On the Formulation of a Minimal Uncertainty Model for Robust Control With Structured Uncertainty

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Symbols

A, B, C, D	appropriately dimensioned state-space realization of $G(s, \delta)$
$\mathbf{B}_{xp}, \mathbf{C}_{qx}, \mathbf{D}_{qp},$ $\mathbf{D}_{qu}, \mathbf{D}_{yp}$	matrices relating state model of $\mathbf{P}(s)$ to uncertain parameters in $\mathbf{\Delta}$ matrix
$G(s, \delta)$	transfer function of uncertain system
$K(s)$	feedback control system
$K_u(\delta), K_y(\delta)$	uncertain scalar gain factor reflected at input and output, respectively
k	number of first-order transfer function blocks in $G(s, \delta)$
ℓ	number of second-order transfer function blocks in $G(s, \delta)$
i	index variable
$\mathbf{M}(s)$	nominal closed-loop plant transfer function matrix
$\mathbf{P}(s)$	nominal open-loop plant transfer function matrix (or interconnection structure)
m	minimal number of uncertain parameters in $\mathbf{\Delta}$ matrix
$\mathbf{p}$	fictitious uncertain parameter input vector to $\mathbf{P}(s)$, $\mathbf{p} \in R^m$
$\mathbf{q}$	fictitious uncertain parameter output vector from $\mathbf{P}(s)$, $\mathbf{q} \in R^m$
$O_A, O_B,$ O_C, O_D	order of highest cross term in A_Δ, B_Δ, C_Δ, and D_Δ, respectively
r	order of highest cross term in A_Δ, B_Δ, C_Δ, and D_Δ collectively
s	Laplace frequency variable
$\mathbf{u}$	control input vector, $\mathbf{u} \in R^{n_u}$
$\mathbf{w}$	vector of exogenous inputs (e.g., noise, disturbances, commands), $\mathbf{w} \in R^{n_w}$
$\mathbf{x}$	state vector, $\mathbf{x} \in R^{n_x}$
$\mathbf{y}$	output measurement vector, $\mathbf{y} \in R^{n_y}$
$\mathbf{z}$	vector of controlled variables (e.g., tracking error and control position and rate), $\mathbf{z} \in R^{n_z}$
δ	vector of real uncertain parameters, $\delta \in R^m = R^{m_I} + R^{m_D}$
δ_i	ith element of δ
$\mathbf{\Delta}, \mathbf{\Delta}\ (\delta)$	diagonal uncertainty matrix with δ along main diagonal, $\mathbf{\Delta} \in R^{m \times m}$
$\mathbf{A}_\Delta, \mathbf{B}_\Delta,$ $\mathbf{C}_\Delta, \mathbf{D}_\Delta$	uncertainty matrix associated with matrix $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, respectively
ϵ	uncertain variable

Subscripts:

i	index variable
o	nominal (unperturbed)
R	real (or first order)

C	complex (or second order)
I	independent
D	dependent (or repeated)

Superscript:

-1	matrix inverse

Special Notation:

$:=$	defined as, assigned to
$\oplus$	modulo m addition over set $\{1, 2, \ldots, m\}$
$\left[\begin{array}{c\|c}\mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D}\end{array}\right]$	this notation represents system transfer function matrix in terms of system state-space matrices, $\mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

Summary

In the design and analysis of robust control systems for uncertain plants, representing the system transfer matrix in the form of what has come to be termed an "M-Δ model" has become widely accepted and applied in the robust control literature. The symbol $\mathbf{M}$ represents a transfer function matrix $\mathbf{M}(s)$ of the nominal closed-loop system, and Δ represents an uncertainty matrix acting on $\mathbf{M}(s)$. The nominal closed-loop system $\mathbf{M}(s)$ results from closing the feedback control system $K(s)$ around a nominal plant interconnection structure $\mathbf{P}(s)$. The uncertainty can arise from various sources, such as structured uncertainty from parameter variations or multiple unstructured uncertainties from unmodeled dynamics and other neglected phenomena. In general, Δ is a block diagonal matrix, but for real parameter variations, Δ is a diagonal matrix of real elements. Conceptually, the M-Δ structure can always be formed for any linear interconnection of inputs, outputs, transfer functions, parameter variations, and perturbations. However, very little of the currently available literature addresses computational methods for obtaining this structure, and none of this literature (to the authors' knowledge) addresses a general methodology for obtaining a **minimal M-Δ** model for a wide class of uncertainty, where the term "minimal" refers to the dimension of the Δ matrix. Since having a minimally dimensioned Δ matrix would improve the efficiency of structured singular value (or multivariable stability margin) computations, a method of obtaining a minimal **M-Δ** model would be useful. Hence, a method of obtaining the interconnection system $\mathbf{P}(s)$ is required. This paper presents (without proof) a generalized procedure for obtaining a minimal **P-Δ** structure for systems with real parameter variations. With this model, the minimal M-Δ model can then be easily obtained by closing the feedback loop. The procedure involves representing the system in a cascade-form state-space realization, determining the minimal uncertainty matrix Δ, and constructing the state-space representation of $\mathbf{P}(s)$. Three examples are presented to illustrate the procedure.

1. Introduction

Robust control theory for both analysis and design has been the subject of a vast amount of research. (See refs. 1 through 35.) In particular, robust stability and performance have been emphasized in much of this work, as, for example, in the development of H^{∞} control theory. (See refs. 10 through 15 and 19 through 23.) Moreover, the development of robust control system design and analysis techniques for unstructured (refs. 1 through 9, 13, 19, and 21) as well as structured (refs. 16 through 35) plant uncertainty continues to be the subject of much research— particularly the latter. Unstructured plant uncertainty arises from unmodeled dynamics (such as actuator or engine dynamics) and other neglected phenomena (such as nonlinearities with conic sector bounds). This uncertainty is called "unstructured" because it is represented as a norm-bounded perturbation with no particular assumed structure. Plant uncertainty is called "structured" when there is real parameter uncertainty in the plant model, or when there are multiple unstructured uncertainties occurring at various points within the system simultaneously. Plant parameter uncertainty can arise from modeling errors (which usually result from assumptions and simplifications made during the modeling process and/or from the unavailability of dynamic data on which the model is based) or from parameter variations that occur during system operation.

Robust control design and analysis methods for systems with unstructured uncertainty are accomplished via singular value techniques (refs. 1 through 9 and 21). For systems with structured plant uncertainty, however, a technique which takes advantage of that structure, such as the structured singular value (SSV) (refs. 16 through 27) or multivariable stability margin (MSM) (refs. 28 through 33), should be used. In order to compute the SSV or MSM, it is required that the system be represented in terms of an **M-Δ** model. The M represents a transfer function matrix $\mathbf{M}(s)$ of the nominal closed-loop system, and Δ represents an uncertainty matrix acting

on $\mathbf{M}(s)$. The system $\mathbf{M}(s)$ is formed by closing the feedback system $K(s)$ around the nominal open-loop plant interconnection structure $\mathbf{P}(s)$, as shown in figure 1.
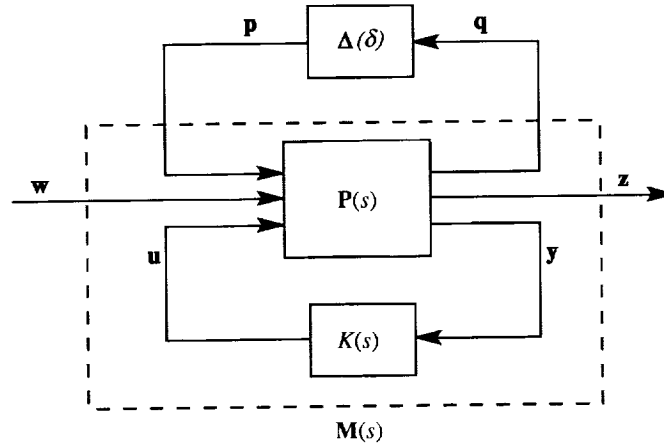

Figure 1. Block diagram of general $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model.

For multiple unstructured uncertainties, $\boldsymbol{\Delta}$ is a block diagonal matrix, and for real parameter uncertainties, the $\boldsymbol{\Delta}$ matrix is diagonal. As indicated in the literature (refs. 17, 18, 20, and 21), this model can always be formed for any linear interconnection of inputs, outputs, transfer functions, parameter variations, and perturbations. However, very little of the literature discusses methods for obtaining an $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model. While formulation of an $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model for unstructured uncertainties does not pose a major problem, forming an $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model for real parameter variations can be very difficult. In reference 29, De Gaston and Safonov present an $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model for a third-order transfer function with uncertainty in the location of its two real nonzero poles and in its gain factor. Although the given $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model is easily obtained for this simple example, other examples do not yield such a straightforward result. A general state model of $\mathbf{M}(s)$ for additive real perturbations in the system $\mathbf{A}$ matrix (where $\mathbf{A}$ is assumed to be closed loop) is discussed in reference 34. Unfortunately, this model is not general enough for many examples, since system uncertainty is restricted to the $\mathbf{A}$ matrix and the uncertainty class is restricted to be linear. Morton and McAfoos (ref. 26) present a general method for obtaining an $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model for linear (affine) real perturbations in the system matrices ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$) of the open-loop plant state model. In this model, the interconnection matrix $\mathbf{P}(s)$ is constructed first for separating the uncertainties from the nominal plant, and then $\mathbf{M}(s)$ is formed by closing the feedback loop. The $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model thus formed can be used in performing robustness analysis of a previously determined control system. Alternatively, if the feedback loop is not closed, μ-synthesis techniques (refs. 19, 20, and 21) can be applied to the $\mathbf{P}\text{-}\boldsymbol{\Delta}$ model for robust control system design of $K(s)$. The result of Morton and McAfoos essentially reduces to that of reference 34 when the perturbations occur only in the $\mathbf{A}$ matrix (and the $\mathbf{A}$ matrix of ref. 34 is assumed to be open loop). An algorithm for easily computing $\mathbf{M}(s)$ based on the result of Morton and McAfoos is presented in reference 35. Although this method of constructing an $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model is adequate for linear uncertainties, many realistic problems require a more general class of uncertainties, since, for real problems, uncertainties can arise in a nonlinear functional form (such as nth-order terms and cross terms). For these cases, it often becomes necessary to have repeated uncertain parameters in the $\boldsymbol{\Delta}$ matrix, as is discussed later. Since the $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model is a nonunique representation, models of various dimensions (due to the repeated parameters) can be constructed depending on how the model is obtained. It is therefore desirable to obtain one of minimal dimension so that the complexity of the SSV or MSM computations during robust control system design or analysis can be minimized. However, none of the literature (to the authors' knowledge) addresses the issue of minimality.

This paper presents a methodology for constructing a **minimal P-Δ** model for single-input, single-output (SISO) systems with real parametric **multilinear** uncertainties, where the terms "minimal" and "multilinear" are defined as follows:

Definition: A **P-Δ** model is **minimal** if the **Δ** matrix is of minimal dimension. It is shown later that **Δ** is minimal when its dimension is as close to the number of independent uncertain parameters as possible, that is, when it contains a minimal number of repeated uncertain parameters (if any).

Definition: A function is **multilinear** if the functional form is linear (affine) when any variable is allowed to vary while the others remain fixed; for example, $f(a, b, c) = a + ab + bc + abc$ is a multilinear function.

The requirement that the **P-Δ** model be minimal provides a means of improving the efficiency of the SSV and/or MSM computations during robust control system design or analysis. The allowance of multilinear functions of the uncertain parameters provides a means of handling certain nonlinear terms in the transfer function coefficients (and, hence, the system matrix elements), namely, **cross-product terms**. In addition, nonlinear nth-order terms can be approximated within the multilinear framework, although this representation is conservative. The proposed procedure determines the minimal **Δ** matrix and the state-space form of **P**(s) given the system transfer function in terms of the uncertain parameters, where any or all the numerator and/or denominator coefficients can be multilinear functions of the uncertain parameters except the leading denominator coefficient. It should be noted, however, that this procedure is presented without formal proof. Moreover, the state-space form used in modeling **P**(s) is an extension of the result of Morton and McAfoos for real parametric linear (affine) uncertainties (ref. 26). An extension of this result to multiple-input, multiple-output (MIMO) systems appears possible and is under study. The paper is divided into the following sections: a formal statement of the problem to be solved is presented in section 2, followed by a discussion of minimality considerations in section 3; the solution structure is presented in section 4, and computational details of the solution are presented in section 5; finally, several examples illustrating the solution are given next in section 6, followed by some concluding remarks in section 7.

