left(\Delta\kappa_{eq}\right)^n$

D is a material parameter
n is a material parameter

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Integrating from the initial crack length to the final crack length gives in the fatigue life

$$T = \int_{a_i}^{a_f} \frac{da}{D\left(\Delta\kappa_{eq}\right)^n}$$

Equivalent Stress Intensity Factor

- Maximum Stress Theory

$$\kappa_{eq} = \Theta^T\kappa$$

$$\Theta = \cos^2\left(\frac{\theta}{2}\right)\begin{bmatrix} \cos\dfrac{\theta}{2} \\[2ex] -3\sin\dfrac{\theta}{2} \end{bmatrix}$$

CRACK DIRECTION LAW

- Maximum Tangential Stress Criterion

- Minimum Strain Energy Density Criterion

- Maximum Energy Release Rate Criterion

Maximum Tangential Stress Criterion

- Crack grows in a radial direction from the crack
- Crack grows in the plane perpendicular to the direction of maximum tension

$\therefore$ Crack grows in the direction where the tangential stress is a maximum and the shear stress is zero.

$$Z(\kappa, \theta) \equiv \Phi^T\kappa = 0$$

$$\Phi = \begin{bmatrix} \sin\theta \\[2ex] 3\cos\theta - 1 \end{bmatrix}$$

For mode I fracture, $\theta = 0°$ and is constant

For mode II is fracture, $\theta = 70.54°$ and changes

Basis for determining the reliability and/or probability of failure depends on the choice of a performance function.

$$p_f = P(g \leq 0)$$

Performance function for fatigue crack growth

$$g = T - T_s$$

T_s is the desired service life
T is the fatigue life

The component fails when the fatigue life is less than the desired service life.

DISCRETIZATION

Discretize the Crack Path into "nint" Integration Points

• Crack Length

$$a_i = \frac{1}{2} \left[(a_f - a_1) \xi_i + (a_f + a_1) \right] \quad i = 1, ..., nint$$

• Crack Growth Law

$$T = \int_{-1}^{+1} \frac{J}{D (\kappa_{eq})^n} \, d\xi \qquad\qquad J = \frac{da}{d\xi}$$
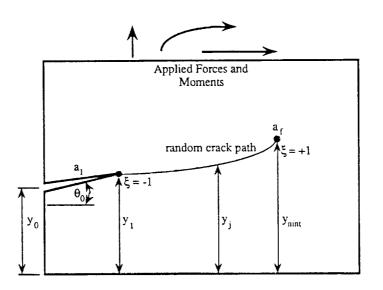
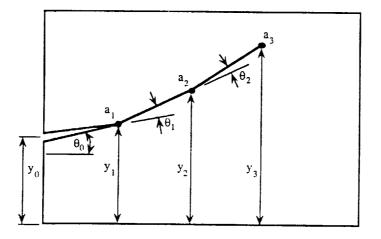• Crack Direction Law

$$Z_k = \Phi_k^T \kappa_k = 0 \qquad\qquad k = 1, ..., nint$$

Crack direction is calculated at each integration point
Crack grows to the next integration point

• Equilibrium

$$K_i \; \delta_i = f_i \qquad\qquad i = 1, ..., nint$$

$$\delta_i = \begin{bmatrix} d \\ \kappa \end{bmatrix}_i$$

Applied Forces

a_f

a_i

a_1

$\xi = -1$ random crack path $\xi = +1$

Applied Forces and
Moments

random crack path

a_f

$\xi = +1$

a_1

θ_0 $\xi = -1$

y_0 y_1 y_j y_{nint}

a_3

a_2 θ_2

a_1 θ_1

θ_0

y_0 y_1 y_2 y_3

RELIABILITY AS AN OPTIMIZATION PROBLEM
FOR FATIGUE CRACK GROWTH

First-Order Probability of Failure

$\rightarrow$ Reliability Index

$\rightarrow$ Constrained Optimization Problem

<u>Minimize</u> the Reliability Index

$$\beta^2 = \sqrt{r^T\, r}$$

Subject to <u>Equality</u> Constraints

Crack growth direction law

$$Z_k = 0 \qquad\qquad k = 1, ..., \text{nint}$$

Equilibrium

$$K_i\, \delta_i = F_i \qquad\qquad i = 1, ..., \text{nint}$$

Subject to <u>Inequality</u> Constraint

Limit-State Surface and Failure State

$$T - T_s \le 0$$

Define a Lagrange Functional

$$\mathcal{L}\ (\mathbf{b}, \mu_1, ..., \mu_{nint}, \delta_1, ..., \delta_{nint}, \alpha, \lambda,$$

$$\varphi_1, ..., \varphi_{nint}, \theta_1, ..., \theta_{nint})$$

$$= \beta^2 + \sum_{i=1}^{nint} \mu_i^T [\, F_i - K_i\ \delta_i \,]$$

$$+ \sum_{k=1}^{nint} \varphi_k [\, Z_k = 0 \,]$$

$$+ \lambda \left\{ T - T_s + \alpha^2 \right\}$$

Necessary Conditions
(derivatives w.r.t. <u>independent</u> variables)

Probabilistic Finite Element Method
(PFEM)

