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Scalars, Vectors, and Tensors

Robert G. Deissler
Lewis Research Center
Cleveland, Ohio

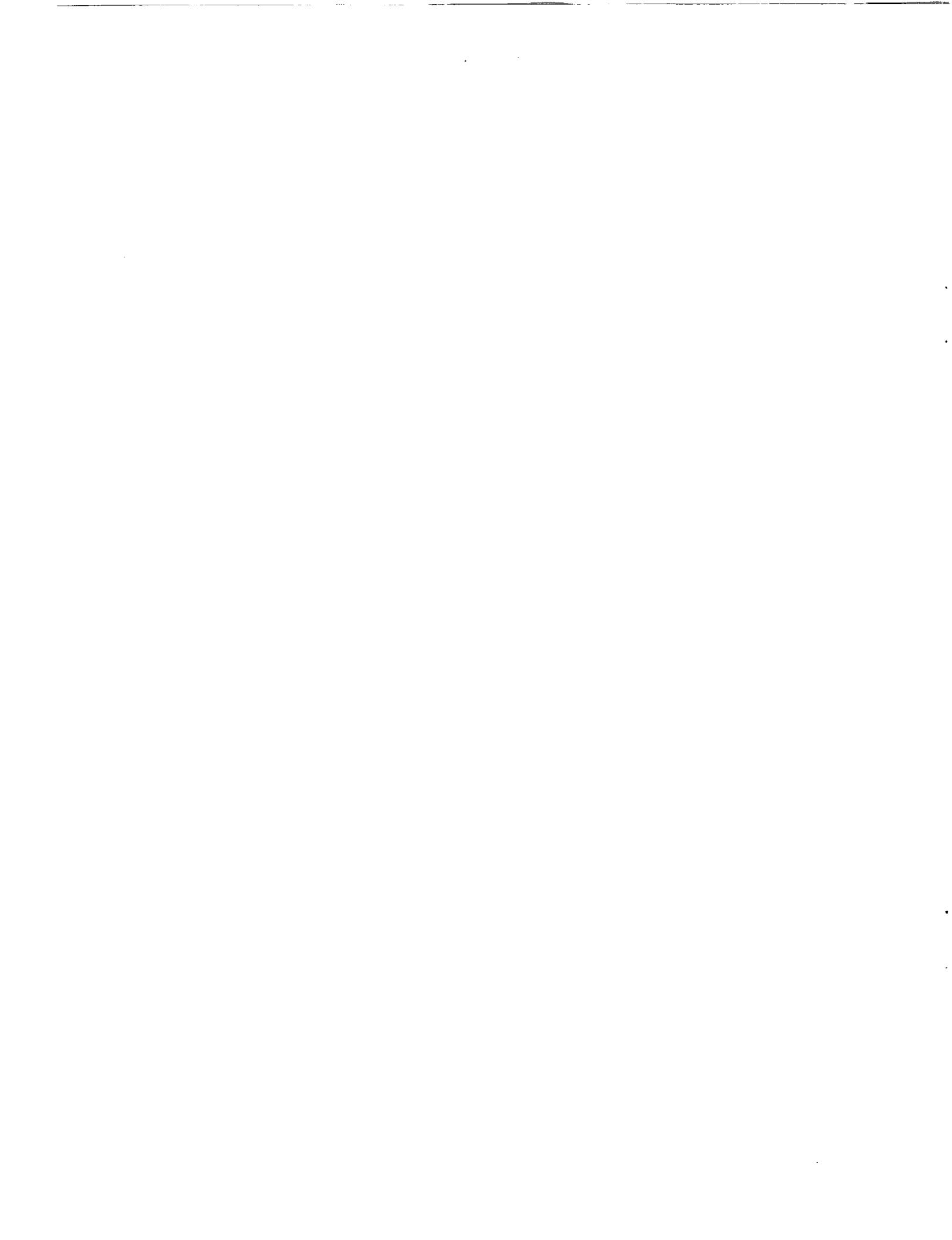July1991

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TURBULENT FLUID MOTION II -

Scalars, Vectors, and Tensors

Robert G. Deissler
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

SUMMARY

Some of the mathematical apparatus used for the representation and study of turbulence is developed.