2. Problem Statement

Given the transfer function of an uncertain system $G(s, \delta)$ as a function of the real uncertain parameters δ, **find** a minimal **P-Δ** model of the form depicted in figure 2 such that

1. The diagonal uncertainty matrix **Δ** is of minimal dimension

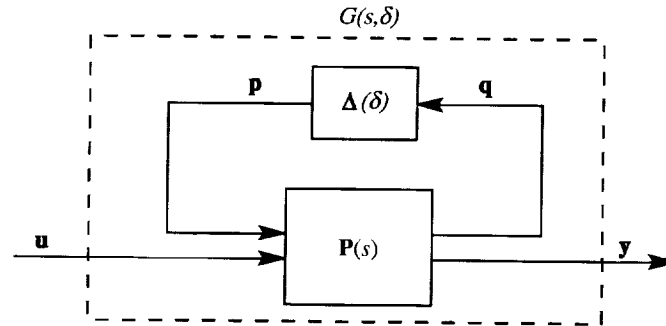2. The model of the nominal open-loop plant **P**(s) is in state-space form

Figure 2. Block diagram of general **P-Δ** model.

The model must handle multilinear uncertainty functions in any or all the transfer function coefficients except the leading denominator coefficient. In order to construct a minimal **P-Δ** model, the dimension of the **Δ** matrix must be minimized. Hence, factors which have been found to affect the dimension of the **P-Δ** model are discussed next, followed by the approach used in forming a solution to this problem.

3. Minimality Considerations

In constructing a **P-Δ** model of an uncertain system, the **Δ** matrix can become unnecessarily large due to repeated uncertain parameters on its main diagonal. It is therefore of interest to examine the factors which can cause this repetition, so that the number of repeated uncertain parameters can be minimized. A factor which can be shown to increase the size of the **Δ** matrix is the particular state-space realization used in representing the system. To illustrate this, consider the following simple examples:

Example 3.1: Consider the system

$$G(s) = \frac{1}{s^2 + (\theta_1 + \theta_2)s + \theta_1\theta_2} = \frac{1}{(s + \theta_1)(s + \theta_2)}$$

where

$$\theta_1 = \theta_{1o} + \delta_1 \qquad \theta_2 = \theta_{2o} + \delta_2$$

$$-1 \le \delta_1 \le 1 \qquad -1 \le \delta_2 \le 1$$

and the system is represented in state-space form as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\theta_1\theta_2 & (-\theta_1 - \theta_2) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

This system has two uncertain real poles, θ_1 and θ_2. The terms δ_1 and δ_2 represent the uncertain parameters associated with the uncertain poles. In block diagram form, this system can be represented directly from the transfer equation
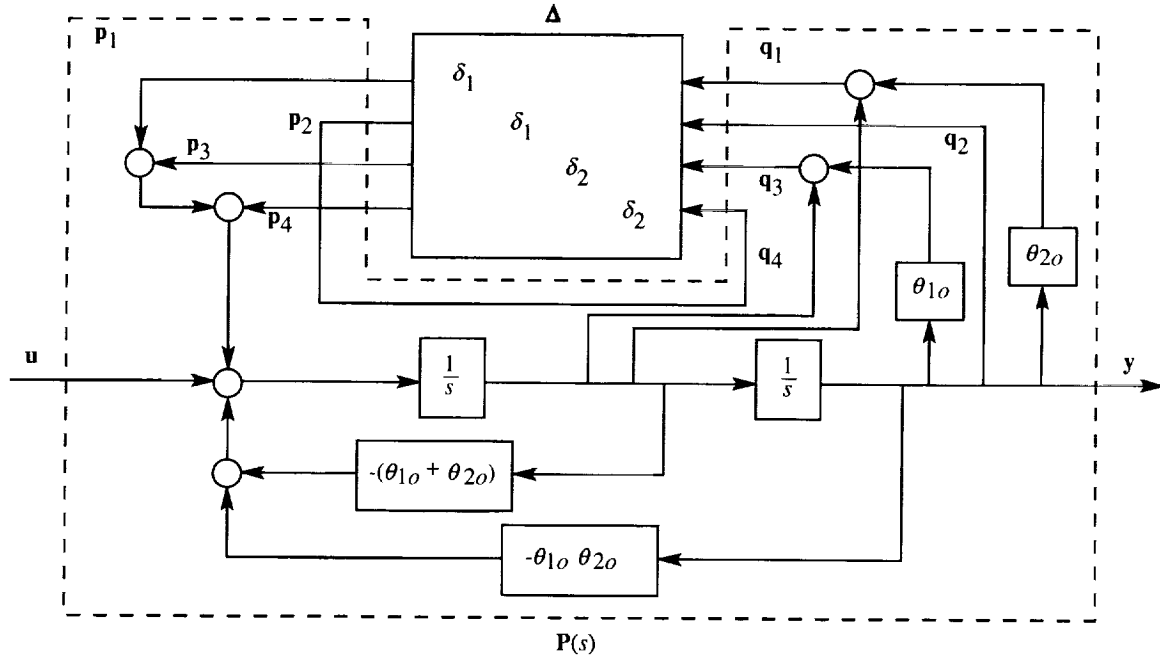
$$s^2\mathbf{y} = -(\theta_1 + \theta_2)s\mathbf{y} - \theta_1\theta_2\mathbf{y} + \mathbf{u}$$
$$= [-(\theta_{1o} + \theta_{2o}) - (\delta_1 + \delta_2)]s\mathbf{y} + (-\theta_{1o}\theta_{2o} - \theta_{2o}\delta_1 - \theta_{1o}\delta_2 - \delta_1\delta_2)\mathbf{y} + \mathbf{u}$$

as shown in sketch A.

Sketch A

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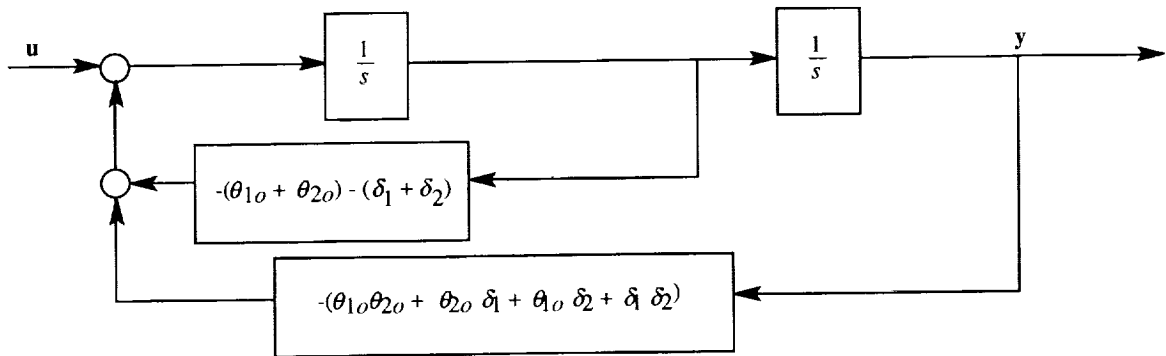
Then the uncertain parameters can be separated out into the **P-Δ** model shown in sketch B.

Sketch B

The **Δ** matrix associated with this model is $\mathbf{\Delta} = \mathrm{diag}\,[\delta_1, \delta_1, \delta_2, \delta_2]$, which is four-dimensional. The nominal system $\mathbf{P}(s)$ can be represented as

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\theta_{1o}\theta_{2o} & -(\theta_{1o} + \theta_{2o}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
$$

$$
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \theta_{2o} & 1 \\ 1 & 0 \\ \theta_{1o} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}
$$

$$
y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

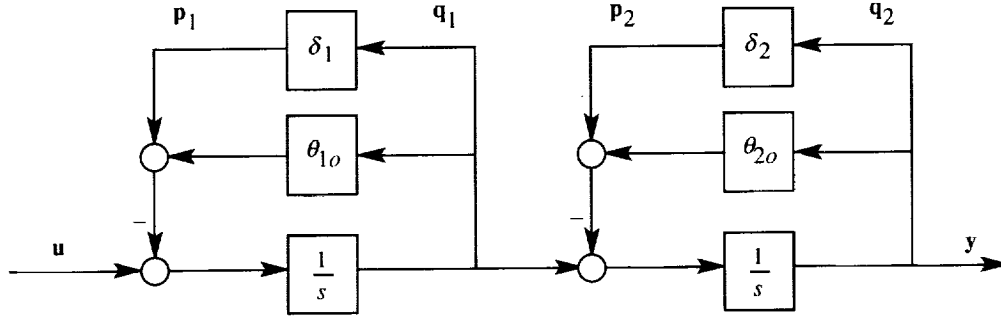**Example 3.2**: The system $G(s)$ in example 3.1 can also be represented by the block diagram in sketch C.

Sketch C

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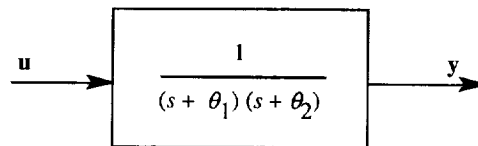
where

$$\theta_1 = \theta_{1o} + \delta_1 \qquad \theta_2 = \theta_{2o} + \delta_2$$
$$-1 \le \delta_1 \le 1 \qquad -1 \le \delta_2 \le 1$$
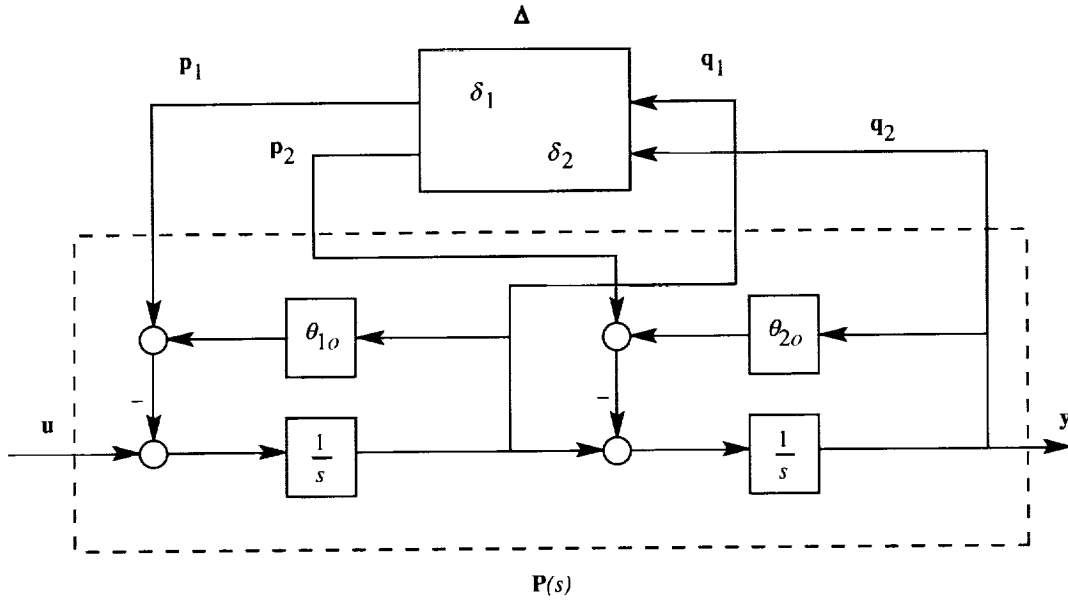
and the state-space representation is given by

$$\mathbf{A} = \begin{bmatrix} -\theta_1 & 0 \\ 1 & -\theta_2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{C} = [0 \ 1]$$

An equivalent block-diagram realization for this system is given directly by the cascaded form in sketch D.

Sketch D

An equivalent $\mathbf{P}$-$\boldsymbol{\Delta}$ model is given in sketch E.

Sketch E

where the $\boldsymbol{\Delta}$ matrix is given by $\boldsymbol{\Delta} = \text{diag}\,[\delta_1, \delta_2]$, which is two-dimensional, and a state-space realization of the nominal plant $\mathbf{P}(s)$ is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\theta_{1o} & 0 \\ 1 & -\theta_{2o} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

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$$\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

The **P-Δ** models obtained in examples 3.1 and 3.2 are very different, even though they both represent the same system with two uncertain parameters. Obviously, the **P-Δ** model of example 3.2 is minimal (**Δ** is two-dimensional) and that of example 3.1 is not minimal (**Δ** is four-dimensional), since for minimality the dimension of **Δ** should be as close to the number of independent uncertain parameters as possible. These examples therefore demonstrate the effect the particular realization has in forming a **P-Δ** model. In particular, it appears that a cascade realization is a desirable form for obtaining a minimal **P-Δ** model. Thus, a general cascade-form realization is part of the approach taken in constructing a minimal **P-Δ** model so that uncertain real poles and zeros can be cascaded whenever possible. A problem arises, however, in that some transfer functions have a form which precludes cascading uncertain real poles or zeros such as

$$G(s, \delta) = \frac{b_1 s^2 + b_2 s + b_3}{(s + \theta_1)(s + \theta_2)} \qquad G(s, \delta) = \frac{(s + \theta_1)(s + \theta_2)}{s^2 + a_1 s + a_2}$$

where θ_1 and θ_2 are assumed to be uncertain (and hence a function of δ). Cascading the poles and zeros for either case would result in improper transfer function blocks being realized. For these cases, it is unavoidable for the minimal **Δ** matrix to have repeated uncertain parameters on the main diagonal. However, for each inseparable pole or zero pair it is only necessary to repeat one uncertain parameter. This issue is addressed in the proposed solution, and a minimal **P-Δ** model for the first transfer function above is given as an example.

