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On the Most General Tensor B_{ij} Which is Zero for $i \neq j$, and the Most General Isotropic Tensor I_{ij}

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ON THE MOST GENERAL TENSOR B_{ij} WHICH IS ZERO FOR $i \neq j$, AND THE MOST GENERAL ISOTROPIC TENSOR I_{ij}.

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ABSTRACT

We show that the most general second-order tensor B_{ij} which is zero for $i \neq j$ is proportional to the Kronecker delta δ_{ij}. By a slight modification of that argument we obtain the known result that the most general second-order *isotropic* tensor is also proportional to δ_{ij}. These results are useful for instance in obtaining the stress tensor for a viscous fluid (NASA TM–104386).

It is sometimes necessary, for example in the derivation of the equations of motion for a viscous fluid [1], to determine the most general second-order tensor B_{ij} that is zero for $i \neq j$ $(i, j = 1,2,3)$. (Herein the word tensor refers to Cartesian tensor.) One cannot know a priori that a particular second-order tensor with the property that it is zero for $i \neq j$, say $B\delta_{ij}$, where B is a scaler and δ_{ij} is the Kronecker delta, is necessarily the most general tensor with that property. For instance, if one is dealing with a fluid, he might imagine that such a tensor could depend on the motion of the fluid, as well as on pressure forces that are not accompanied by motion.

Thus we need to determine the form of B_{ij}, or of B^*_{ij} in a rotated coordinate system, that being the most general second-order tensor for which

$$B_{ij} = B^*_{ij} = 0 \quad \text{for} \quad i \neq j. \tag{1}$$

Note that we have not said anything about B_{ij} for $i = j$; that will be considered in what follows. We first encountered the problem of the determination of the form of B_{ij} when attempting to obtain the stress tensor for a viscous fluid by assuming only its form for a simple shear [1]. Since we have not seen a satisfactory determination of B_{ij}, we give such a determination here.

Since B_{ij} is a second-order tensor, its law for transformation to a rotated coordinate system is [2,3]

$$B^*_{ij} = a_{ik}a_{j\ell}B_{k\ell}, \tag{2}$$

where the summation convention is operative, a star designates a quantity in the rotated coordinate system x^*_i, and the a_{ij} form a set of nine constants which define a transformation from the unrotated coordinate system x_i to the system x^*_i. The necessary and sufficient condition that lengths remain invariant under the transformation is

$$a_{i\ell}a_{im} = a_{\ell i}a_{mi} = \delta_{\ell m} \tag{3}$$

where δ_{ij}, the Kronecker delta, is 1 for $i = j$ and 0 for $i \neq j$. Multiplying Eq. (2) by a_{jm}, and using Eq. (3) and the definition of δ_{ij}, give

$$a_{jm}B^*_{ij} = a_{ik}a_{j\ell}a_{jm}B_{k\ell} = a_{ik}\delta_{\ell m}B_{k\ell} = a_{ik}B_{km}.$$

Rewriting the first and last members of this equation,

$$a_{jm}B^*_{ij} = a_{ik}B_{km}. \tag{4}$$

The unrepeated (assignable) subscripts in Eq. (4) are m and i. Since Eq. (4) is true for any m, i = 1,2,3 one can set $m = i = 1$. Then we carry out the summations on the repeated subscripts j and k, and, with q and r as general subscripts, we let $B_{qr} = B^*_{qr} = 0$ for $q \neq r$ (Eq. (1)). Equation (4) then becomes

$$a_{11}(B^*_{11} - B_{11}) = 0,$$

or

$$B^*_{11} = B_{11}. \tag{5}$$

Similarly, by setting $m = 2, i = 1$, Eq. (4) becomes

$$B^*_{11} = B_{22}, \tag{6}$$

and by setting $m = 3, i = 1$,

$$B^*_{11} = B_{33}. \tag{7}$$

Then, from Eqs. (5) to (7),

$$B_{11} = B_{22} = B_{33}. \tag{8}$$

Thus, starting from Eq. (1), we have shown that $B_{11} = B_{22} = B_{33}$ if $B_{ij} = B^*_{ij} = 0$ for $i \neq j$. From Eqs. (1) and (8), and the definition of the Kronecker delta (given after Eq.(3)), we have, finally,

$$B_{ij} = B\delta_{ij}, \tag{9}$$

where B is a scalar. We have placed no restrictions on B_{ij} other than that it be a second-order tensor with the property that it and B^*_{ij} are zero for $i \neq j$. Therefore the expression for B_{ij} in Eq. (9) is the most general second-order tensor for which $B_{ij} = B^*_{ij} = 0$ for $i \neq j$. Note that B_{ij} turns out to be an *isotropic* tensor, although we have not explicitly made that assumption.

According to our analysis, the definition of the Kronecker delta δ_{ij} could be given by a weaker statement than the usual one. The usual statement that $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$ could be replaced by the weaker statement that $\delta_{ij} = 1$ for $i, j = 1$ and $\delta_{ij} = 0$ for $i \neq j$; the stronger statement is implied by the weaker one (see the sentence following Eq. (8)).

Consider now the most general second-order *isotropic* tensor I_{ij}. Although the form of I_{ij} is known from previous work, e.g., [2], it can be obtained more directly by a simple modification of Eq. (4). Equation (4) becomes, on replacing B_{ij} by I_{ij} and omitting the star, since I_{ij} is isotropic,

$$a_{jm}I_{ij} = a_{ik}I_{km}. \tag{10}$$

This equation can be written as

$$\delta_{km}a_{jk}I_{ij} = \delta_{ij}a_{jk}I_{km}$$

or

$$a_{jk}(\delta_{km}I_{ij} - \delta_{ij}I_{km}) = 0. \tag{11}$$

Since the relation for I_{ij} cannot depend on the a_{jk} (on the orientation of the coordinate axes), I_{ij} being isotropic, the quantity in parentheses in Eq. (11) is zero, and we get, after contracting the indices k and m,

$$I_{ij} = (I_{kk}/3)\delta_{ij} \tag{12}$$

where I_{kk} is a scalar. Any value of I_{kk} satisfies Eq. (12), as can be seen by contracting the indices i and j, so that

$$I_{ij} = I\delta_{ij}. \tag{13}$$

That is, any (the most general) second-order isotropic tensor can be written as $I\delta_{ij}$, where I is an arbitrary scalar. From Eqs. (9) and (13),

$$B_{ij} = (B/I)I_{ij}, \tag{14}$$

or the most general second-order tensor B_{ij} that is zero for $i \neq j$ is proportional to the most general second-order *isotropic* tensor.

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[3] Deissler, R.G.; Turbulent Fluid Motion II—Scalars, Vectors, and Tensors. NASA TM-103756 (1991).

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