2.1 INTRODUCTION

In the study of turbulence one encounters physical quantities such as densities, temperatures, forces, velocities, velocity products, and stresses. We notice that these quantities are not all of the same type. For instance a density requires only one number to represent it, but a velocity requires three numbers, or it has three components. A quantity which is the product of two velocities, where each component of one velocity is multiplied in turn by each component of the other, has nine components.

A density is an example of a scalar, or of a tensor of order zero. It can be represented by a symbol with no subscripts, say ρ. A velocity is an example of a vector and is called a tensor of order one. It can be represented by a symbol with one subscript, say u_i, where $i = 1, 2,$ or 3. The three values of i correspond to directions in space parallel to the directions of the three perpendicular coordinate axes x_i ($i = 1, 2, 3$). The x_i form a right-handed coordinate system, and are often written as (x, y, z). A quantity which is the product of two velocities is an example of a second-order tensor. It can be represented by a symbol with two subscripts, say $T_{ij} = u_i u_j$, where $i = 1, 2, 3$; $j = 1, 2, 3$. Similarly, products of more than two velocities are tensors of higher order. Thus $u_i u_j u_k$ ($i, j, k = 1, 2,$ or 3) is a third order tensor, etc. Averaged values of velocity products, written as $\overline{u_i u_j}$, $\overline{u_i u_j u_k}$ etc. and called velocity correlations (the overbars indicate averaged values), are important quantities in the theory of turbulence.

2.2 ROTATION OF COORDINATE SYSTEMS

It should not be assumed that all quantities represented by symbols with a given number of subscripts (0, 1, 2, . . . subscripts) are tensors as in the examples of the last paragraph (see e.g., section 2.8). To be called a tensor, it is necessary that a quantity obey a certain transformation law when referred to a rotated coordinate system.[1] In considering the transformation laws of tensors, we first note that a rectangular coordinate system transforms under a rotation according to the law

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[1] We shall consider here only rectangular Cartesian coordinate systems. Tensors defined in terms of the transformation laws of such coordinate systems are called Cartesian tensors. When we use the term "tensor" in this series "Cartesian tensor" will be understood.

$$x_i^* = a_{ij}x_j, \tag{2-1}$$

where x_i^* is a coordinate of a point in the rotated coordinate system x_i^*, and x_i is a coordinate of the same point in the unrotated system x_i. (Note that x_i^* or x_i can designate either a coordinate system or a coordinate, since a coordinate system is given by its coordinates.) The a_{ij} form a set of nine constants. The conditions under which equation (2-1) makes sense as a transformation law will be considered later in this section.

Equation (2-1) uses the Einstein summation convention, where a repeated subscript in a term designates a sum of terms, with the subscript successively taking on the values 1, 2, and 3. This convention will be used throughout the text, except where otherwise indicated. Note that the symbol used for a repeated subscript is immaterial, so that such a subscript is often called a dummy subscript. The expression for x_1^*, written out, is

$$x_1^* = a_{11}x_1 + a_{12}x_2 + a_{13}x_3.$$

The symbols used for the subscripts which are unrepeated in a term of an equation (see, e.g., equation (2-1)) are also immaterial, so long as the same symbols are used in all terms, and so long as they differ from those used for other subscripts in the equation. Thus in substituting one equation into another, the symbols used for some of the subscripts must frequently be changed in order to avoid confusion. If a subscript occurs more than twice in a term the equation is generally ambiguous. Also, the same unrepeated subscripts must occur in all terms of an equation. These points, while possibly obvious once they have been mentioned, are important for carrying out meaningful tensor manipulations.

The square of the distance ds between two neighboring points is given in the x_i-coordinate system by

$$ds^2 = dx_i dx_i = \delta_{ij} dx_i dx_j \tag{2-2}$$

where δ_{ij} is called the Kronecker delta, defined by

$$\left. \begin{array}{l} \delta_{ij} = 1 \quad \text{for} \quad i = j, \\ \text{and} \\ \delta_{ij} = 0 \quad \text{for} \quad i \neq j. \end{array} \right\} \tag{2-3}$$