Another factor which affects the dimension of the **P-Δ** model is the form of the coefficient in the system transfer function. If any of the coefficients are nonlinear functions of the uncertain parameters instead of multilinear functions (e.g., there are nth-order uncertain terms in any of the coefficients), then extra dependent uncertain parameters must be defined in order to represent these terms in a multilinear form. For example, δ_1^2 would be represented as $\delta_1 \delta_2$, where $\delta_2 = \delta_1$, and both δ_1 and δ_2 would appear in the **Δ** matrix. Thus, for this case, it is again necessary that the minimal **Δ** matrix contain repeated uncertain parameters on its main diagonal. An example illustrating this situation is presented later.

These issues are addressed in the solution presented herein for constructing a minimal **P-Δ** model. The approach taken in forming this solution is described in the next section.

4. Solution Structure

Based on the problem definition and the minimality considerations outlined previously, several issues are addressed in forming a solution to the problem of constructing a minimal **P-Δ** model given the transfer function of an uncertain system. First, a general cascade-form realization is found which can be used to obtain a minimal **P-Δ** model. Second, the minimal **Δ** matrix is determined for any uncertain system such that extra dependent parameters are assigned to account for inseparable pairs of uncertain real poles or zeros as well as nonmultilinear (e.g., squared) terms. (As stated previously, it should be noted that the representation of nth-order terms in a multilinear form is a conservative approximation.) Third, a method of obtaining

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a state-space realization of $\mathbf{P}(s)$ for any uncertain system is found. Therefore, the proposed approach for constructing a minimal $\mathbf{P}$-$\mathbf{\Delta}$ model is given as follows:

1. Obtain a cascade-form realization of the system so that the state-space uncertain model can be written as

$$\left.\begin{array}{l} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} + D\mathbf{u} \end{array}\right\} \tag{1}$$

where

$$\left.\begin{array}{ll} A = A_o + A_\Delta & B = B_o + B_\Delta \\ C = C_o + C_\Delta & D = D_o + D_\Delta \end{array}\right\} \tag{2}$$

The terms with the subscript o (A_o, B_o, C_o, and D_o) represent the nominal matrix components, and the terms with the subscript Δ (A_Δ, B_Δ, C_Δ, and D_Δ) represent the uncertain matrix components.

2. Determine the minimal uncertainty matrix $\mathbf{\Delta}$ as described in the problem definition and depicted in figure 2, where $\mathbf{\Delta}$ is defined as follows:

$$\mathbf{\Delta} = \text{diag}\,[\delta_1, \delta_2, \delta_3, \ldots, \delta_m] = \text{diag}\,[\delta_I, \delta_D] = \text{diag}\,[\delta] \tag{3}$$

where

$$\mathbf{\Delta} \in R^{m \times m} \qquad \delta_I \in R^{m_I} \qquad \delta_D \in R^{m_D} \qquad \delta \in R^m$$

The vector of uncertain parameters is

$$\delta = [\delta_1, \delta_2, \delta_3, \ldots, \delta_m]$$

the partition of δ containing independent parameters is

$$\delta_I = [\delta_1, \delta_2, \delta_3, \ldots, \delta_{m_I}]$$

and the partition of δ containing dependent parameters is

$$\delta_D = [\delta_{m_I+1}, \delta_{m_I+2}, \delta_{m_I+3}, \ldots, \delta_{m_D}]$$

where

m minimal number of uncertain parameters

m_I number of independent parameters given in $G(s, \delta)$

m_D minimal number of dependent (or repeated) parameters

Also

$$\mathbf{p} = \Delta \mathbf{q} \tag{4}$$

where

$\mathbf{p}$ uncertain parameters input to $\mathbf{P}(s)$, $\mathbf{p} \in R^m$

$\mathbf{q}$ uncertain variables output from $\mathbf{P}(s)$, $\mathbf{q} \in R^m$

Thus, since m_I is given and fixed, the minimal $\mathbf{\Delta}$ matrix results when m_D (i.e., the number of dependent (or repeated) parameters in the $\mathbf{\Delta}$ matrix) is minimal (or zero, if possible).

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3. Determine the state-space model of the nominal open-loop plant $\mathbf{P}(s)$ having the following form:

$$\left.\begin{aligned}\dot{\mathbf{x}} &= A_o x + \begin{bmatrix}\mathbf{B}_{xp} & B_o\end{bmatrix}\begin{bmatrix}\mathbf{p}\\\mathbf{u}\end{bmatrix}\\[2mm]\begin{bmatrix}\mathbf{q}\\\mathbf{y}\end{bmatrix} &= \begin{bmatrix}\mathbf{C}_{qx}\\\mathbf{C}_o\end{bmatrix}x + \begin{bmatrix}\mathbf{D}_{qp} & \mathbf{D}_{qu}\\\mathbf{D}_{yp} & \mathbf{D}_o\end{bmatrix}\begin{bmatrix}\mathbf{p}\\\mathbf{u}\end{bmatrix}\end{aligned}\right\} \tag{5}$$

where $\mathbf{B}_{xp}$, $\mathbf{C}_{qx}$, $\mathbf{D}_{qp}$, $\mathbf{D}_{qu}$, and $\mathbf{D}_{yp}$ are constant matrices. Thus, $\mathbf{P}(s)$ can also be written in the equivalent shorthand notation defined as follows:

$$\mathbf{P}(s) = \begin{bmatrix}\mathbf{P}_{11}(s) & \mathbf{P}_{12}(s)\\\mathbf{P}_{21}(s) & \mathbf{P}_{22}(s)\end{bmatrix} := \left[\begin{array}{c|cc}A_o & \mathbf{B}_{xp} & B_o\\\hline\mathbf{C}_{qx} & \mathbf{D}_{qp} & \mathbf{D}_{qu}\\C_o & \mathbf{D}_{yp} & D_o\end{array}\right] \tag{6}$$

where

$$\left.\begin{aligned}\mathbf{P}_{11}(s) &= \frac{\mathbf{q}(s)}{\mathbf{p}(s)} = \mathbf{C}_{qx}(sI - A_o)^{-1}\mathbf{B}_{xp} + \mathbf{D}_{qp}\\[1mm]\mathbf{P}_{12}(s) &= \frac{\mathbf{y}(s)}{\mathbf{p}(s)} = \mathbf{C}_{qx}(sI - A_o)^{-1}B_o + \mathbf{D}_{qu}\\[1mm]\mathbf{P}_{21}(s) &= \frac{\mathbf{q}(s)}{\mathbf{u}(s)} = C_o(sI - A_o)^{-1}\mathbf{B}_{xp} + \mathbf{D}_{yp}\\[1mm]\mathbf{P}_{22}(s) &= \frac{\mathbf{y}(s)}{\mathbf{u}(s)} = C_o(sI - A_o)^{-1}B_o + D_o\end{aligned}\right\} \tag{7}$$

and with the notation of equation (6), the individual transfer function matrices $\mathbf{P}_{ij}(s)$ can be expressed as

$$\left.\begin{aligned}\mathbf{P}_{11}(s) &= \left[\begin{array}{c|c}A_o & \mathbf{B}_{xp}\\\hline\mathbf{C}_{qx} & \mathbf{D}_{qp}\end{array}\right] & \mathbf{P}_{12}(s) &= \left[\begin{array}{c|c}A_o & B_o\\\hline\mathbf{C}_{qx} & \mathbf{D}_{qu}\end{array}\right]\\[3mm]\mathbf{P}_{21}(s) &= \left[\begin{array}{c|c}A_o & \mathbf{B}_{xp}\\\hline C_o & \mathbf{D}_{yp}\end{array}\right] & \mathbf{P}_{22}(s) &= \left[\begin{array}{c|c}A_o & B_o\\\hline C_o & D_o\end{array}\right]\end{aligned}\right\} \tag{8}$$

This notation is used in this paper (and in the literature) to conveniently represent a transfer function matrix in terms of the state-space matrices of the system realization. The last term in equation (6) should not be viewed as a partitioned constant matrix but as the partitioned transfer function matrices defined by equations (7) and (8). This distinction is made in the notation through the use of solid (as opposed to dashed) partitioning lines.

It should be noted that this model is an extension of the result of Morton and McAfoos (ref. 26), where the $\mathbf{D}_{qp}$ matrix was required to be zero. In this paper, however, $\mathbf{D}_{qp}$ is allowed to be nonzero in order to model the multilinear (cross-product) uncertain terms.

The results for constructing a minimal $\mathbf{P}$-$\mathbf{\Delta}$ model via this approach are presented in the next section.

5. Computational Details of Solution

The proposed solution is presented in four parts. The results for obtaining a cascade-form realization of the uncertain system are summarized first, followed by the results for obtaining

a minimal $\boldsymbol{\Delta}$ matrix and a state-space realization of $\mathbf{P}(s)$. Then, a summary of the overall procedure is presented.

5.1. Cascade-Form Realization

Given the transfer function of an uncertain system in terms of its uncertain parameters $G(s, \delta)$, it is desired to realize the system in a cascade form of first- and second-order subsystems. Thus, if the transfer function is given in unfactored form, the numerator and denominator polynomials must be factored into first- and second-order subsystems, where the second-order terms are only used to represent complex conjugate and inseparable pole/zero pairs. The given transfer function can then be represented as follows:

$$G(s, \delta) = K_y(\delta) \ G_C(s, \delta) \ G_R(s, \delta) \ K_u(\delta) \tag{9}$$

where K_u and K_y represent input and output gain terms, respectively, and G_R and G_C represent the first-order and second-order transfer function components, respectively. Then

$$G_R(s, \delta) = G_{R_k}(s, \delta) \ G_{R_{k-1}}(s, \delta) \ldots G_{R_2}(s, \delta) \ G_{R_1}(s, \delta) \tag{10}$$

$$G_C(s, \delta) = G_{C_\ell}(s, \delta) \ G_{C_{\ell-1}}(s, \delta) \ldots G_{C_2}(s, \delta) \ G_{C_1}(s, \delta) \tag{11}$$

$$G_{R_i}(s, \delta) = \frac{\beta_{2i-1}s + \beta_{2i}}{s + \alpha_i} \tag{12}$$

$$G_{C_i}(s, \delta) = \frac{b_{3i-2}s^2 + b_{3i-1}s + b_{3i}}{s^2 + a_{2i-1}s + a_{2i}} \tag{13}$$

and

k number of first-order blocks

ℓ number of second-order blocks

Any or all these transfer function coefficients may be uncertain. The uncertainty may arise from either the coefficient itself being uncertain or from the coefficient being a multilinear function of one or more uncertain variables. Therefore, for either case, any of the coefficients may be a function of δ. Furthermore, the uncertain variables may have either an additive, $\epsilon = \epsilon_0 + \delta_\epsilon$, or multiplicative, $\epsilon = \epsilon_0(1 + \delta_\epsilon)$, form.

