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Exact and Approximate Solutions to the Oblique Shock Equations for Real-Time Applications

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Exact and Approximate Solutions to the Oblique Shock Equations for Real-Time Applications*

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Abstract

This note is concerned with the derivation of exact solutions for determining the characteristics of an oblique shock wave in a supersonic flow. Specifically, an explicit expression for the oblique shock angle in terms of the free-stream Mach number, the centerbody deflection angle, and the ratio of the specific heats, is derived. A simpler approximate solution is obtained and compared to the exact solution. The primary objectives of obtaining these solutions is to provide a fast algorithm that can run in a real-time environment.

Introduction

Oblique shocks usually form when a supersonic flow is deflected and turned into itself [1], as shown in Figure 1. The flow before the wedge is supersonic, i.e., $M_1 > 1.0$. At point A the centerbody surface is deflected upward through an angle δ. This will force the flow streamlines to be deflected upward through the main bulk of the flow above the centerbody surface. A shock wave is then formed which is oblique to the free-stream flow direction. This is a two dimensional interaction of the flow field. Although numerous tables and charts are available for determining the characteristics of the oblique shock, to the best of the authors knowledge, no closed-form solutions have been explicitly reported.

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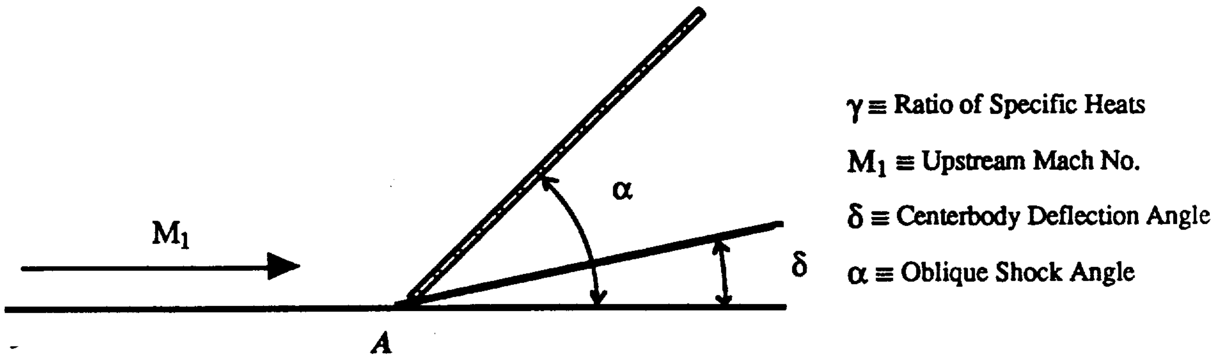

Figure 1

In this note, we formulate a closed-form solution for the oblique shock angle, α, in terms of the free-stream Mach number, M_1, the centerbody deflection angle, δ, and the ratio of the specific heats, γ. This information can in turn be used for design and control purposes. It should be remembered that the primary objective of obtaining the solutions is to implement them in a real-time environment

Derivation of an explicit expression for Oblique Shock Angle

In order to find an explicit expression for the oblique shock angle resulting from the incidence of a supersonic flow upon an uptended centerbody, some straight-forward but messy mathematical manipulations must be performed.

The relationship between the centerbody angle, δ, and the oblique shock angle, α, can be easily determined from the basic thermodynamic relationships. That is (cf. reference [1]; equation 4.16);

$$\frac{\tan(\alpha - \delta)}{\tan\alpha} = \frac{2 + (\gamma - 1)M_1^2 \sin^2\alpha}{(\gamma + 1)M_1^2 \sin^2\alpha} \tag{1}$$

Cross multiplying, squaring, and simplifying, results in the following cubic equation in terms of $[\sin(\alpha)]^2$;

$$X^3 + bX^2 + cX + d = 0 \tag{2}$$

where

$$X \equiv [\sin(\alpha)]^2,$$

$$b = -\left[\frac{M_1^2 + 2}{M_1^2} + \gamma \sin^2 \delta\right] \tag{3}$$

$$c = \frac{2M_1^2 + 1}{M_1^4} + \left(\frac{(\gamma + 1)^2}{4} + \frac{\gamma - 1}{M_1^2}\right) \sin^2 \delta, \text{ and} \tag{4}$$

$$d = -\frac{\cos^2 \delta}{M_1^4}. \tag{5}$$

This equation was derived and presented long ago [3]. Apparently, however, no one has since published an exact solution to this cubic equation. This now follows. By the general solution of the cubic [4] define;

$$Q = \frac{3c - b^2}{9},$$

$$R = \frac{9bc - 27d - 2b^3}{54}, \text{ and}$$

$$D = Q^3 + R^2. \tag{6}$$

The deflected shock exists only if $D < 0$, as equation (2) has two complex conjugate solutions for $D > 0$ (this will never occur in a real problem). There now remain two solutions for the shock angle, α. The larger shock angle is called the *strong shock angle*, α_s, and the other one is referred to as the *weak shock angle*, α_w. These shock angles are determined as follows:

$$\alpha_s = \tan^{-1}\left[\sqrt{\frac{\chi_s}{1 - \chi_s}}\right] \text{ and } \alpha_w = \tan^{-1}\left[\sqrt{\frac{\chi_w}{1 - \chi_w}}\right], \tag{7}$$

where

$$\chi_s = -\frac{b}{3} + 2\sqrt{-Q}\,\cos\phi\ , \tag{8a}$$

$$\chi_w = -\frac{b}{3} - \sqrt{-Q}\,\left(\cos\phi - \sqrt{3}\,\sin\phi\right)\ , \tag{8b}$$

$$\phi = \frac{1}{3}\left(\tan^{-1}\left(\frac{\sqrt{-D}}{R}\right) + \Delta\right),\ \text{and} \tag{9}$$

$$\Delta = \begin{cases} 0 & \text{if } R \geq 0 \\ \pi & \text{if } R < 0 \end{cases}\ . \tag{10}$$