The truth of equation (2-2) can be seen by writing out the three terms of $dx_i dx_i$ (summation on i) and the nine terms of $\delta_{ij} dx_i dx_j$ (summation on i and j) and using (2-3). Note that $\delta_{ij} = \delta_{ji}$. In order that the transformation (2-1) make physical or geometrical sense, it is necessary that the distance ds be the same in the x_i^*- and the x_i-coordinate systems. Thus

$$ds^2 = dx_i^* dx_i^* = a_{ij} a_{ik} dx_j dx_k \tag{2-4}$$

$$= \delta_{jk} dx_j dx_k$$

where equation (2-1) in differential form ($dx_i^* = a_{ij}dx_j$) is applied twice, and equation (2-2) is used. Equation (2-4) gives

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$$(a_{ij}a_{ik} - \delta_{jk})dx_j dx_k = 0$$

for all values of the dx_i, or

$$a_{ij}a_{ik} = \delta_{jk}. \qquad (2-5)$$

Equation (2-5) gives the nine relations (six of which are different) which must be satisfied by the a_{ij} if equation (2-1) is to be a sensible transformation law (lengths remain invariant under the transformation). Multiplication of equation (2-1) by a_{ik} and use of equation (2-5) and (2-3) give

$$a_{ik}x_i^* = a_{ik}a_{ij}x_j = \delta_{jk}x_j = x_k, \qquad (2-6)$$

where the last step can be verified by writing out the terms and using equation (2-3). In general, multiplication of a quantity containing say a subscript j or a subscript k (or both) by δ_{jk} changes the subscript j to k, or k to j, in that quantity.

Considering only the first and last terms of equation (2-6), and changing the subscript k to j on both sides of the equation, gives

$$x_j = a_{ij}x_i^*. \qquad (2-7)$$

If we differentiate equation (2-1) with respect to x_j and equation (2-7) with respect to x_i^*, we get

$$\frac{\partial x_i^*}{\partial x_j} = \frac{\partial x_j}{\partial x_i^*} = a_{ij}. \qquad (2-8)$$

A set of equations equivalent to (2-5), but of slightly different form, can be obtained as follows:

$$ds^2 = dx_i dx_i = a_{ji}dx_j^* a_{ki}dx_k^* = dx_i^* dx_i^* = \delta_{jk}dx_j^* dx_k^*$$

where equation (2-7) in differential form is used. Therefore

$$a_{ji}a_{ki} = \delta_{jk}. \qquad (2-5a)$$

2.3 VECTORS (FIRST-ORDER TENSORS)

A quantity u_i is said to be a vector, or a first-order tensor, if it obeys the transformation law

$$u_i^* = a_{ij}u_j, \qquad (2-9)$$

where u_i^* is a component of a vector in the rotated coordinate system x_i^* and u_i is a component of the same vector in the unrotated system x_i. (Note that u_i (or u_i^*) can designate either a vector component or the vector itself, since a vector can be specified by specifying its components.)

We note by comparing equations (2-1) and (2-9) that a vector transforms according to the same law as the coordinates of a point. Thus the coordinates of a point form a vector. They define a directed-line segment drawn from the origin to the

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point x_i. That vector is usually called a position or displacement vector. Similarly any vector defined by equation (2-9) can be interpreted as a directed-line segment. So the definition given by equation (2-9) agrees with the perhaps more familiar definition of a vector as a directed-line segment, or as a quantity with both magnitude and direction (displacement, velocity, area, force, etc.).

It is easy to show that the sum or difference of two vectors, say u_i and v_i is a vector. For, from equation (2-9),

$$u_i^* + v_i^* = a_{ij}u_j + a_{ij}v_j = a_{ij}(u_j + v_j) \qquad (2\text{-}10)$$

which shows that $u_i + v_i$ obeys the transformation law for a vector. Equation (2-10) is another way of stating the familiar addition law for vectors represented by directed-line segments; instead of adding directed-line segments geometrically, we add corresponding components in either the x_i or x_i^* coordinate system.

2.4 SECOND-ORDER TENSORS

2.4.1 Definition and Simple Examples

A second-order tensor u_{ij} is defined by generalization of equation (2-9) as a quantity that obeys the transformation law

$$u_{ij}^* = a_{ik}a_{jl}u_{kl} \qquad (2\text{-}11)$$

where, as usual, repeated subscripts (in this case, k and l) indicate summations. Thus equation (2-11) represents nine equations, each with nine terms on the right side. Tensor notation affords, among other things, considerable economy in writing.