The following cascade-form state-space realization of this system is proposed:

$$G(s, \delta) = \left[\begin{array}{cc|c} \mathbf{A}_R & \mathbf{0} & \mathbf{B}_R K_u \\ \mathbf{B}_C \mathbf{C}_R & \mathbf{A}_C & \mathbf{B}_C \mathbf{D}_R K_u \\ \hline K_y \mathbf{D}_C \mathbf{C}_R & K_y \mathbf{C}_C & K_y \mathbf{D}_C \mathbf{D}_R K_u \end{array} \right] \tag{14}$$

where

$$\mathbf{A}_R = \left[\begin{array}{ccccc} \mathbf{A}_{R_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{R_2}\mathbf{C}_{R_1} & \mathbf{A}_{R_2} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{R_3}\mathbf{D}_{R_2}\mathbf{C}_{R_1} & \mathbf{B}_{R_3}\mathbf{C}_{R_2} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{R_4}\mathbf{D}_{R_3}\mathbf{D}_{R_2}\mathbf{C}_{R_1} & \mathbf{B}_{R_4}\mathbf{D}_{R_3}\mathbf{C}_{R_2} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{B}_{R_{k-1}}\mathbf{D}_{R_{k-2}}\cdots\mathbf{D}_{R_2}\mathbf{C}_{R_1} & \mathbf{B}_{R_{k-1}}\mathbf{D}_{R_{k-2}}\cdots\mathbf{D}_{R_3}\mathbf{C}_{R_2} & \cdots & \mathbf{A}_{R_{k-1}} & \mathbf{0} \\ \mathbf{B}_{R_k}\mathbf{D}_{R_{k-1}}\cdots\mathbf{D}_{R_2}\mathbf{C}_{R_1} & \mathbf{B}_{R_k}\mathbf{D}_{R_{k-1}}\cdots\mathbf{D}_{R_3}\mathbf{C}_{R_2} & \cdots & \mathbf{B}_{R_k}\mathbf{C}_{R_{k-1}} & \mathbf{A}_{R_k} \end{array} \right] \tag{15}$$

$$\mathbf{B}_R = \begin{bmatrix} \mathbf{B}_{R_1} \\ \mathbf{B}_{R_2}\mathbf{D}_{R_1} \\ \mathbf{B}_{R_3}\mathbf{D}_{R_2}\mathbf{D}_{R_1} \\ \mathbf{B}_{R_4}\mathbf{D}_{R_3}\mathbf{D}_{R_2}\mathbf{D}_{R_1} \\ \vdots \\ \mathbf{B}_{R_{k-1}}\mathbf{D}_{R_{k-2}}\cdots\mathbf{D}_{R_2}\mathbf{D}_{R_1} \\ \mathbf{B}_{R_k}\mathbf{D}_{R_{k-1}}\cdots\mathbf{D}_{R_2}\mathbf{D}_{R_1} \end{bmatrix} \tag{16}$$

$$\mathbf{C}_R = \left[\left(\mathbf{D}_{R_k}\mathbf{D}_{R_{k-1}}\cdots\mathbf{D}_{R_2}\mathbf{C}_{R_1}\right) \left(\mathbf{D}_{R_k}\mathbf{D}_{R_{k-1}}\cdots\mathbf{D}_{R_3}\mathbf{C}_{R_2}\right) \cdots \left(\mathbf{D}_{R_k}\mathbf{C}_{R_{k-1}}\right) \left(\mathbf{C}_{R_k}\right) \right] \tag{17}$$

$$\mathbf{D}_R = \left[\mathbf{D}_{R_k}\mathbf{D}_{R_{k-1}}\cdots\mathbf{D}_{R_2}\mathbf{D}_{R_1} \right] \tag{18}$$

The $\mathbf{A}_C$, $\mathbf{B}_C$, $\mathbf{C}_C$, and $\mathbf{D}_C$ matrices have the exact same form as equations (15) through (18), except that the subscripts R and k are replaced by C and ℓ, respectively. The submatrices are defined from equations (12) and (13) as follows:

$$\left.\begin{aligned} \mathbf{A}_{R_i} &= -\alpha_i & \mathbf{B}_{R_i} &= 1 \\ \mathbf{C}_{R_i} &= \beta_{2i} - \alpha_i\beta_{2i-1} & \mathbf{D}_{R_i} &= \beta_{2i-1} \end{aligned}\right\} \tag{19}$$

where the terms α and β are defined in equation (12), and

$$\left.\begin{aligned} \mathbf{A}_{C_j} &= \begin{bmatrix} 0 & 1 \\ -a_{2j} & -a_{2j-1} \end{bmatrix} & \mathbf{B}_{C_j} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathbf{C}_{C_j} &= \left[\left(b_{3j} - a_{2j}b_{3j-2}\right) \left(b_{3j-1} - a_{2j-1}b_{3j-2}\right) \right] & \mathbf{D}_{C_j} &= b_{3j-2} \end{aligned}\right\} \tag{20}$$

where the terms a and b are defined in equation (13). The state-space realizations $\left\{\mathbf{A}_{R_i}, \mathbf{B}_{R_i}, \mathbf{C}_{R_i}, \mathbf{D}_{R_i}\right\}$ and $\left\{\mathbf{A}_{C_j}, \mathbf{B}_{C_j}, \mathbf{C}_{C_j}, \mathbf{D}_{C_j}\right\}$ lead to the ith first-order and jth second-order transfer matrices $\mathbf{G}_{R_i}(s,\delta)$ and $\mathbf{G}_{C_j}(s,\delta)$, respectively, where $i = 1, 2, \ldots, k$, and $j = 1, 2, \ldots, \ell$.

The resulting cascade-form realization of the uncertain system is therefore obtained from equation (14) as

$$\left.\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{A}_R & 0 \\ \mathbf{B}_C\mathbf{C}_R & \mathbf{A}_C \end{bmatrix} & \mathbf{B} &= \begin{bmatrix} \mathbf{B}_R K_u \\ \mathbf{B}_C\mathbf{D}_R K_u \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} K_y\mathbf{D}_C\mathbf{C}_R & K_y\mathbf{C}_C \end{bmatrix} & \mathbf{D} &= \begin{bmatrix} K_y\mathbf{D}_C\mathbf{D}_R K_u \end{bmatrix} \end{aligned}\right\} \tag{21}$$

This model is a general cascade-form realization for any uncertain open-loop SISO transfer function. The model does not, however, handle nonmonic denominator polynomials with uncertain leading coefficients. This would result in fractional (i.e., rational) matrix elements in the realization with uncertain parameters in the denominator of these elements. For real uncertain poles or zeros, two factors determine whether the first-order (real) or second-order (complex) block form should be used. The first is the nature of the uncertainty associated with

these terms, and the second is the form of the transfer function. If the real pole or zero locations are the uncertain parameters and the transfer function form allows these poles or zeros to be separated out, then the real block form should be used. If the transfer function form does not allow this separation, then the complex block form must be used. Furthermore, if there is a **pair** of uncertain poles or zeros that cannot be cascaded, then the resulting minimum Δ matrix will have a repeated parameter on the main diagonal for each inseparable pole or zero pair. Alternatively, if the coefficients of the second-order polynomial associated with the real poles are the uncertain parameters, then the complex block form should be used. These cases are illustrated in section 6 of this paper. The formulation of the minimal Δ matrix is presented next.

5.2. Minimal Δ Matrix

In formulating the minimal **P-Δ** model, the minimal Δ matrix must be determined first. The minimal Δ matrix is defined as in equation (3) with

$$m = m_I + m_D \tag{22}$$

where m_I is the number of independent uncertain parameters and m_D is the number of dependent uncertain parameters that must be added. The independent uncertain parameters are those defined in $G(s, \delta)$. However, as discussed previously, the dependent uncertain parameters are those independent parameters that must be repeated due to nonmultilinear terms in the transfer function coefficients and/or pairs of uncertain real poles or zeros that cannot be cascaded. Thus, for Δ to be minimal, m_D (the dimension of δ_D) should be minimized. It can be shown that if the system transfer function is formed from a given minimal **P-Δ** model of an uncertain system, the coefficients of the numerator and denominator polynomials will be multilinear functions of the uncertain parameters. Unfortunately, the converse is not generally true because of the dependence of the **P-Δ** model on the realization used for the plant. However, if the general cascade-form realization posed in this paper is used, the multilinear form of the transfer function coefficients can be used to establish that $m = m_I$ (i.e., $m_D = 0$) unless there are real uncertain pairs of poles or zeros that cannot be cascaded. Furthermore, it can be shown that if the coefficients of all the **factors** of the numerator and denominator polynomials are multilinear functions, then the coefficients of the expanded polynomials will also be multilinear. However, if there are nonmultilinear uncertain terms in the transfer function, then dependent parameters must be defined (and included in Δ) to represent the nonmultilinear term in a multilinear form. Moreover, if the nonmultilinear term is of the form δ^n, then $n - 1$ dependent parameters must be defined. If there are pairs of uncertain real poles or zeros that cannot be cascaded, then one additional dependent parameter must be added for each pair, and the dependent parameter can be either of the uncertain real parameters in the pair. Therefore, the number m, as determined by these rules, is the minimal dimension of the Δ matrix for the uncertainty class considered in this paper. Once this minimal dimension is determined, the Δ matrix can be defined as a diagonal matrix, as in equation (3), with the specified uncertain parameters on the main diagonal. Examples which illustrate these cases are presented later in section 6.

5.3. State-Space Realization of $P(s)$

Once the cascade-form realization has been determined, the system can be modeled as in equations (1) and (2), where the elements of A_Δ, B_Δ, C_Δ, and D_Δ are known functions of the uncertain parameters. Since any nonmultilinear terms have been redefined in a multilinear form when the minimal Δ matrix is determined, these matrices are multilinear functions of the parameters. In order to obtain a state-space model for $P(s)$ as defined in equations (5),

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expressions for these uncertainty matrices must be determined in terms of the matrices $\mathbf{B}_{xp}$, $\mathbf{C}_{qx}$, $\mathbf{D}_{qp}$, $\mathbf{D}_{qu}$, and $\mathbf{D}_{yp}$ from the model. With equations (4) and (5), these expressions can be determined as follows:

$$\left.\begin{aligned}
\mathbf{A}_\Delta &= \mathbf{B}_{xp}\,\mathbf{\Delta}\,(I - \mathbf{D}_{qp}\,\mathbf{\Delta})^{-1}\mathbf{C}_{qx} &&= \mathbf{B}_{xp}(I - \mathbf{\Delta}\,\mathbf{D}_{qp})^{-1}\,\mathbf{\Delta}\,\mathbf{C}_{qx} \\
\mathbf{B}_\Delta &= \mathbf{B}_{xp}\,\mathbf{\Delta}\,(I - \mathbf{D}_{qp}\,\mathbf{\Delta})^{-1}\mathbf{D}_{qu} &&= \mathbf{B}_{xp}(I - \mathbf{\Delta}\,\mathbf{D}_{qp})^{-1}\,\mathbf{\Delta}\,\mathbf{D}_{qu} \\
\mathbf{C}_\Delta &= \mathbf{D}_{yp}\,\mathbf{\Delta}\,(I - \mathbf{D}_{qp}\,\mathbf{\Delta})^{-1}\mathbf{C}_{qx} &&= \mathbf{D}_{yp}(I - \mathbf{\Delta}\,\mathbf{D}_{qp})^{-1}\,\mathbf{\Delta}\,\mathbf{C}_{qx} \\
\mathbf{D}_\Delta &= \mathbf{D}_{yp}\,\mathbf{\Delta}\,(I - \mathbf{D}_{qp}\,\mathbf{\Delta})^{-1}\mathbf{D}_{qu} &&= \mathbf{D}_{yp}(I - \mathbf{\Delta}\,\mathbf{D}_{qp})^{-1}\,\mathbf{\Delta}\,\mathbf{D}_{qu}
\end{aligned}\right\} \tag{23}$$

The inverse term makes computation of $\mathbf{D}_{qp}$ very difficult. Furthermore, the matrix inversion can cause $\mathbf{A}_\Delta$, $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$, and $\mathbf{D}_\Delta$ to have fractional (i.e., rational) elements with uncertain parameters in the denominator, which is not allowed in the uncertainty class being considered. Thus, it is desirable to represent this factor in terms of its Neumann expansion (ref. 36):

$$(I - \mathbf{\Delta}\,\mathbf{D}_{qp})^{-1} = I + \mathbf{\Delta}\,\mathbf{D}_{qp} + (\mathbf{\Delta}\,\mathbf{D}_{qp})^2 + (\mathbf{\Delta}\,\mathbf{D}_{qp})^3 + \ldots \tag{24}$$

where the latter form in equations (23) has been assumed. Then equations (23) can be rewritten as

$$\left.\begin{aligned}
\mathbf{A}_\Delta &= \mathbf{B}_{xp}\,\mathbf{\Delta}\,\mathbf{C}_{qx} + \mathbf{B}_{xp}\left\{\mathbf{\Delta}\,\mathbf{D}_{qp} + (\mathbf{\Delta}\,\mathbf{D}_{qp})^2 + (\mathbf{\Delta}\,\mathbf{D}_{qp})^3 + \ldots\right\}\mathbf{\Delta}\,\mathbf{C}_{qx} \\
\mathbf{B}_\Delta &= \mathbf{B}_{xp}\,\mathbf{\Delta}\,\mathbf{D}_{qu} + \mathbf{B}_{xp}\left\{\mathbf{\Delta}\,\mathbf{D}_{qp} + (\mathbf{\Delta}\,\mathbf{D}_{qp})^2 + (\mathbf{\Delta}\,\mathbf{D}_{qp})^3 + \ldots\right\}\mathbf{\Delta}\,\mathbf{D}_{qu} \\
\mathbf{C}_\Delta &= \mathbf{D}_{yp}\,\mathbf{\Delta}\,\mathbf{C}_{qx} + \mathbf{D}_{yp}\left\{\mathbf{\Delta}\,\mathbf{D}_{qp} + (\mathbf{\Delta}\,\mathbf{D}_{qp})^2 + (\mathbf{\Delta}\,\mathbf{D}_{qp})^3 + \ldots\right\}\mathbf{\Delta}\,\mathbf{C}_{qx} \\
\mathbf{D}_\Delta &= \mathbf{D}_{yp}\,\mathbf{\Delta}\,\mathbf{D}_{qu} + \mathbf{D}_{yp}\left\{\mathbf{\Delta}\,\mathbf{D}_{qp} + (\mathbf{\Delta}\,\mathbf{D}_{qp})^2 + (\mathbf{\Delta}\,\mathbf{D}_{qp})^3 + \ldots\right\}\mathbf{\Delta}\,\mathbf{D}_{qu}
\end{aligned}\right\} \tag{25}$$

