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A Restricted Signature Normal Form for Hermitian Matrices, Quasi-Spectral Decompositions, and Applications

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ABSTRACT

In recent years, a number of results on the relationships between the inertias of Hermitian matrices and the inertias of their principal submatrices appeared in the literature. In this paper, we study restricted congruence transformations of Hermitian matrices M which, at the same time, induce a congruence transformation of a given principal submatrix A of M. Such transformations lead to the concept of the resctricted signature normal form of M. In particular, by means of this normal form, we obtain short proofs of most of the known inertia theorems and also derive some new results of this type. For some applications, a special class of "almost" unitary restricted congruence transformations turns

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out to be useful. We show that, with such transformations, M can be reduced to a quasi-diagonal form which, in particular, displays the eigenvalues of A. Finally, applications of this quasi-spectral decomposition to generalized inverses and Hermitian matrix pencils are discussed.

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1. INTRODUCTION

In recent years, there has been considerable interest [2,4,5,6,9,10,12,13,14,15] in studying connections between the inertias $\mathrm{in}(M)$ of Hermitian matrices M and the inertias of their principal submatrices. Here and in the sequel,

$$\mathrm{in}(M) := (\pi(M), \nu(M), \delta(M))$$

where $\pi(M)$, $\nu(M)$, and $\delta(M)$ denotes the number (counted according to their multiplicities) of positive, negative, and zero eigenvalues of M, respectively. A typical result of this type is the following

Theorem A (Dancis [6]). *Let M be a Hermitian $n \times n$ matrix and M_1 any $m \times m$ principal submatrix of M. Then, with $d := \dim\,(\ker(M) \cap \ker(M_1))$,*

$$\pi(M_1) + \delta(M_1) - d \leq \pi(M) \leq \pi(M_1) + n - m - \delta(M) + d. \tag{1.1}$$

It turns out that most of the results in [2,4,5,6,9,10,12,13,14,15] can be easily derived in a uniform manner by means of the *restricted signature normal form* for Hermitian matrices which was introduced in [7] in connection with extension problems for Toeplitz matrices (see also [8]).

Throughout this paper, let M be an $n \times n$ Hermitian matrix, $1 \leq m < n$, and A any $m \times m$ principal submatrix. It is always assumed that the rows and columns of M have been permuted such that A is a leading submatrix of M. Hence, M can be partitioned in the form

$$M = \begin{pmatrix} A & B \\ B^H & C \end{pmatrix}. \tag{1.2}$$

We call $T^H M T$ a *restricted congruence transformation* of M if T is a nonsingular matrix of the form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \quad \text{with } T_{11} \text{ an } m \times m \text{ matrix.} \tag{1.3}$$

Note that such a restricted congruence transformation induces the congruence transformation $T_{11}^H A T_{11}$ of A. Because of the zero block in T, in general it is not possible to reduce

M to a signature matrix by restricted congruence transformations. However, M can be transformed into a *restricted signature matrix* of the type

$$\Sigma = \left(\begin{array}{cccc|cccc} I_{\pi_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{\nu_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{d_1} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\pi_0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I_{\nu_0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0_{d_0} \end{array}\right). \tag{1.4}$$

Here and in the sequel, I_j resp. 0_j denotes the $j \times j$ identity resp. zero matrix. Moreover, the lines in (1.4) correspond to the partitioning (1.2) of M, i.e. the block to the left of the vertical line and above the horizontal line is $m \times m$.

With these notations, our result on the restricted signature normal form reads as follows.

Theorem B ([7, Lemma 1]). *Let M be a Hermitian matrix of the form (1.2). Then, there exists a restricted congruence transformation $T^H MT = \Sigma$ where Σ is a uniquely determined restricted signature matrix of the type (1.4). Moreover, the sizes of the blocks in (1.4) are determined by*

$$\begin{aligned} &\pi_1 = \pi(A), \quad \nu_1 = \nu(A), \quad k = \operatorname{rank}(A \quad B) - \operatorname{rank} A, \quad d_1 = \delta(A) - k, \\ &\pi_0 = \pi(M) - \pi(A) - k, \quad \nu_0 = \nu(M) - \nu(A) - k, \quad d_0 = \delta(M) - d_1. \end{aligned} \tag{1.5}$$

The restricted signature normal form is an efficient tool for obtaining results on the inertia of partitioned Hermitian matrices. For example, (1.1) (with M_1 replaced by A) is a consequence of the relations

$$\pi(M) = \pi_1 + k + \pi_0, \quad k = \delta(A) - d, \quad k + \pi_0 \leq n - m - \delta(M) + d,$$

which readily follow from (1.4).