From the definition given by equation (2-11) it follows that the product of two vectors u_i and v_j is a second-order tensor. For, from equation (2-9),

$$u_i^* v_j^* = a_{ik}a_{jl}u_k v_l , \qquad (2\text{-}12)$$

which shows that $u_i v_j$ obeys the transformation law for a second-order tensor (equation (2-11)). The product $u_i v_j$, where the subscript on one vector is not repeated on the other one, is called an outer product.

Another example of a second-order tensor is the spatial derivative, or gradient, of a vector $\partial u_i / \partial x_j$. We can show that $\partial u_i / \partial x_j$ is a second-order tensor as follows: Following the rules of partial differentiation, we obtain

$$\frac{\partial u_i^*}{\partial x_j^*} = \frac{\partial u_i^*}{\partial x_k}\frac{\partial x_k}{\partial x_j^*} = \frac{\partial x_k}{\partial x_j^*}\frac{\partial}{\partial x_k}(a_{jl}u_l) , \qquad (2\text{-}13)$$

where u_i^* is assumed to be a function of x_k, and equation (2-9) is used. Using equation (2-8) for $\partial x_k / \partial x_j^*$ we get

$$\frac{\partial u_i^*}{\partial x_j^*} = a_{il}a_{jk}\frac{\partial u_l}{\partial x_k} , \qquad (2\text{-}14)$$

which shows that $\partial u_i / \partial x_j$ obeys the transformation law (2-11) and is thus a second-order tensor.

2.4.2 Stress and the Quotient Law

Still another second-order tensor is the stress σ_{ij} defined by

$$\sigma_{ij}\Delta A_{(i)} = \Delta F_j \quad \text{(no sum on i)}, \tag{2-15}$$

where ΔF_j is the force component in the x_j-direction acting on the small area element ΔA_i, whose normal is in the x_i-direction.

To show that σ_{ij} is a tensor, first write a sum of forces of the type ΔF_j:

$$\sigma_{ij}\Delta A_i = \Delta F_j' \quad \text{(sum on i)} \tag{2-16}$$

A product of two tensors such as that in equation (2-16), where a subscript is repeated, is called an inner product. But the area ΔA_i (where we designate the area by its components) is a vector, since it can be represented by a directed-line segment normal to the plane of the area (it has magnitude and direction). Similarly the force ΔF_j has magnitude and direction and is thus a vector. The quantity $\Delta F_j'$ is also a vector, since it is a sum of vectors of the type ΔF_j (equation (2-10)).

Next write equation (2-16) in the transformed coordinate system x_i^*:

$$\sigma_{ij}^* \Delta A_i^* = \Delta F_j'^*. \tag{2-17}$$

Since ΔA_i and $\Delta F_j'$ are vectors, we have, according to equation (2-9),

$$\sigma_{ij}^* a_{ik}\Delta A_k = a_{jk}\Delta F_k'$$

$$= a_{jk}\sigma_{ik}\Delta A_i = a_{ji}\sigma_{ki}\Delta A_k,$$

where equation (2-16) is used in the next to last term, and the dummy subscripts i and k are interchanged in the last term. Then, from the first and last terms,

$$\Delta A_k(\sigma_{ij}^* a_{ik} - a_{ji}\sigma_{ki}) = 0. \tag{2-18}$$

Since equation (2-18) holds for all values of ΔA_k,

$$a_{ik}\sigma_{ij}^* = a_{ji}\sigma_{ki}. \tag{2-19}$$

To get this equation into the form of equation (2-11), multiply it by $a_{\ell k}$, or

$$a_{ik}a_{\ell k}\sigma_{ij}^* = a_{ji}a_{\ell k}\sigma_{ki}.$$

Finally, using equation (2-5a),

$$\delta_{i\ell}\sigma_{ij}^* = a_{ji}a_{\ell k}\sigma_{ki}$$

or

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$$\sigma_{\ell j}^{*} = a_{\ell k} a_{j i} \sigma_{k i}. \qquad (2\text{-}20)$$

Comparing equation (2-20) with (2-11) shows that σ_{ij} is a second-order tensor.