The second group of terms add in the cross terms of the multilinear uncertainty functions. Each term in the series adds a higher order cross-product term. Since $\mathbf{A}_\Delta$, $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$, and $\mathbf{D}_\Delta$ are multilinear functions with a finite number of terms, the $\mathbf{D}_{qp}$ matrix can be defined to have a special structure such that the infinite series of equation (24) can be replaced by a finite series. Hence, convergence of the infinite series of equation (24) need not be considered. This special structure for $\mathbf{D}_{qp}$ is given as follows:

$$(\mathbf{\Delta}\,\mathbf{D}_{qp})^{r+1} = 0 \tag{26}$$

Thus,

$$(I - \mathbf{\Delta}\,\mathbf{D}_{qp})^{-1} = I + \mathbf{\Delta}\,\mathbf{D}_{qp} + (\mathbf{\Delta}\,\mathbf{D}_{qp})^2 + \ldots (\mathbf{\Delta}\,\mathbf{D}_{qp})^r \tag{27}$$

where r is the order of the highest cross term occurring in $\mathbf{A}_\Delta$, $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$, and $\mathbf{D}_\Delta$; that is,

$$r = \max\,(O_A,\ O_B,\ O_C,\ O_D) \tag{28}$$

and O_A, O_B, O_C, and O_D represent the order of the highest order cross-product term in $\mathbf{A}_\Delta$, $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$, and $\mathbf{D}_\Delta$, respectively. That is, for a general uncertain $m \times n$ matrix $\mathbf{M}$,

$$O_\mathbf{M} = \max\,[\text{order}\,(m_{ij});\ \text{for all } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n] \tag{29a}$$

where the order of each m_{ij} is the order of its highest order cross-product term, and cross-product term order is defined as

$$\text{order } (\delta_1 \ \delta_2 \ \delta_3 \ldots \delta_i) = i - 1 \qquad (i = 1, 2, \ldots, m) \tag{29b}$$

Thus, the maximum value of r is $r_{\max} = m - 1$, where m is the dimension of the $\boldsymbol{\Delta}$ matrix. The required structure for $\mathbf{D}_{qp}$ to satisfy equations (26) and (27) is given as follows:

$$\left.\begin{array}{l} \text{1. } d_{ii} = 0 \qquad (i = 1, 2, \ldots, m) \\[4pt] \text{2. If } d_{ij} \neq 0, \text{ then for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, m: \\[4pt] \qquad \text{a. } d_{ji} = 0 \\[4pt] \qquad \text{b. } d_{i\oplus 1, j\oplus 1} = 0 \text{ or } d_{i\oplus 2, j\oplus 2} = 0 \text{ or } d_{i\oplus(m-1), j\oplus(m-1)} = 0 \end{array}\right\} \tag{30}$$

where $\oplus$ represents modulo m addition (ref. 37) over the set $\{1, 2, \ldots, m\}$; that is,

$$a \oplus b = \left\{ \begin{array}{ll} a + b & (a + b \leq m) \\ a + b - m & (a + b > m) \end{array} \right\}$$

The desired equations can therefore be written as

$$\left.\begin{array}{l} \mathbf{A}_\Delta = \mathbf{B}_{xp} \, \boldsymbol{\Delta} \, \mathbf{C}_{qx} + \mathbf{B}_{xp} \left\{ \boldsymbol{\Delta} \, \mathbf{D}_{qp} + (\boldsymbol{\Delta} \, \mathbf{D}_{qp})^2 + \ldots (\boldsymbol{\Delta} \, \mathbf{D}_{qp})^r \right\} \boldsymbol{\Delta} \, \mathbf{C}_{qx} \\[12pt] \mathbf{B}_\Delta = \mathbf{B}_{xp} \, \boldsymbol{\Delta} \, \mathbf{D}_{qu} + \mathbf{B}_{xp} \left\{ \boldsymbol{\Delta} \, \mathbf{D}_{qp} + (\boldsymbol{\Delta} \, \mathbf{D}_{qp})^2 + \ldots (\boldsymbol{\Delta} \, \mathbf{D}_{qp})^r \right\} \boldsymbol{\Delta} \, \mathbf{D}_{qu} \\[12pt] \mathbf{C}_\Delta = \mathbf{D}_{yp} \, \boldsymbol{\Delta} \, \mathbf{C}_{qx} + \mathbf{D}_{yp} \left\{ \boldsymbol{\Delta} \, \mathbf{D}_{qp} + (\boldsymbol{\Delta} \, \mathbf{D}_{qp})^2 + \ldots (\boldsymbol{\Delta} \, \mathbf{D}_{qp})^r \right\} \boldsymbol{\Delta} \, \mathbf{C}_{qx} \\[12pt] \mathbf{D}_\Delta = \mathbf{D}_{yp} \, \boldsymbol{\Delta} \, \mathbf{D}_{qu} + \mathbf{D}_{yp} \left\{ \boldsymbol{\Delta} \, \mathbf{D}_{qp} + (\boldsymbol{\Delta} \, \mathbf{D}_{qp})^2 + \ldots (\boldsymbol{\Delta} \, \mathbf{D}_{qp})^r \right\} \boldsymbol{\Delta} \, \mathbf{D}_{qu} \end{array}\right\} \tag{31}$$

where r is defined by equation (28). Since the $\mathbf{A}_\Delta$, $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$, $\mathbf{D}_\Delta$, and $\boldsymbol{\Delta}$ matrices are known for the given system, equations (31) can be used to determine $\mathbf{B}_{xp}$, $\mathbf{C}_{qx}$, $\mathbf{D}_{qu}$, $\mathbf{D}_{yp}$, and $\mathbf{D}_{qp}$. Once these matrices are obtained, the state-space model of $\mathbf{P}(s)$ is determined. Hence, a minimal $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model can also be formed.

A procedure which summarizes the necessary steps in obtaining a minimal $\mathbf{P}\text{-}\boldsymbol{\Delta}$ model using these results is presented next.

5.4. Summary of Procedure

The following is a summary of the procedure implied by the preceding proposed approach for forming a minimal $\mathbf{P}\text{-}\boldsymbol{\Delta}$ model of a given uncertain system:

1. Obtain the system transfer function in factored form. The coefficients of each factor should be a multilinear function of the uncertain parameters. If necessary, define new dependent parameters to represent any nonmultilinear terms in a multilinear form.

2. Define the number of parameters in the $\boldsymbol{\Delta}$ matrix, m, using equation (22). In so doing, determine if any new parameters are required to model inseparable uncertain real pole or zero pairs. If there are inseparable real pairs, either uncertain parameter in the pair must be repeated.

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3. Define the minimal $\mathbf{\Delta}$ matrix as in equation (3), using the independent parameters defined in the given transfer function as well as those defined in steps 1 and 2.

4. Obtain a cascade-form realization for the system as a function of the uncertain parameters.

5. Express the system matrices as in equations (2).

6. Determine the maximum order of cross-product terms r in $\mathbf{A}_{\Delta}$, $\mathbf{B}_{\Delta}$, $\mathbf{C}_{\Delta}$, and $\mathbf{D}_{\Delta}$ as defined by equations (28) and (29). Then $\mathbf{A}_{\Delta}$, $\mathbf{B}_{\Delta}$, $\mathbf{C}_{\Delta}$, and $\mathbf{D}_{\Delta}$ have the form represented in equations (31), where $\mathbf{D}_{qp}$ has the special structure of equation (30) required by equation (26).

7. Express $\mathbf{A}_{\Delta}$, $\mathbf{B}_{\Delta}$, $\mathbf{C}_{\Delta}$, and $\mathbf{D}_{\Delta}$ as

$$
\left.
\begin{aligned}
\mathbf{A}_{\Delta} &= \mathbf{A}_{\Delta 0} + \mathbf{A}_{\Delta 1} + \mathbf{A}_{\Delta 2} + \ldots + \mathbf{A}_{\Delta r} \\
\mathbf{B}_{\Delta} &= \mathbf{B}_{\Delta 0} + \mathbf{B}_{\Delta 1} + \mathbf{B}_{\Delta 2} + \ldots + \mathbf{B}_{\Delta r} \\
\mathbf{C}_{\Delta} &= \mathbf{C}_{\Delta 0} + \mathbf{C}_{\Delta 1} + \mathbf{C}_{\Delta 2} + \ldots + \mathbf{C}_{\Delta r} \\
\mathbf{D}_{\Delta} &= \mathbf{D}_{\Delta 0} + \mathbf{D}_{\Delta 1} + \mathbf{D}_{\Delta 2} + \ldots + \mathbf{D}_{\Delta r}
\end{aligned}
\right\}
\tag{32}
$$

where the subscript i represents the cross terms of ith order in each uncertainty matrix.

8. The $\mathbf{B}_{xp}$, $\mathbf{C}_{qx}$, $\mathbf{D}_{yp}$, and $\mathbf{D}_{qu}$ matrices are found with the expansion described in reference 26 for the uncertainty matrices having zero-order cross-product terms; that is, define

$$
\mathbf{M} =
\begin{bmatrix}
\mathbf{A}_{\Delta 0} & | & \mathbf{B}_{\Delta 0} \\
\hline
\mathbf{C}_{\Delta 0} & | & \mathbf{D}_{\Delta 0}
\end{bmatrix}
= \mathbf{M}_1 \delta_1 + \mathbf{M}_2 \delta_2 + \ldots + \mathbf{M}_m \delta_m
\tag{33}
$$

where the $\mathbf{M}_i$ matrices are appropriately partitioned. For the case of repeated parameters (due to inseparable real poles or zeros, or due to nonmultilinear functions), the $\mathbf{M}_i$ matrix associated with the repeated parameter must be **nonzero**. These matrices can be decomposed into the product of appropriately partitioned column and row matrices as follows:

$$
\mathbf{M}_i =
\begin{bmatrix}
\mathbf{M}_{B_i} \\
--- \\
\mathbf{M}_{D_{1i}}
\end{bmatrix}
\begin{bmatrix}
\mathbf{M}_{C_i} & | & \mathbf{M}_{D_{2i}}
\end{bmatrix}
\tag{34}
$$

where $\mathbf{M}_{B_i}$ forms the ith column of $\mathbf{B}_{xp}$, $\mathbf{M}_{D_{1i}}$ forms the ith column of $\mathbf{D}_{yp}$, $\mathbf{M}_{C_i}$ forms the ith row of $\mathbf{C}_{qx}$, and $\mathbf{M}_{D_{2i}}$ forms the ith row of $\mathbf{D}_{qu}$. Thus,

$$
\left.
\begin{aligned}
\mathbf{B}_{xp} &= [\mathbf{M}_{B_1} \quad \mathbf{M}_{B_2} \quad \ldots \quad \mathbf{M}_{B_m}] \\
\mathbf{D}_{yp} &= [\mathbf{M}_{D_{11}} \quad \mathbf{M}_{D_{12}} \quad \ldots \quad \mathbf{M}_{D_{1m}}] \\
\mathbf{C}_{qx}^T &= \left[\mathbf{M}_{C_1}^T \quad \mathbf{M}_{C_2}^T \quad \ldots \quad \mathbf{M}_{C_m}^T\right]^T \\
\mathbf{D}_{qu}^T &= \left[\mathbf{M}_{D_{21}}^T \quad \mathbf{M}_{D_{22}}^T \quad \ldots \quad \mathbf{M}_{D_{2m}}^T\right]^T
\end{aligned}
\right\}
\tag{35}
$$

9. Use the higher order cross terms of $\mathbf{A}_{\Delta}$, $\mathbf{B}_{\Delta}$, $\mathbf{C}_{\Delta}$, and $\mathbf{D}_{\Delta}$, as in equations (32), to determine the elements of the $\mathbf{D}_{qp}$ matrix. An augmented matrix equation can be formed

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with equations (31). Begin with the first-order terms and specify as many elements as possible. Continue with the second-order terms, and proceed until all elements of $\mathbf{D}_{qp}$ are specified. Check $\mathbf{D}_{qp}$ to ensure that the required special structure of equation (30) and, hence, equation (26) is satisfied.