The weak shock is the one which is of particular interest. However, the formation of the strong or weak shock is determined by the back pressure. For the strong shock solution, the flow after the shock is subsonic, whereas for the weak shock solution the flow after the shock is still supersonic.

Although the present equation for α, equation (7), would be rather formidable if put explicitly in terms of M, δ, and γ ; a simpler solution for programming is suggested as follows:

1) Given M, δ, and γ ; compute the cubic equation coefficients from equations (3), (4), and (5).
2) Compute the solution parameters from equation (6).
3) Then obtain the solution by computing in order equations (10), (9), (8), and (7).

Approximate Solution

There exist many tables from which the shock angle can be determined as the function of M_1, δ, and γ, e.g. [2]. For on-line computational use of such information, table look-up algorithms are usually implemented. This procedure is very fast especially if the table is described by one variable. The major problems for a table look-up procedure are the inaccuracy introduced by interpolating the data and the volume of data to be stored.

As an alternative approximation, one can use the Least-Squares algorithm to fit a polynomial to the now available exact solution. This approach is faster to compute on line than the exact solution, but probably slower than the table look-up. Although not necessarily more accurate than the table look-up, it does generate the solution from a compact predetermined formula which is much easier to program. The resulting polynomial is subsequently evaluated to obtain the approximate solution. This procedure does require some off-line calculation for determining the coefficients of the polynomial. The amount of the on-line calculation is directly dependent upon the order of the polynomial, that is, the desired accuracy. The procedure can be described as follows. Suppose that for a fixed γ and δ a set of data V and $\overline{\alpha}$, in vector form, is given where vector V corresponds to the freestream Mach numbers and vector $\overline{\alpha}$ contains the shock angle associated with the corresponding Mach number. Then one can fit a polynomial P(M) of degree n to the given data by:

$$P(M) = a_0 + \frac{a_1}{M} + \cdots + \frac{a_n}{M^n} \equiv \alpha(M) \tag{11}$$

The n+1 coefficients of the polynomial are determined by solving a system of simultaneous equations, i.e.

$$A\,a = \overline{\alpha} \tag{12}$$

or in matrix form

$$\begin{bmatrix} 1 & \frac{1}{M_1} & \cdots & \frac{1}{(M_1)^n} \\ 1 & \frac{1}{M_2} & \cdots & \frac{1}{(M_2)^n} \\ 1 & \frac{1}{M_3} & \cdots & \frac{1}{(M_3)^n} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1}{M_k} & \cdots & \frac{1}{(M_k)^n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_k \end{bmatrix} \text{ with } k > n. \tag{13}$$

The subscripts on the Mach number, M_i, correspond to different input freestream Mach numbers of interest in the specific application with the corresponding shock angle α_i. The Least-Squares solution to this overdetermined system of equations is known to be [4] :

$$a = \left(A^T A \right)^{-1} A^T \overline{\alpha} \tag{14}$$

The above approximation is carried out for the case in which $\delta = 17°$ and k=19. The approximation results for several different degrees of accuracy are tabulated and compared to the exact results as shown in Table 1. It should noted that the on-line computation associated with equation (11) is significantly less than that for the exact solution, equations (3)-(10), and generally more easily programmed than table lookup.

Conclusion

This paper contains an exact solution to the oblique shock equations. Specifically, an exact solution is given for the oblique shock angle as a function of centerbody deflection angle, freestream Mach number, and the ratio of specific heats. A technique for obtaining a simplified approximation is also given which is faster for real-time implementation.

References

[1] Anderson, J.D., "Modern Compressible Flow With Historical Perspective", McGraw-Hill Book Co., New York 1982.

[2] The Ames Tables, "Equations, Tables, and Charts for Compressible Flow", National Advisory Committee for Aeronautics, Report 1135, 1953.

[3] Thompson, M.J., "A Note on the Calculation of Oblique Shock- Wave Characteristics", Journal of the Aeronautical Sciences, November 1950.

[4] Spiegel, M.R., "Mathematical Handbook of Formulas and Tables", Schaum's Outline Series in Mathematics, McGraw-Hill Book Co., New York.

Table 1

n	Coefficients of polynomial						Max. error in $\alpha°$
	a_0	a_1	a_2	a_3	a_4	a_5	
1	13.6112	67.0661	—	—	—	—	4.3298
2	20.3487	12.3061	88.2963	—	—	—	0.7626
3	19.3487	27.8523	27.6956	66.5591	—	—	0.4295
4	21.0962	-11.9379	287.3568	-575.0471	530.7294	—	0.1202
5	20.4147	8.6005	91.2243	224.0812	-918.7301	962.3091	0.0876

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