We have shown, starting from equation (2-16), that if inner multiplication of a quantity σ_{ij} by a vector with arbitrary components gives another vector, then σ_{ij} is a second-order tensor. This is one form of the quotient law. Once it has been established, as has been done here, it provides in some cases a simple test for determining whether a quantity is a tensor. Further discussion of the quotient law will be given in section 2.5.2.

2.4.3 The Kronecker Delta, a Tensor

In the foregoing paragraphs we showed that a product of two velocities (or vectors), a gradient of a velocity, and a stress, while representing different physical entities, are all alike in that they are second-order tensors. Next we ask whether the Kronecker delta δ_{ij} is a second-order tensor. According to the definition in equation (2-3) the components of δ_{ij} do not depend on the orientation of coordinate axes. Thus

$$\delta_{ij}^{*} = \delta_{ij}.$$

Using equations (2-5a) and (2-3),

$$\delta_{ij}^{*} = \delta_{ij} = a_{ik} a_{jk} = a_{ik} a_{j\ell} \delta_{k\ell}. \qquad (2\text{-}21)$$

Comparing the first and last members of equations (2-21) with (2-11) shows that δ_{ij} is a second-order tensor.

The Kronecker delta δ_{ij} is an example of an isotropic tensor. That is, its components remain invariant with rotation of coordinate axes. An isotropic tensor is sometimes called a numerical tensor, since its components have the same numerical values for all rotations of the coordinate axes.

We now show that the most general second-order isotropic tensor is $I\delta_{ij}$, or that any second-order isotropic (numerical) tensor can be written as $I\delta_{ij}$, where I is a scalar. The transformation law for a second-order tensor is given by equation (2-11). Let I_{ij} be any (the most general) second-order isotropic tensor, so that $I_{ij}^{*} = I_{ij}$. Then equation (2-11) becomes

$$I_{ij}^{*} = I_{ij} = a_{ik} a_{j\ell} I_{k\ell}.$$

Multiplying the last two members of this equation by a_{jm} and using equation (2-5) give

$$a_{jm} I_{ij} = a_{ik} a_{j\ell} a_{jm} I_{k\ell} = a_{ik} \delta_{\ell m} I_{k\ell} = a_{ik} I_{km}.$$

The first and last members of this equation give

$$\delta_{km} a_{jk} I_{ij} = \delta_{ij} a_{jk} I_{km},$$

or

$$a_{jk} (\delta_{km} I_{ij} - \delta_{ij} I_{km}) = 0.$$

Since the relation for I_{ij} cannot depend on the a_{jk} (on the orientation of the coordinate axes), the quantity in parentheses is zero,[2] and we get, after contracting the indices k and m (setting $m = k$),

$$I_{ij} = (I_{kk}/3) \delta_{ij}$$

where I_{kk} is a scalar (see section 2.6). Any value of I_{kk} satisfies this equation, as can be seen by contracting the indices i and j, so that

$$I_{ij} = I \delta_{ij}.$$

That is, any (the most general) second-order isotropic tensor can be written as $I\delta_{ij}$, where I is an arbitrary scaler.

2.5 THIRD- and HIGHER-ORDER TENSORS

The generalization of equation (2-11) to tensors of higher order is obvious. For instance a third-order tensor u_{ijk} is defined by

$$u_{ijk}^{*} = a_{i\ell} a_{jm} a_{kn} u_{\ell mn} \qquad (2\text{-}22)$$

which represents 27 equations, each with 27 terms on the right side. An example of a third-order tensor is the product of three velocities $u_i u_j u_k$. The product of four velocities forms a fourth-order tensor $u_i u_j u_k u_\ell$, etc.