10. Form the minimal $\mathbf{P}\text{-}\boldsymbol{\Delta}$ model as given in equations (3), (5), and (6) and depicted in figure 2. If the $\mathbf{M}\text{-}\boldsymbol{\Delta}$ model is desired, the feedback control system $K(s)$ can be closed as discussed previously.

It should be noted that the matrices $\mathbf{M}_{B_i}$, $\mathbf{M}_{C_i}$, $\mathbf{M}_{D_{1i}}$, and $\mathbf{M}_{D_{2i}}$, obtained in decomposing the $\mathbf{M}_i$ matrices in equation (34), are not necessarily unique. A method of formalizing this decomposition for computer implementation is not addressed in this paper. However, an algorithm is presented in reference 35 which accomplishes this decomposition as an extension to reference 36. Some examples are given in the next section to illustrate these results.

6. Examples

The following examples illustrate the proposed procedure presented in section 5.4 for the various cases discussed in the preceding sections.

Example 6.1: This example illustrates the construction of a minimal $\mathbf{P}\text{-}\boldsymbol{\Delta}$ model for an uncertain system whose transfer function contains an uncertain coefficient which is a **multilinear function** of the uncertain parameters as well as a coefficient which is a **nonmultilinear function** of an uncertain parameter:

$$G(s, \delta) = \frac{1}{s^2 + 2\zeta\omega s + \omega^2}$$

where

$$\zeta = \zeta_0 + \delta_\zeta \qquad \omega = \omega_0 + \delta_\omega$$

This is a second-order system with uncertain complex poles. The uncertainty appears in the damping and frequency characteristics of the complex poles. The procedure in section 5.4 is used to determine a minimal $\mathbf{P}\text{-}\boldsymbol{\Delta}$ model as follows:

1. As given, $G(s, \delta)$ is in factored form. However, the constant coefficient ω^2 is not a multilinear function of the uncertain parameters. Substituting for ω in the above transfer function therefore yields the problematic term δ_ω^2. In order to represent this equation in multilinear form, the following dependent variable is defined:

$$\delta_3 = \delta_\omega$$

so that

$$\delta_\omega^2 = \delta_\omega \delta_3$$

2. There are two independent parameters, δ_ω and δ_ζ, and one dependent parameter, δ_3, due to the nonmultilinear term. Thus, from equation (22),

$$m = m_I + m_D = 2 + 1 = 3$$

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3. The minimal Δ matrix can be defined as follows:

$$\Delta = \text{diag}[\delta_\zeta, \ \delta_\omega, \ \delta_3]$$

although ordering of the uncertain parameters in Δ is arbitrary.

4. The cascade realization using equations (13) and (20) is determined to be

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

Thus, it can be seen that for this example, the uncertain parameters occur only in the $\mathbf{A}$ matrix.

5. Separate the nominal and uncertain parts of the system matrices as in equation (2):

$$\mathbf{A} = A_o + A_\Delta = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & -2\zeta_o\omega_o \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ (-2\omega_o\delta_\omega + \delta_\omega\delta_3) & -2(\omega_o\delta_\zeta + \zeta_o\delta_\omega + \delta_\zeta\delta_\omega) \end{bmatrix}$$

$$\mathbf{B} = B_o + B_\Delta = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = C_o + C_\Delta = \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = D_o + D_\Delta = \begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix}$$

6. Find the maximum cross-term order, r:

$$O_A = \text{order } (\delta_\omega\delta_3) = \text{order } (\delta_\zeta\delta_\omega) = 1$$
$$O_B = O_C = O_D = 0$$
$$r = \max (1, 0, 0, 0) = 1 \Rightarrow r = 1$$

7. Expand $\mathbf{A}_\Delta$ as in equations (32):

$$\mathbf{A}_\Delta = A_{\Delta 0} + A_{\Delta 1} = \begin{bmatrix} 0 & 0 \\ -2\omega_o\delta_\omega & -2(\omega_o\delta_\zeta + \zeta_o\delta_\omega) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \delta_\omega\delta_3 & -2\delta_\zeta\delta_\omega \end{bmatrix}$$

8. Find $\mathbf{B}_{xp}$, $\mathbf{C}_{qx}$, $\mathbf{D}_{yp}$, and $\mathbf{D}_{qu}$ by using equations (33)–(35):

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{A}_{\Delta 0} & \mathbf{B}_{\Delta 0} \\ \hline \mathbf{C}_{\Delta 0} & \mathbf{D}_{\Delta 0} \end{array} \right] = \mathbf{M}_1\delta_\zeta + \mathbf{M}_2\delta_\omega + \mathbf{M}_3\delta_3$$

where

$$\mathbf{M}_1 = \begin{bmatrix} 0 & 0 & \vdots & 0 \\ 0 & -2\omega_o & \vdots & 0 \\ \hline 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\mathbf{M}_2 = \begin{bmatrix} 0 & 0 & \vdots & 0 \\ -2\omega_o & 0 & \vdots & 0 \\ \hline 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\mathbf{M}_3 = \begin{bmatrix} 0 & 0 & \vdots & 0 \\ 0 & -2\zeta_o & \vdots & 0 \\ \hline 0 & 0 & \vdots & 0 \end{bmatrix}$$

It is noted that $\mathbf{M}_2$ and $\mathbf{M}_3$ are the coefficient matrices associated with the repeated parameter δ_ω. As indicated in the procedure, neither of these matrices can be a null matrix. Thus, the nonzero elements of the coefficient matrix associated with δ_ω have been split columnwise to form $\mathbf{M}_2$ and $\mathbf{M}_3$.

Note: This is equivalent to reassigning the repeated parameters in $\mathbf{A}_{\Delta 0}$, $\mathbf{B}_{\Delta 0}$, $\mathbf{C}_{\Delta 0}$, and $\mathbf{D}_{\Delta 0}$, to be columnwise independent prior to determining the $\mathbf{M}_i$ matrices. For this example, the reassignment can be accomplished as follows:

$$\mathbf{A}_{\Delta 0} = \begin{bmatrix} 0 & 0 \\ -2\omega_o\delta_\omega & -2(\omega_o\delta_\zeta + \zeta_o\delta_3) \end{bmatrix}$$

in order to be consistent with the $\mathbf{M}_2$ and $\mathbf{M}_3$ matrices determined above.

Then

$$\mathbf{M}_1 = \begin{bmatrix} 0 \\ -2 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 0 & \omega_o & \vdots & 0 \end{bmatrix}$$

$$\mathbf{M}_2 = \begin{bmatrix} 0 \\ -2 \\ \hline 0 \end{bmatrix} \begin{bmatrix} \omega_o & 0 & \vdots & 0 \end{bmatrix}$$

$$\mathbf{M}_3 = \begin{bmatrix} 0 \\ -2 \\ \hline 0 \end{bmatrix} \begin{bmatrix} 0 & \zeta_o & \vdots & 0 \end{bmatrix}$$

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Thus

$$\mathbf{B}_{xp} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix} \qquad \mathbf{D}_{yp} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_{qx} = \begin{bmatrix} 0 & \omega_o \\ \omega_o & 0 \\ 0 & \zeta_o \end{bmatrix} \qquad \mathbf{D}_{qu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

9. Find $\mathbf{D}_{qp}$ by using equations (31) and (32). Since $r = 1$ for this example, equations (31) can be written as

$$\mathbf{A}_{\Delta 1} = \mathbf{B}_{xp}[\Delta \ \mathbf{D}_{qp}] \ \Delta \ \mathbf{C}_{qx}$$
$$\mathbf{B}_{\Delta 1} = \mathbf{B}_{xp}[\Delta \ \mathbf{D}_{qp}] \ \Delta \ \mathbf{D}_{qu}$$
$$\mathbf{C}_{\Delta 1} = \mathbf{D}_{yp}[\Delta \ \mathbf{D}_{qp}] \ \Delta \ \mathbf{C}_{qx}$$
$$\mathbf{D}_{\Delta 1} = \mathbf{D}_{yp}[\Delta \ \mathbf{D}_{qp}] \ \Delta \ \mathbf{D}_{qu}$$

These equations can be combined into an augmented matrix equation as follows:

$$\left[\begin{array}{c|c} \mathbf{A}_{\Delta 1} & \mathbf{B}_{\Delta 1} \\ \hline \mathbf{C}_{\Delta 1} & \mathbf{D}_{\Delta 1} \end{array} \right] = \begin{bmatrix} \mathbf{B}_{xp} \\ \mathbf{D}_{yp} \end{bmatrix} \begin{bmatrix} \Delta \ \mathbf{D}_{qp} \end{bmatrix} \Delta \ \left[\begin{array}{c|c} \mathbf{C}_{qx} & \mathbf{D}_{qu} \end{array} \right]$$

Substituting into this equation for $\mathbf{A}_{\Delta 1}$, $\mathbf{B}_{\Delta 1}$, $\mathbf{C}_{\Delta 1}$, $\mathbf{D}_{\Delta 1}$, $\mathbf{B}_{xp}$, $\mathbf{D}_{yp}$, $\mathbf{C}_{qx}$, $\mathbf{D}_{qu}$, and Δ and solving for $\mathbf{D}_{qp}$ yields

$$\mathbf{D}_{qp} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\omega_o} & 0 & 0 \\ 0 & \frac{-1}{2\omega_o} & 0 \end{bmatrix}$$

which is consistent with the special structure required by equation (30). To see this, consider equation (30). As required by condition 1, $d_{11} = 0$, $d_{22} = 0$, and $d_{33} = 0$. Now consider condition 2. First, $d_{21} \neq 0$ requires that $d_{12} = 0$ and that $d_{32} = 0$ or $d_{13} = 0$. Since $d_{12} = 0$ and $d_{13} = 0$, this condition is satisfied. Similarly, $d_{32} \neq 0$ requires that $d_{23} = 0$ and that $d_{13} = 0$ or $d_{21} = 0$. Since $d_{23} = 0$ and $d_{13} = 0$, the condition is satisfied. Hence, the special structure of equation (30) is satisfied.

10. The minimal $\mathbf{P}$-Δ model as shown in figure 2 can now be constructed by using equations (3), (5), and (6). The realization of $\mathbf{P}(s)$ for the resulting model can be depicted as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & -2\zeta_o\omega_o \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_o \\ \omega_o & 0 \\ 0 & \zeta_o \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\omega_o} & 0 & 0 \\ 0 & \frac{-1}{2\omega_o} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The Δ matrix is given by

$$\Delta = \text{diag}\begin{bmatrix} \delta_\zeta, & \delta_\omega, & \delta_\omega \end{bmatrix}$$

Example 6.2: This example illustrates the construction of a minimal $\mathbf{P}$-$\mathbf{\Delta}$ model for an uncertain system whose transfer function contains an inseparable uncertain real pole pair:

$$G(s,\delta) = \frac{b_1 s^2 + b_2 s + b_3}{(s + \theta_1)(s + \theta_2)}$$

where

$$b_1 = b_{1o} + \delta_{b1} \qquad b_2 = b_{2o} + \delta_{b2} \qquad b_3 = b_{3o} + \delta_{b3}$$

$$\theta_1 = \theta_{1o} + \delta_{\theta 1} \qquad \theta_2 = \theta_{2o} + \delta_{\theta 2}$$

1. Since the numerator is second order with uncertain coefficients, the uncertain real poles in the denominator cannot be separated into the real cascade form. The denominator must therefore be expanded, and the second-order (complex) block must be used in the realization; that is,

$$G(s,\delta) = \frac{b_1 s^2 + b_2 s + b_3}{s^2 + (\theta_1 + \theta_2)s + \theta_1\theta_2}$$

2. There are five independent uncertain parameters in this system which must all appear in the $\mathbf{\Delta}$ matrix, namely,

$$b_1, \ b_2, \ b_3, \ \theta_1, \ \text{and} \ \theta_2$$

In addition, since θ_1 and θ_2 are an inseparable pole pair, either θ_1 or θ_2 must be repeated in the $\mathbf{\Delta}$ matrix. (It can be shown that if $\delta_{\theta 1}$ or $\delta_{\theta 2}$ is not repeated, the $\mathbf{D}_{qp}$ matrix will not have the required structure of equation (30) and the higher order cross terms will not be modeled correctly.) Thus, from equation (22)

$$m = 5 + 1 = 6$$

3. The $\mathbf{\Delta}$ matrix can be defined as

$$\Delta = \text{diag}\begin{bmatrix} \delta_{\theta 1}, & \delta_2, & \delta_{\theta 2}, & \delta_{b1}, & \delta_{b2}, & \delta_{b3} \end{bmatrix}$$

where

$$\delta_2 = \delta_{\theta 1}$$

4. The cascade realization can be constructed by using equations (13) and (20) and is given as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\theta_1\theta_2 & -(\theta_1 + \theta_2) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} (b_3 - \theta_1\theta_2 b_1) & (b_2 - (\theta_1 + \theta_2)b_1) \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} b_1 \end{bmatrix}$$

For this example, the uncertain parameters occur in the $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{D}$ matrices.