Eliminate Lagrange Multipliers

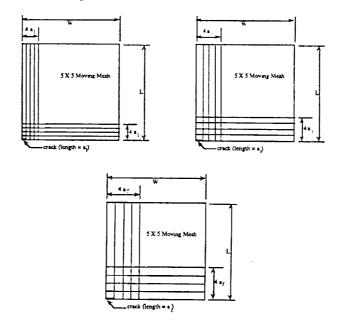Final Form of Optimization for Reliability

$$L^\beta + \lambda\ L^g = 0$$

$$L^\beta = \frac{\partial \beta^2}{\partial \mathbf{b}}$$

$$L^g = \sum_{i=1}^{nint} \left[\sum_{k=1}^{i} \left(\frac{T,_{\theta_k}}{(Z_k),_{\theta_k}} (Z_k),_{\kappa_i} \right) + T,_{\kappa_i} \right] \frac{\partial \kappa_i}{\partial \mathbf{b}}$$

$$+ \sum_{k=1}^{nint} \left(\frac{T,_{\theta_k}}{(Z_k),_{\theta_k}} (Z_k),_{\mathbf{b}} \right) + T,_{\mathbf{b}}$$

Solve for **b** via an Optimization Algorithm

- Lagrange Multipliers
- Gradient Projection Algorithm
- HL-RF Algorithm (Tangent Matching)
- Modified HL-RF Algorithm

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CRACK GROWTH MOVING MESH

- Variation of ALE formulation
  - Pseudo-Static (neglect transport term)

- Moving Element Mesh with Constraints
  - Aspect ratio of elements must remain one

EXAMPLE

Single-Edged Cracked Plate with an Applied Load

$$b = [\, a_1\,, a_f\,, D\,, n\,, \sigma\,]^T$$
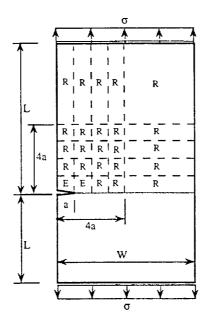
a_1 is the initial crack length

a_f is the final crack length

D is a material parameter

n is a material parameter
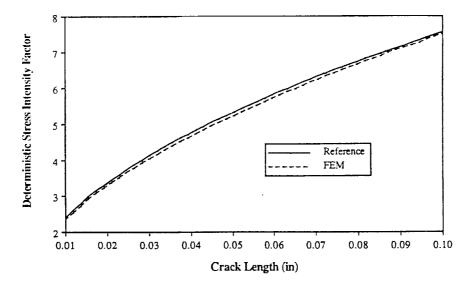
σ is the applied stress

Parameter	Mean	Standard Deviation	Percent
Length (L)	10.0 in	0.0	0.0
Width (W)	4.0 in	0.0	0.0
Thickness (t)	1.0 in	0.0	0.0
Young's Modulus (E)	30,000.0 ksi	0.0	0.0
Poisson's Ratio (v)	0.3	0.0	0.0
Applied Stress (σ)	12.0 ksi	3.0 ksi	25.0
Initial Crack Length (a_1)	0.01 in	0.01 in	100.0
Final crack Length (a_f)	0.1 in	0.01 in	10.0
Fatigue Parameter (D)	1.0×10^{-10}	3.0×10^{-11}	30.0
Fatigue Parameter (n)	3.25	0.08	2.5

- Variations in the parameters are reasonable.

- nint = 50

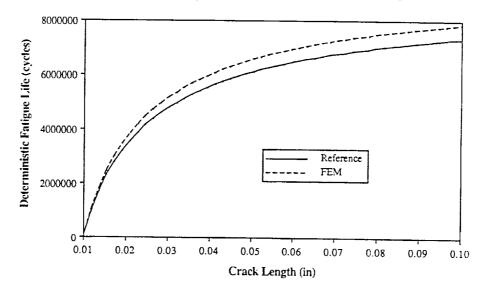- Deterministic fatigue life, T = 7,500,000 cycles

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Reference Solution

$$\kappa_I = \gamma \; \sigma \; \sqrt{a}$$

$$\gamma = 1.99 - 0.41\frac{a}{W} + 18.7\left(\frac{a}{W}\right)^2 - 38.48\left(\frac{a}{W}\right)^3 + 53.85\left(\frac{a}{W}\right)^4$$

Deterministic Stress Intensity Factor for Reference and FEM Solutions

Deterministic Fatigue Life for Reference and FEM Solutions

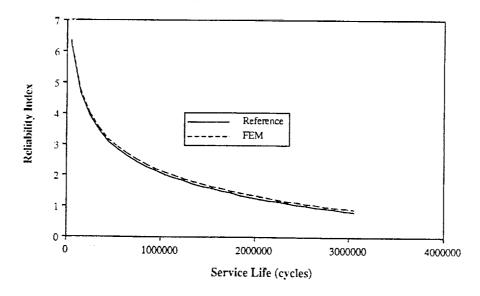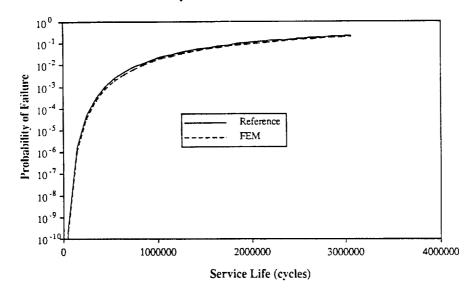## Reliability Index for Combined Randomness

Probability of Failure for Combined Randomness