2.5.1 Vorticity and the Alternating Tensor

An important third-order tensor is the alternating tensor ε_{ijk}, where

$$\varepsilon_{ijk} = \delta_{i1} \delta_{j2} \delta_{k3} + \delta_{i2} \delta_{j3} \delta_{k1} + \delta_{i3} \delta_{j1} \delta_{k2} \qquad (2\text{-}23)$$

$$- \delta_{i1} \delta_{j3} \delta_{k2} - \delta_{i2} \delta_{j1} \delta_{k3} - \delta_{i3} \delta_{j2} \delta_{k1}.$$

(We call ε_{ijk} a tensor in anticipation of showing that it is such later in this section.) Evaluation of equation (2-23) shows that

$$\left. \begin{array}{l} \varepsilon_{ijk} = 0 \quad \text{if two subscripts are equal,} \\[2mm] \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \\[2mm] \text{and} \quad \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1. \end{array} \right\} \qquad (2\text{-}23\text{a})$$

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[2] If, however, I_{ij} were not an isotropic tensor, then $a_{jk} (\delta_{km} I_{ij}^{*} - \delta_{ij} I_{km}) = 0$, and we could not set the quantity in parentheses equal to zero; the relation between I_{ij}^{*} and I_{km} would not be independent of the a_{kj}.

The alternating tensor ε_{ijk} is usually defined by equation (2-23a), but equation (2-23) is more convenient for our purposes (and may generally be preferable).

We define a quantity w_i as follows:

$$w_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j},\tag{2-24}$$

where u_k is a vector and x_j is a coordinate (also a vector). Equation (2-24) becomes, on using (2-23),

$$w_i = (\delta_{i1}\delta_{j2}\delta_{k3} - \delta_{i1}\delta_{j3}\delta_{k2} + \delta_{i2}\delta_{j3}\delta_{k1} - \delta_{i2}\delta_{j1}\delta_{k3}$$

$$+ \delta_{i3}\delta_{j1}\delta_{k2} - \delta_{i3}\delta_{j2}\delta_{k1})\,\frac{\partial u_k}{\partial x_j}$$

$$= \delta_{i1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \delta_{i2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \delta_{i3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).\tag{2-25}$$

Therefore, the three components of the quantity w_i are

$$w_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3},\qquad w_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1},\tag{2-26}$$

$$\text{and}\quad w_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

If u_i is the velocity at a point in a fluid, w_i is called the vorticity and is a measure of the local swirl or rotation. Equations (2-26) show that each component w_i has the magnitude of the rotation of the fluid in an $x_j - x_k$ plane ($j, k \neq i$) and is perpendicular to that plane. Thus the quantity w_i has both magnitude and direction and so is a vector. The vector w_i (or w) is also called the curl of u_i and, in vector notation, is written as $\nabla \times u$. The quantity $\partial u_k/\partial x_j$ in equation (2-24) is the gradient of a vector and, as shown in section 2.4.1, is a second-order tensor. Since w_i is a vector, ε_{ijk} is a third-order tensor by a slightly more general form of the quotient law than that in section 2.4.2. (See the next section.) By equation (2-23) it is isotropic (or numerical), since its components remain invariant with rotation of coordinate axes.

2.5.2 A More General Quotient Law

In order to prove the quotient law used in the last section, we note that equation (2-24) is of the form

$$v_i = v_{ijk}v_{kj}, \qquad (2\text{-}27)$$

where v_i is a vector and v_{kj} is an arbitrary second-order tensor (its components can have arbitrary values). We want to prove that v_{ijk} is a third-order tensor. In a rotated coordinate system equation (2-27) becomes

$$v_i^* = v_{ijk}^* v_{kj}^*, \qquad (2\text{-}28)$$

or, since v_i and v_{kj} are tensors, we can write, using equations (2-9) and (2-11),

$$a_{i\ell}v_\ell = v_{ijk}^* a_{k\ell}a_{jm}v_{\ell m}.$$

Substituting for v_ℓ from equation (2-27),

$$v_{ijk}^* a_{k\ell}a_{jm}v_{\ell m} - a_{i\ell}v_{\ell jk}v_{kj} = 0. \qquad (2\text{-}29)$$

We factor out $v_{\ell m}$ after making a change of dummy subscripts in the second term of equation (2-29). This gives

$$(a_{k\ell}a_{jm}v_{ijk}^* - a_{in}v_{nm\ell})v_{\ell m} = 0.$$

Since $v_{\ell m}$ is arbitrary, the quantity in parentheses is zero, or

$$a_{k\ell}a_{jm}v_{ijk}^* = a_{in}v_{nm\ell}.$$

To get this equation into the form of equation (2-22) we multiply it by $a_{r\ell}a_{qm}$, or

$$a_{k\ell}a_{r\ell}a_{jm}a_{qm}v_{ijk}^* = a_{in}a_{qm}a_{r\ell}v_{nm\ell}. \qquad (2\text{-}30)$$