5. These system matrices can be expanded as in equations (2) to yield:

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$$\mathbf{A}_o = \begin{bmatrix} 0 & 1 \\ -\theta_{1o}\theta_{2o} & -(\theta_{1o} + \theta_{2o}) \end{bmatrix} \qquad \mathbf{B}_o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C}_o = \begin{bmatrix} (b_{3o} - \theta_{1o}\theta_{2o}b_{1o}) & b_2 - (\theta_{1o} + \theta_{2o})b_{1o} \end{bmatrix} \qquad \mathbf{D}_o = \begin{bmatrix} b_{1o} \end{bmatrix}$$

$$\mathbf{A}_\Delta = \begin{bmatrix} 0 & 0 \\ -(\theta_{2o}\delta_{\theta 1} + \theta_{1o}\delta_{\theta 2} + \delta_{\theta 1}\delta_{\theta 2}) & -(\delta_{\theta 1} + \delta_{\theta 2}) \end{bmatrix} \qquad \mathbf{B}_\Delta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C}_\Delta = \begin{bmatrix} c_{\Delta 11} & c_{\Delta 12} \end{bmatrix} \qquad \mathbf{D}_\Delta = \begin{bmatrix} \delta_{b1} \end{bmatrix}$$

where

$$c_{\Delta 11} = \delta_{b3} - \theta_{2o}b_{1o}\delta_{\theta 1} - \theta_{1o}b_{1o}\delta_{\theta 2} - \theta_{1o}\theta_{2o}\delta_{b1} - b_{1o}\delta_{\theta 1}\delta_{\theta 2} - \theta_{1o}\delta_{\theta 2}\delta_{b1} - \theta_{2o}\delta_{\theta 1}\delta_{b1} - \delta_{\theta 1}\delta_{\theta 2}\delta_{b1}$$

$$c_{\Delta 12} = \delta_{b2} - b_{1o}\delta_{\theta 1} - b_{1o}\delta_{\theta 2} - (\theta_{1o} + \theta_{2o})\delta_{b1} - \delta_{\theta 1}\delta_{b1} - \delta_{\theta 2}\delta_{b1}$$

6. Find the maximum cross-term order r using equations (28) and (29):

$$O_A = \text{order}\,(\delta_{\theta 1}\delta_{\theta 2}) = 1 \qquad O_B = \text{order}\,(0) = 0$$
$$O_C = \text{order}\,(\delta_{\theta 1}\delta_{\theta 2}\delta_{b1}) = 2 \qquad O_D = \text{order}\,(\delta_{b1}) = 0$$

$$r = \max\,(1, 0, 2, 0) = 2 \Rightarrow r = 2$$

7. Expand $\mathbf{A}_\Delta$, $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$, and $\mathbf{D}_\Delta$ as in equations (32):

$$\mathbf{A}_\Delta = \mathbf{A}_{\Delta 0} + \mathbf{A}_{\Delta 1} + \mathbf{A}_{\Delta 2} \qquad \mathbf{B}_\Delta = \mathbf{B}_{\Delta 0} + \mathbf{B}_{\Delta 1} + \mathbf{B}_{\Delta 2}$$
$$\mathbf{C}_\Delta = \mathbf{C}_{\Delta 0} + \mathbf{C}_{\Delta 1} + \mathbf{C}_{\Delta 2} \qquad \mathbf{D}_\Delta = \mathbf{D}_{\Delta 0} + \mathbf{D}_{\Delta 1} + \mathbf{D}_{\Delta 2}$$

where

$$\mathbf{A}_{\Delta 0} = \begin{bmatrix} 0 & 0 \\ -(\theta_{2o}\delta_{\theta 1} + \theta_{1o}\delta_{\theta 2}) & -(\delta_{\theta 1} + \delta_{\theta 2}) \end{bmatrix}$$

$$\mathbf{A}_{\Delta 1} = \begin{bmatrix} 0 & 0 \\ -\delta_{\theta 1}\delta_{\theta 2} & 0 \end{bmatrix}$$

$$\mathbf{A}_{\Delta 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_{\Delta 0} = \mathbf{B}_{\Delta 1} = \mathbf{B}_{\Delta 2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C}_{\Delta 0} = [\{\delta_{b3} - \theta_{2o}b_{1o}\delta_{\theta 1} - \theta_{1o}b_{1o}\delta_{\theta 2} - \theta_{1o}\theta_{2o}\delta_{b1}\} \quad \{\delta_{b2} - b_{1o}\delta_{\theta 1} - b_{1o}\delta_{\theta 2} - (\theta_{1o} + \theta_{2o})\delta_{b1}\}]$$

$$\mathbf{C}_{\Delta 1} = [-(b_{1o}\delta_{\theta 1}\delta_{\theta 2} + \theta_{2o}\delta_{\theta 1}\delta_{b1} + \theta_{1o}\delta_{\theta 2}\delta_{b1}) \qquad -(\delta_{\theta 1}\delta_{b1} + \delta_{\theta 2}\delta_{b1})]$$

$$\mathbf{C}_{\Delta 2} = [-\delta_{\theta 1}\delta_{\theta 2}\delta_{b1} \quad 0]$$

$$\mathbf{D}_{\Delta 0} = [\delta_{b1}] \qquad \mathbf{D}_{\Delta 1} = [0] \qquad \mathbf{D}_{\Delta 2} = [0]$$

8. Solve for $\mathbf{B}_{xp}$, $\mathbf{C}_{qx}$, $\mathbf{D}_{yp}$, and $\mathbf{D}_{qu}$ using equations (33)–(35):

$$\mathbf{M} = \mathbf{M}_1 \delta_{\theta 1} + \mathbf{M}_2 \delta_2 + \mathbf{M}_3 \delta_{\theta 2} + \mathbf{M}_4 \delta_{b_1} + \mathbf{M}_5 \delta_{b_2} + \mathbf{M}_6 \delta_{b_3}$$

where

$$\mathbf{M}_1 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -\theta_{2o} & 0 & 0 \\ \hline -\theta_{2o} b_{1o} & 0 & 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ -1 \\ \hline -b_{1o} \end{array}\right] \left[\begin{array}{cc|c} \theta_{2o} & 0 & 0 \end{array}\right]$$

$$\mathbf{M}_2 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & b_{1o} & 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ -1 \\ \hline -b_{1o} \end{array}\right] \left[\begin{array}{cc|c} 0 & 1 & 0 \end{array}\right]$$

$$\mathbf{M}_3 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -\theta_{1o} & -1 & 0 \\ \hline -\theta_{1o} b_{1o} & -b_{1o} & 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ -1 \\ \hline -b_{1o} \end{array}\right] \left[\begin{array}{cc|c} \theta_{1o} & 1 & 0 \end{array}\right]$$

$$\mathbf{M}_4 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline -\theta_{1o}\theta_{2o} & -(\theta_{1o} + \theta_{2o}) & 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline -1 \end{array}\right] \left[\begin{array}{cc|c} \theta_{1o}\theta_{2o} & (\theta_{1o} + \theta_{2o}) & -1 \end{array}\right]$$

$$\mathbf{M}_5 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \end{array}\right] \left[\begin{array}{cc|c} 0 & 1 & 0 \end{array}\right]$$

$$\mathbf{M}_6 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right]$$

It is noted that $\mathbf{M}_1$ and $\mathbf{M}_2$ are associated with the repeated parameter $\delta_{\theta 1}$ in $\mathbf{\Delta}$. As discussed in example 6.1, neither $\mathbf{M}_1$ nor $\mathbf{M}_2$ can be a null matrix. Thus, as in example 6.1, $\mathbf{M}_1$ and $\mathbf{M}_2$ were formed by splitting up the columns of the coefficient matrix associated with $\delta_{\theta 1}$. (It can be shown that if $\mathbf{M}_1$ and $\mathbf{M}_2$ are not both required to be nonzero, $\mathbf{D}_{qp}$ will not have the required structure, and hence the higher order cross-product terms will not be modeled correctly.)

Note: As mentioned in the previous example, this is equivalent to reassigning the repeated parameters in $\mathbf{A}_{\Delta 0}$, $\mathbf{B}_{\Delta 0}$, $\mathbf{C}_{\Delta 0}$, and $\mathbf{D}_{\Delta 0}$ to be columnwise independent prior to determining the $\mathbf{M}_i$ matrices. For this example, the reassignment can be accomplished as follows:

$$\mathbf{A}_{\Delta 0} = \begin{bmatrix} 0 & 0 \\ -(\theta_{2o}\delta_{\theta 1} + \theta_{1o}\delta_{\theta 2}) & -(\delta_2 + \delta_{\theta 2}) \end{bmatrix}$$

$$\mathbf{C}_{\Delta 0} = \begin{bmatrix} \{\delta_{b3} - \theta_{2o}b_{1o}\delta_{\theta 1} - \theta_{1o}b_{1o}\delta_{\theta 2} - \theta_{1o}\theta_{2o}\delta_{b1}\} & \{\delta_{b2} - b_{1o}\delta_2 - b_{1o}\delta_{\theta 2} - (\theta_{1o} + \theta_{2o})\delta_{b1}\} \end{bmatrix}$$

in order to be consistent with the $\mathbf{M}_1$ and $\mathbf{M}_2$ matrices determined above.

Thus, based on the above, the following results are obtained:

$$\mathbf{B}_{xp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{D}_{yp} = \begin{bmatrix} -b_{1o} & -b_{1o} & -b_{1o} & -1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{C}_{qx} = \begin{bmatrix} \theta_{2o} & 0 \\ 0 & 1 \\ \theta_{1o} & 1 \\ \theta_{1o}\theta_{2o} & (\theta_{1o} + \theta_{2o}) \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{D}_{qu} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

9. The higher order cross terms are used as in equations (31) and (32) for $r = 2$ to determine $\mathbf{D}_{qp}$ as follows:

$$\mathbf{A}_{\Delta 1} + \mathbf{A}_{\Delta 2} = \mathbf{B}_{xp}\left[\Delta\, \mathbf{D}_{qp} + (\Delta\, \mathbf{D}_{qp})^2\right]\Delta\, \mathbf{C}_{qx}$$

$$\mathbf{B}_{\Delta 1} + \mathbf{B}_{\Delta 2} = \mathbf{B}_{xp}\left[\Delta\, \mathbf{D}_{qp} + (\Delta\, \mathbf{D}_{qp})^2\right]\Delta\, \mathbf{D}_{qu}$$

$$\mathbf{C}_{\Delta 1} + \mathbf{C}_{\Delta 2} = \mathbf{D}_{yp}\left[\Delta\, \mathbf{D}_{qp} + (\Delta\, \mathbf{D}_{qp})^2\right]\Delta\, \mathbf{C}_{qx}$$

$$\mathbf{D}_{\Delta 1} + \mathbf{D}_{\Delta 2} = \mathbf{D}_{yp}\left[\Delta\, \mathbf{D}_{qp} + (\Delta\, \mathbf{D}_{qp})^2\right]\Delta\, \mathbf{D}_{qu}$$

Substituting for these known matrices and solving for $\mathbf{D}_{qp}$ yields the following result:

$$\mathbf{D}_{qp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\theta_{2o}} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which satisfies the required structure of equation (30).