The purpose of this paper is twofold. First, we investigate in Section 2 congruence transformations $T^H MT$ with matrices T of the form (1.3) whose diagonal blocks are in addition required to be unitary. It turns out that, under this restriction, M can still be transformed into a matrix with the same zero structure as (1.4). Since such matrices T are "almost" unitary, we refer to the resulting factorization as *quasi-spectral decomposition* of M. In particular, Theorem B is an immediate consequence of Theorem 2.1 on quasi-spectral decompositions.

Secondly, using the restricted signature normal form resp. quasi-spectral decomposition, we deduce some new results and also obtain short proofs of a number of known results. More precisely, in Section 3, connections with generalized inverses are pointed out. In Section 4, we are concerned with inertia theorems. Section 5 deals with applications to Hermitian matrix pencils. Finally, inequalities for inertias of M and its submatrices are collected in Section 6.

Throughout this paper, the following notations are used. $X^\dagger$ is the Moore-Penrose inverse (e.g. [1, p. 7]) of the matrix X. For partitioned matrices M of the type (1.2)

$$M/A := C - B^H A^\dagger B$$

is the generalized Schur complement of A in M (see [3]). Furthermore, the function $\delta(X)$ is extended to arbitrary matrices X by setting

$$\delta(X) := \dim(\ker X).$$

Finally, $X > 0$ resp. $X \geq 0$ indicates that a Hermitian matrix X is positive definite resp. positive semidefinite.

2. QUASI-SPECTRAL DECOMPOSITIONS OF HERMITIAN MATRICES

In this section, we investigate transformations $T^H M T$ of partitioned matrices (1.2) where T is of the form

$$T = \begin{pmatrix} U & X \\ 0 & V \end{pmatrix} \quad \text{with } U \text{ resp. } V \text{ unitary } m \times m \text{ resp. } (n-m) \times (n-m) \text{ matrices.} \tag{2.1}$$

The spectral theorem for Hermitian matrices states that there exists a unitary matrix T such that $T^H M T$ is diagonal. With the restricted class of transformations (2.1), it is possible to reduce M to the *quasi-diagonal matrix*

$$\Lambda = \left(\begin{array}{ccc|ccc} \Lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_k & 0 & 0 \\ 0 & 0 & 0_{d_1} & 0 & 0 & 0 \\ \hline 0 & D_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{d_0} \end{array}\right), \tag{2.2}$$

with Λ_1, Λ_0 nonsingular diagonal matrices and $D_k > 0$ a $k \times k$ diagonal matrix.

More precisely, we have the following

Theorem 2.1. *Let M be a Hermitian matrix of the form (1.2). Then, there exists a matrix T of type (2.1) such that*

$$T^H M T = \Lambda \quad \text{with } \Lambda \text{ a quasi-diagonal matrix (2.2).} \tag{2.3}$$

Moreover, the nonzero blocks Λ_1, D_k, Λ_0 of all quasi-diagonal matrices Λ and U, V, X of all transformations T of the form (2.1) which satisfy (2.3) are given by

$$AU_r = U_r \Lambda_1, \quad AU_s = 0, \quad U = (\,U_r \quad U_s\,), \tag{2.4}$$

$$U_s^H B = \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_r^H \\ V_s^H \end{pmatrix}, \quad V_s^H (M/A) V_s = \begin{pmatrix} \Lambda_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V = (\,V_r \quad V_s\,), \tag{2.5}$$

$$X = -A^{\dagger} B V + U_s \begin{pmatrix} D_k^{-1}(S - \frac{1}{2} G_r) & -D_k^{-1} G_s \\ Z_r & Z_s \end{pmatrix} \tag{2.6}$$

$$\text{with} \quad G_j := V_r^H (M/A) V_j, \quad j = r, s, \tag{2.7}$$

and arbitrary matrices Z_r, Z_s, and skew–Hermitian $S = -S^H$.