Finally, using equation (2-5a) on the left side of (2-30) gives

$$\delta_{kr}\delta_{jq}v_{ijk}^* = a_{in}a_{qm}a_{r\ell}v_{nm\ell}$$

or

$$v_{iqr}^* = a_{in}a_{qm}a_{r\ell}v_{nm\ell}. \qquad (2\text{-}31)$$

Comparison of equation (2-31) with (2-22) shows that v_{ijk} is a third-order tensor. Thus if, in equation (2-27), v_i is a vector (first-order tensor) and v_{kj} is an arbitrary tensor, then v_{ijk} is a tensor. This proves a rather general form of the quotient law. As mentioned earlier, a product such as that in equation (2-27), in which a subscript or subscripts in one factor is repeated in the other one, is called an inner product. In general the quotient law states that a quantity is a tensor if an inner multiplication of that quantity with an arbitrary tensor (its components can have arbitrary values) is itself a tensor.

2.6 ZERO-ORDER TENSORS AND CONTRACTION

We notice in the definitions of first-, second-, and third-order tensors (equations (2-9), (2-11), and (2-22)), that the number of a_{ij}'s in the transformation law equals the order of the tensor. Also, of course, the number of subscripts

on a tensor equals its order. Thus for a tensor of order zero or a scalar, say u, we should have in place of equation (2-9) or (2-11),

$$u^* = u. \tag{2-32}$$

So a zero-order tensor, or a scalar, has the same value in the coordinate system x_i^* as in x_i. For that reason it is often called an invariant. Since u in equation (2-32) can be any unsubscripted quantity, we can say that any unsubscripted quantity is a tensor of order zero.

Multiplication of a tensor by a scalar gives a tensor of the same order. For instance multiplication of equation (2-11) (for a second-order tensor) by a scalar u gives

$$u^* u_{ij}^* = a_{ik} a_{jl} u u_{kl}, \tag{2-33}$$

where equation (2-32) was used. Thus $u u_{kl}$ transforms as a second-order tensor.

In the second-order tensor u_{ij} we can set $j = i$. That process is called contraction. Then, according to equation (2-11), we have

$$u_{ii}^* = a_{ik} a_{il} u_{kl} = \delta_{kl} u_{kl} = u_{kk} = u_{ii} \tag{2-34}$$

where equation (2-5) is used. Comparison of equation (2-34) with (2-32) shows that u_{ii} is a zero-order tensor, or a scalar. Thus contraction of the subscripts i and j lowered the order of the tensor by two. In general the process of contraction lowers the order of a tensor by two. As another example we contract the second-order tensor $\partial u_i / \partial x_j$ to form the scalar $\partial u_i / \partial x_i$, which is called the divergence of u_i, and which, according to equations (2-14) and (2-5) is a scalar. In vector notation the divergence of u_i is written as $\nabla \cdot u$. As a final example of contraction, contract the second-order tensor $u_i v_j$ to form $u_i v_i$. The quantity $u_i v_i$ is a scalar, since

$$u_i^* v_i^* = a_{ik} a_{il} u_k v_l = \delta_{kl} u_k v_l = u_k v_k = u_i v_i. \tag{2-35}$$

It is called the dot or inner product of the vectors u and v and is often written as u·v.

We can show that the gradient of a scalar , say u, is a vector. For, proceeding as in obtaining equations (2-13) and (2-14),

$$\frac{\partial u^*}{\partial x_j^*} = \frac{\partial u^*}{\partial x_k} \frac{\partial x_k}{\partial x_j^*} = a_{jk} \frac{\partial u}{\partial x_k}, \tag{2-36}$$

which is the transformation law for a vector. In vector notation the gradient of u is written as ∇u.