10. Thus, from equations (3), (5), and (6), the following **P-Δ** model can be formed as depicted in figure 2 where the state-space form of $\mathbf{P}(s)$ is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\theta_{1o}\theta_{2o} & -(\theta_{1o} + \theta_{2o}) \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix} p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}$$

$$\mathbf{q} = \begin{bmatrix} \theta_{2o} & 0 \\ 0 & 1 \\ \theta_{1o} & 1 \\ \theta_{1o}\theta_{2o} & \theta_{1o} + \theta_{2o} \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \mathbf{u} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\theta_{2o}} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} p$$

$$\mathbf{y} = [b_{3o} - \theta_{1o}\theta_{1o}b_{1o} \quad b_{2o} - (\theta_{1o} + \theta_{1o}) b_{1o}] \mathbf{x} + [b_{1o}] \mathbf{u}$$
$$+ [-b_{1o} \quad -b_{1o} \quad -b_{1o} \quad -1 \quad 1 \quad 1] p$$

and the Δ matrix is given by:

$$\Delta = \text{diag} \begin{bmatrix} \delta_{\theta 1}, & \delta_{\theta 1}, & \delta_{\theta 2}, & \delta_{b1}, & \delta_{b2}, & \delta_{b3} \end{bmatrix}$$

Example 6.3: This example illustrates the construction of a minimal **P-Δ** model for an uncertain system which has many independent uncertain parameters:

$$G(s, \delta) = \frac{(\beta_1 s + \beta_2)(\beta_3 s + \beta_4)\left(b_1 s^2 + b_2 s + b_3\right)}{(s + \alpha_1)(s + \alpha_2)(s^2 + a_1 s + a_2)}$$

where

$$\alpha_1 = \alpha_{1o} + \delta_{\alpha 1} \quad \alpha_2 = \alpha_{2o} + \delta_{\alpha 2} \quad a_1 = a_{1o} + \delta_{a1} \quad a_2 = a_{2o} + \delta_{a2}$$
$$\beta_1 = \beta_{1o} + \delta_{\beta 1} \quad \beta_2 = \beta_{2o} + \delta_{\beta 2} \quad \beta_3 = \beta_{3o} + \delta_{\beta 3} \quad \beta_4 = \beta_{4o} + \delta_{\beta 4}$$
$$b_1 = b_{1o} + \delta_{b1} \quad b_2 = b_{2o} + \delta_{b2} \quad b_3 = b_{3o} + \delta_{b3}$$

1. The given transfer function is in correct form, all coefficients are multilinear, and all pole-zero pairs can be cascaded.

2. This system has 11 independent uncertain parameters. Since there are no nonmultilinear terms in the transfer function coefficients and no inseparable real uncertain pole or zero pairs, no dependent parameters need to be defined. Thus, m can be defined with equation (22) as follows:
$$m = m_I + m_D = 11 + 0 = 11 \Rightarrow m = 11$$

3. The minimal Δ matrix can be defined as
$$\Delta = \text{diag} \begin{bmatrix} \delta_{\alpha 1}, & \delta_{\alpha 2}, & \delta_{a1}, & \delta_{a2}, & \delta_{\beta 1}, & \delta_{\beta 2}, & \delta_{\beta 3}, & \delta_{\beta 4}, & \delta_{b1}, & \delta_{b2}, & \delta_{b3} \end{bmatrix}$$

4. The cascade-form realization of this example is determined as follows:
$$G_C = \frac{b_1 s^2 + b_2 s + b_3}{s^2 + a_1 s + a_2} \qquad G_R = G_{R_1} G_{R_2} = \begin{bmatrix} \frac{\beta_1 s + \beta_2}{s + \alpha_1} \end{bmatrix} \begin{bmatrix} \frac{\beta_3 s + \beta_4}{s + \alpha_2} \end{bmatrix}$$

The realization of G_C is determined from equations (13) and (20) for $\ell = 1$:

$$\mathbf{A}_C = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \qquad \mathbf{B}_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C}_C = \begin{bmatrix} b_3 - a_2 b_1 & b_2 - a_1 b_1 \end{bmatrix} \qquad \mathbf{D}_C = \begin{bmatrix} b_1 \end{bmatrix}$$

and the realization of G_R is determined from equations (12) and (15)–(19) for $k = 2$:

$$\mathbf{A}_R = \begin{bmatrix} -\alpha_1 & 1 \\ \beta_2 - \alpha_1 \beta_1 & -\alpha_2 \end{bmatrix} \qquad \mathbf{B}_R = \begin{bmatrix} 1 \\ \beta_1 \end{bmatrix}$$

$$\mathbf{C}_R = \begin{bmatrix} \beta_3 (\beta_2 - \alpha_1 \beta_1) & \beta_4 - \alpha_2 \beta_3 \end{bmatrix} \qquad \mathbf{D}_R = \begin{bmatrix} \beta_3 \beta_1 \end{bmatrix}$$

Then the cascade realization is given by equations (14) and (21) for unity gains at the input and output:

$$\mathbf{A} = \begin{bmatrix} A_R & 0 \\ B_C C_R & A_C \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} B_R \\ B_C D_R \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} D_C C_R & C_C \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} D_C D_R \end{bmatrix}$$

Thus

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & 0 & 0 & 0 \\ \beta_2 - \alpha_1 \beta_1 & -\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \beta_3(\beta_2 - \alpha_1 \beta_1) & \beta_4 - \alpha_2 \beta_3 & -a_2 & -a_1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ \beta_1 \\ 0 \\ \beta_1 \beta_3 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} b_1 \beta_3(\beta_2 - \alpha_1 \beta_1) & b_1(\beta_4 - \alpha_2 \beta_3) & (b_3 - a_2 b_1) & (b_2 - a_1 b_1) \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \beta_1 \beta_3 b_1 \end{bmatrix}$$

5-9. For this example, uncertainty arises in the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ matrices. The $\mathbf{A}_\Delta$, $\mathbf{B}_\Delta$, $\mathbf{C}_\Delta$, and $\mathbf{D}_\Delta$ matrices are fairly complicated for this example and are therefore not given. The order of the highest cross-product term is 3, so that $r = 3$. Following the procedure outlined in section 5, the results are determined in a straightforward manner to be

$$\mathbf{B}_{xp} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{1o} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{1o}\beta_{3o} & \beta_{3o} & 1 & 1 & \beta_{3o} & \beta_{3o} & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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$$\mathbf{D}_{yp} = \begin{bmatrix} b_{1o}\beta_{1o}\beta_{3o} & b_{1o}\beta_{3o} & b_{1o} & b_{1o} & b_{1o}\beta_{3o} & b_{1o}\beta_{3o} & b_{1o} & b_{1o} & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{C}_{qx}^T = \begin{bmatrix} -1 & 0 & 0 & 0 & -\alpha_{1o} & 1 & (\beta_{2o} - \alpha_{1o}\beta_{1o}) & 0 & \beta_{3o}(\beta_{2o} - \alpha_{1o}\beta_{1o}) & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -\alpha_{2o} & 1 & (\beta_{4o} - \alpha_{2o}\beta_{3o}) & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -\alpha_{2o} & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha_{1o} & 1 & 0 \end{bmatrix}$$

$$\mathbf{D}_{qu}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & \beta_{1o} & 0 & \beta_{1o}\beta_{3o} & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_{qp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{1o} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{1o}\beta_{3o} & \beta_{3o} & 1 & 1 & \beta_{3o} & \beta_{3o} & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10. Equations (5) and (6) can now be used to obtain the state-space model of $\mathbf{P}(s)$ and the Δ matrix determined as step 3 of this example. This yields the desired $\mathbf{P}$-Δ model as depicted in figure 2.

These examples illustrate the proposed procedure for forming a minimal $\mathbf{P}$-Δ model of an uncertain system. Although all the steps involved in obtaining these results have not been included (particularly in example 6.3), the stated results should provide a guide in performing the steps of the proposed procedure. It should be noted that, for ease of hand computation, examples 6.2 and 6.3 included only the simplistic (and less realistic) case in which the coefficients themselves are the uncertain parameters. However, it is emphasized that, as illustrated in example 6.1, the proposed procedure does handle the more realistic case in which the uncertain transfer function coefficients are multilinear functions of the uncertain parameters.

7. Concluding Remarks

A proposed procedure is presented for forming a $\mathbf{P}$-Δ model of an uncertain system which appears to be of minimal dimension, given its transfer function in terms of the uncertain parameters. The uncertainty class considered in this paper allows the transfer function coefficients to be **multilinear** functions of the uncertain parameters, and the uncertainties may arise in any or all of the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ matrices of the system model. Although no proofs are presented regarding minimality, the resulting models appear to be minimal in dimension for all examples worked thus far. Moreover, even if some counterexample exists for which the resulting

$\mathbf{P}$-$\boldsymbol{\Delta}$ model is not minimal, the outlined procedure does provide a means of handling the more realistic uncertainty class which includes multilinear functions of the uncertain parameters. This procedure involves realizing the system in a cascade form, determining the minimal $\boldsymbol{\Delta}$ matrix of uncertain parameters, and obtaining a state-space model for the nominal open-loop system $\mathbf{P}(s)$. As stated previously, the minimal $\mathbf{M}$-$\boldsymbol{\Delta}$ model can then be easily obtained by closing the feedback loop. Three examples were given to illustrate the proposed procedure. The first example had uncertainty in the $\mathbf{A}$ matrix only and illustrated how this method handles uncertain system transfer functions having coefficients that are multilinear functions of the uncertain parameters as well as coefficients that are nonmultilinear functions of the uncertain parameters, for which a repeated uncertain parameter had to be used. This example is representative of typical problems that can arise in the mathematical modeling of realistic dynamic systems. The second example had uncertain parameters arising in the $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{D}$ matrices only. This example illustrated the formulation of a minimal $\mathbf{P}$-$\boldsymbol{\Delta}$ model for a system with inseparable real uncertain poles, and also involved repeating an uncertain parameter in the $\boldsymbol{\Delta}$ matrix. The third example had 11 independent uncertain parameters, which arose in the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ matrices of the system realization. This example was included to demonstrate this method for a large number of uncertain parameters which occur in all the system matrices. Also, for this example none of the uncertain parameters had to be repeated. Thus, the minimal $\boldsymbol{\Delta}$ matrix contained only the independent uncertain parameters given in the problem.

Further work on the proposed procedure could include systems having a nonmonic characteristic polynomial with an uncertain leading coefficient, as well as systems having an inner feedback loop which may or may not have uncertainties. The former case might require extending the uncertainty class to include rational expressions containing multilinear functions of the uncertain parameters in the numerator and denominator, and the latter case might require a modification in the formulation of the cascade realization. Although the procedure presented in this paper is for single-input, single-output systems, an extension to multiple-input, multiple-output systems appears possible and should primarily involve modifying the cascade-form realization. Other areas of future work include development of a means of proving the minimality of a given $\mathbf{P}$-$\boldsymbol{\Delta}$ model, development of a proof that the above procedure yields a minimal $\mathbf{P}$-$\boldsymbol{\Delta}$ model, and development of a method of reducing a nonminimal $\mathbf{P}$-$\boldsymbol{\Delta}$ model to a minimal form.

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May 10, 1991

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Report Documentation Page

1. Report No. NASA TP-3094	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle On the Formulation of a Minimal Uncertainty Model for Robust Control With Structured Uncertainty		5. Report Date September 1991
		6. Performing Organization Code
7. Author(s) Christine M. Belcastro, B.-C. Chang, and Robert Fischl		8. Performing Organization Report No. L-16893
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No. 505-66-01-02
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546-0001		13. Type of Report and Period Covered Technical Paper
		14. Sponsoring Agency Code

15. Supplementary Notes

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16. Abstract

In the design and analysis of robust control systems for uncertain plants, representing the system transfer matrix in an "M-Δ" model has become widely accepted and applied in the robust control literature. Conceptually, the M-Δ structure can always be formed for any linear interconnection of inputs, outputs, transfer functions, parameter variations, and perturbations. However, very little of the currently available literature addresses computational methods for obtaining this structure, and none of this literature (to the authors' knowledge) addresses a general methodology for obtaining an M-Δ model of **minimal** dimension for a wide class of uncertainty. Since having a minimally dimensioned Δ matrix would improve the efficiency of structured singular value (or multivariable stability margin) computations, a method of obtaining a minimal M-Δ model would be useful. This paper presents (without proof) a generalized procedure for obtaining the interconnection structure $P(s)$ required in forming a minimal M-Δ model for systems with real parameter variations. The procedure involves representing the system in a cascade-form state-space realization, determining the minimal uncertainty matrix Δ, and constructing the state-space representation of $P(s)$. Three examples are presented to illustrate the procedure.

17. Key Words (Suggested by Author(s)) Robust control Structured uncertainty Uncertainty model M-Δ model	18. Distribution Statement Unclassified—Unlimited Subject Category 08
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 32	22. Price A03