In particular, the diagonal entries of Λ_1 and Λ_0 are the nonzero eigenvalues of A and $V_s^H (M/A) V_s$, respectively. The diagonal elements of D_k are the positive singular values of $U_s^H B$.

Remark 2.2. Clearly, the quasi-diagonal matrix Λ in (2.3) is uniquely determined up to permutations of the diagonal entries of Λ_1, D_k, and Λ_0 respectively.

Remark 2.3. The zero structure of Λ in (2.3) is identical to that of the restricted signature normal form (1.4) of M. In particular, Theorem B is just a corollary to Theorem 2.1. Also, note that (cf. (1.5))

$$\pi(\Lambda_1) = \pi(A), \quad \nu(\Lambda_1) = \nu(A), \quad k = \operatorname{rank}(\,A \quad B\,) - \operatorname{rank} A.$$

Remark 2.4. Since the diagonal blocks U and V in (2.1) are unitary, we have

$$\det(T^H T) = 1 \quad \text{and} \quad \det(M) = \det(\Lambda) \tag{2.8}$$

for any quasi-spectral decomposition (2.3).

Proof of Theorem 2.1. Let T resp. Λ be an arbitrary matrix of the form (2.1) resp. (2.2). First, note that (2.3) is equivalent to $MT = T^{-H}\Lambda$ where

$$T^{-H} = \begin{pmatrix} U & 0 \\ -VX^H U & V \end{pmatrix}.$$

Therefore, T and Λ satisfy (2.3) iff the following four equations are fulfilled:

$$(i) \qquad AU = U \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(ii) \qquad B^H U = -VX^H U \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + V \begin{pmatrix} 0 & D_k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(iii) \qquad AX + BV = U \begin{pmatrix} 0 & 0 & 0 \\ D_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(iv) \qquad B^H X + CV = -VX^H U \begin{pmatrix} 0 & 0 & 0 \\ D_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + V \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, (i) is equivalent to (2.4). Next, consider (ii). Using the partition (2.4) of U and $V^{-1} = V^H$, (ii) can be rewritten in the form

$$U_s^H B = \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix} V^H \quad \text{and} \quad U_r^H X = -\Lambda_1^{-1} U_r^H BV \quad \left(= -U_r^H A^\dagger BV\right). \tag{2.9}$$

The first relation in (2.9) is the same as in (2.5). Note that, for the last identity in (2.9), we have used that, in view of (2.4), $A^\dagger = U_r \Lambda_1^{-1} U_r^H$. Since $U = (\,U_r \quad U_s\,)$ is unitary, the second part of (2.9) implies that X is of the form

$$X = -A^\dagger BV + U_s Z \tag{2.10}$$

where Z is still arbitrary. It remains to fulfill (iii) and (iv). By means of (2.4) and (2.10), one easily verifies that (iii) is equivalent to

$$(I - U_r U_r^H) BV = U_s \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.11}$$

However, since $I - U_r U_r^H = U_s U_s^H$, (2.11) just leads to the first identity in (2.5). Finally, we turn to condition (iv). Substituting the ansatz (2.10) for X into (iv) and using $V^H V = I$, $A^\dagger U_s = 0$, and the first relation in (2.5), one obtains

$$\begin{pmatrix} V_r^H \\ V_s^H \end{pmatrix} \left(M/A\right) (\,V_r \quad V_s\,) = -\begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix} Z - Z^H \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0_k & 0 & 0 \\ 0 & \Lambda_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.12}$$

Next, we partition Z conformally with the matrices on the right-hand side of (2.12):

$$Z = \begin{pmatrix} Y_r & Y_s \\ Z_r & Z_s \end{pmatrix} \quad \text{with } Y_r \text{ a } k \times k \text{ matrix.} \tag{2.13}$$

A straightforward calculation then shows that (2.12) (and hence (iv)) is satisfied iff the second identity in (2.5) holds and

$$Y_r = D_k^{-1}(S - \frac{1}{2}G_r) \text{ with } S = -S^H, \quad Y_s = -D_k^{-1}G_s. \tag{2.14}$$