2.7 OUTER AND INNER PRODUCTS OF TENSORS OF HIGHER ORDER

It is shown in sections 2.4.1 and 2.6 respectively that outer and inner products of vectors are tensors of some order. It is straightforward to show that outer and inner products of tensors of any order are tensors. For example the outer product $u_i u_{jk}$ is a third order tensor, since

$$u_i^* u_{jk}^* = a_{j\ell} a_{jm} a_{kn} u_\ell u_{mn} \qquad (2-37)$$

Also, the inner product $u_i u_{ik}$ is a first-order tensor (vector), since, using equation (2-5),

$$u_i^* u_{ik}^* = a_{i\ell} a_{im} a_{kn} u_\ell u_{mn} = \delta_{\ell m} a_{kn} u_\ell u_{mn}$$

$$= a_{kn} u_m u_{mn}.$$

2.8 SUBSCRIPTED QUANTITIES THAT ARE NOT TENSORS

In order to give a better understanding of what tensors are, we give here some examples of quantities which, although subscripted, are not tensors. First consider the quantity a_{ij}. Recall that the a_{ij} form a set of nine constants defined by equation (2-1). Assume first that a_{ij} is a second-order tensor. Then

$$a_{ij}^* = a_{ik} a_{j\ell} a_{k\ell} = a_{ik} \delta_{jk} = a_{ij} \qquad (2-38)$$

where equations (2-11), (2-5a), and (2-3) were used. Thus if a_{ij} is a tensor, it must be isotropic. But we showed in section 2.4.3 that the most general second-order isotropic tensor is $I\delta_{ij}$, where I is a scalar. So a_{ij} is a tensor only if it is equal to $I\delta_{ij}$. If $a_{ij} = I\delta_{ij}$, equation (2-1) becomes

$$x_i^* = I\delta_{ij} x_j = Ix_i.$$

But in contrast to the statement of this equation the x_i^* are not proportional to the x_i for arbitrary rotations, so that $a_{ij} \neq I\delta_{ij}$, and according to our argument a_{ij} is not a tensor.

We showed in the last paragraph that a_{ij} is not a tensor, since $a_{ij} \neq I\delta_{ij}$. More generally we can say that any quantity h_{ij} whose components have the same numerical values in all rectangular coordinate systems is a nontensor if $h_{ij} \neq I\delta_{ij}$, since $I\delta_{ij}$ is the most general isotropic (numerical) tensor (section 2.4.3).

As another example consider a quantity $w_i = u_{(i)} v_i$ (no sum on i), where u_i and v_i are both vectors. One might imagine that $w_i = u_{(i)} v_i$ is also a vector (first order tensor), since it has one assignable subscript. But $u_i^* = a_{ij} u_j$ and $v_i^* = a_{ik} v_k$, so that

$$w_i^* = u_{(i)}^* v_i^* = a_{(i)j} a_{ik} u_j v_k. \qquad (2-39)$$

If $w_i = u_{(i)} v_i$ were a first-order tensor it would transform as $w_i^* = a_{ij} w_j = a_{ij} u_{(j)} v_j$, which is considerably different from equation (2-39). Since both of these expressions cannot be true, $w_i = u_{(i)} v_i$ is not a tensor. Similarly quantities such as $w_{ij} = u_{(i)} u_{ij}$ and $w_{ijk} = u_{(i)} u_{ijk}$ are not tensors. Note that all of these quantities are products which are neither inner nor outer.

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2.9 CLOSING REMARKS

It has been shown (eq. (2-10)) that the sum or difference of two vectors is a vector. Similarly the sum of any two tensors of the same order is a tensor of that order. No meaning is attached to the sum of tensors of different orders, say $u_i + u_{ij}$; that is not a tensor.

In general, an equation containing tensors has meaning only if all the terms in the equation are tensors of the same order, and if the same unrepeated subscripts appear in all the terms. These facts will be used in obtaining appropriate equations for fluid turbulence.

This explanation of Cartesian tensors should contain what is needed for our purposes. It is hoped that it has been reasonably clear. Other treatments of tensors are given, for instance, in books by Jeffreys, Spain, Arfken, Lass, Langlois, and Goodbody (refs. 1 to 6).

With the foregoing background the derivation of appropriate continuum equations for turbulence should be straightforward. Before deriving them, however, a justification for calling the fluid a continuum for the study of turbulence will be given.

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16. Abstract

Some of the mathematical apparatus used for the representation and study of turbulence is developed.

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