Here G_r and G_s are the matrices defined in (2.7). Note that the blocks Z_r and Z_s in (2.13) are arbitrary. By (2.10), (2.13), and (2.14), X is indeed of the form (2.6), and this concludes the proof. □

3. CONNECTIONS WITH GENERALIZED INVERSES

The quasi-spectral decomposition (2.3) naturally gives rise to a generalized inverse of M. Let T and Λ be matrices of the form (2.1) and (2.2), respectively, such that (2.3) holds. Then, we define

$$M^\sharp := T\Lambda^\dagger T^H. \tag{3.1}$$

Next, let $\Lambda_\star$ be the matrix which is obtained by deleting the $d_1 + d_0$ zero columns and rows in (2.2). Remark that $\Lambda_\star$ is nonsingular and

$$\Lambda_\star^{-1} = \left(\begin{array}{cc|cc} \Lambda_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & D_k^{-1} & 0 \\ \hline 0 & D_k^{-1} & 0 & 0 \\ 0 & 0 & 0 & \Lambda_0^{-1} \end{array}\right). \tag{3.2}$$

Similarly, we denote by $T_\star$ resp. $S_\star$ the matrix which is obtained by deleting the columns with the numbers $m - d_1 + 1, \ldots, m$ and $n - d_0 + 1, \ldots, n$ in T resp. T^{-H}. With these notations, (3.1) can be rewritten in the form

$$M^\sharp = T_\star \Lambda_\star^{-1} T_\star^H. \tag{3.3}$$

Analogously, (2.3) can be stated as follows:

$$M = S_\star \Lambda_\star S_\star^H. \tag{3.4}$$

Since $S_\star$ has full column rank (3.4) immediately (see [1, p. 24]) leads to the representation

$$M^\dagger = (S_\star^\dagger)^H \Lambda_\star^{-1} S_\star^\dagger, \quad \text{where} \quad S_\star^\dagger = (S_\star^H S_\star)^{-1} S_\star^H, \tag{3.5}$$

of the classical Moore-Penrose inverse of M.

In the following theorem, we collect some properties of $M^\sharp$.

Theorem 3.1. *Let M be a Hermitian matrix of the form (1.2) and $M^\sharp$ be defined by (3.1). Then:*

(i) $M^\sharp$ is an 1,2-inverse of M, i.e. $M^\sharp M M^\sharp = M^\sharp$ and $M M^\sharp M = M$. (see [1, p. 8]).

(ii) $M^\sharp$ is the weighted inverse $M^{(1,2)}_{(W,U)}$ of M for $W = TT^H$ and $U = W^{-1}$, i.e. $(WMM^\sharp)^H = WMM^\sharp$ and $(UM^\sharp M)^H = UM^\sharp M$ (see [1, p. 123]).

(iii) Let T in (2.3) be chosen such that $Z_r = Z_s = 0$ in (2.6). Then, $M^\sharp = M^\dagger$ if, and only if,

$$\begin{pmatrix} A^\dagger B \\ C \end{pmatrix} V_s \begin{pmatrix} 0 \\ I_{d_0} \end{pmatrix} = 0 \tag{3.6}$$

with d_0 defined in (2.2).

Proof. With (2.3) and (3.1), one readily verifies (i) and (ii). We now turn to part (iii). In view of (i) and the usual [1, p. 7] definition, $M^\sharp$ and $M^\dagger$ are identical if, and only if, $MM^\sharp$ and $M^\sharp M$ are both Hermitian. From our definition of $T_\star$ and $S_\star$, it is obvious that $T_\star^H S_\star = I$. Thus, with (3.3) and (3.4), we obtain

$$M^\sharp M = T_\star S_\star^H \quad \text{and} \quad MM^\sharp = S_\star T_\star^H = \left(T_\star S_\star^H\right)^H.$$

Therefore, it remains to show that the condition (3.6) is equivalent to $T_\star S_\star^H$ being Hermitian. For this purpose, denote by

$$\tilde{U} := U \begin{pmatrix} 0 \\ I_{d_1} \end{pmatrix} = U_s \begin{pmatrix} 0 \\ I_{d_1} \end{pmatrix}, \quad \tilde{X} := X \begin{pmatrix} 0 \\ I_{d_0} \end{pmatrix}, \quad \tilde{V} := V \begin{pmatrix} 0 \\ I_{d_0} \end{pmatrix} = V_s \begin{pmatrix} 0 \\ I_{d_0} \end{pmatrix} \tag{3.7}$$

the parts of U, X, and V whose deletion from T just yields $T_\star$. Note that U_s and V_s are the matrices defined in the partitions of U and V in (2.4) and (2.5), respectively. With (3.7), one readily verifies that

$$I = TT^{-1} = T_\star S_\star^H + \begin{pmatrix} \tilde{U}\tilde{U}^H & -\tilde{U}\tilde{U}^H X V^H + \tilde{X}\tilde{V}^H \\ 0 & \tilde{V}\tilde{V}^H \end{pmatrix}. \tag{3.8}$$

Moreover, with (2.4), (2.6), (3.7), and our assumption $Z_r = Z_s = 0$, it follows that

$$\tilde{U}^H X = (\, Z_r \quad Z_s \,) = 0. \tag{3.9}$$

Using (3.8), (3.9), and the fact that $\tilde{V}$ fas full column rank, we conclude that $T_\star S_\star^H$ is Hermitian iff $\tilde{X} = 0$. However, by (3.7), (2.6), and (2.7),

$$\tilde{X} = -A^\dagger B \tilde{V} - U_s \begin{pmatrix} D_k^{-1} \\ 0 \end{pmatrix} V_r (M/A) \tilde{V}. \tag{3.10}$$

Since, by (2.4), the columns of U_s are orthogonal to range $A^\dagger$, (3.10) implies that $\bar{X} = 0$ is equivalent to

$$A^\dagger B\bar{V} = 0, \quad V_r(M/A)\bar{V} = 0. \tag{3.11}$$

Finally, note that, by (2.5), $V_s(M/A)\bar{V} = 0$, and therefore the second condition in (3.11) can be rewritten as $(M/A)\bar{V} = 0$. Thus (3.11) is equivalent to (3.6), and this concludes the proof. □

Remark 3.2. In general, (3.6) is not fulfilled and hence $M^\dagger$ and $M^\sharp$ are different. For example, consider the family of 3×3 matrices

$$M_\alpha = \left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 2\alpha & 0 \\ 1 & 0 & -2\alpha \end{array}\right), \quad \alpha \in \mathbb{R}.$$

All quasi-spectral decompositions $T_\alpha^H M_\alpha T_\alpha = \Lambda$ are given by

$$\Lambda = \left(\begin{array}{c|cc} 0 & \sqrt{2} & 0 \\ \hline \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad T_\alpha = \frac{1}{\sqrt{2}}\left(\begin{array}{c|cc} \sqrt{2} & i\sigma & 2\alpha \\ \hline 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right) \quad \sigma \in \mathbb{R} \text{ arbitrary}.$$

Note that Λ does not depend on the parameter α. M_α has the generalized inverses

$$M_\alpha^\sharp = \frac{1}{2}\left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right) \quad \text{and} \quad M_\alpha^\dagger = \frac{1}{2(2\alpha^2+1)}\left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 2\alpha & 0 \\ 1 & 0 & -2\alpha \end{array}\right)$$

which coincide only if $\alpha = 0$.

By means of $M^\sharp$, one can obtain a generalization of a signature formula due to Lazutkin [13]. First, we partition $M^\sharp$ conformally with M:

$$M^\sharp = \begin{pmatrix} P & Q \\ Q^H & R \end{pmatrix} \quad \text{with } P \text{ an } m \times m \text{ matrix.} \tag{3.12}$$

Moreover, denote by $\operatorname{sgn}(X) := \pi(X) - \nu(X)$ the signature of the Hermitian matrix X. Then, we have the following

Theorem 3.3. *Let M be a Hermitian matrix (1.2) and R be defined by (3.12). Then,*

$$\operatorname{sgn} M = \operatorname{sgn} A + \operatorname{sgn} R.$$

Proof. First, we remark that, as an immediate consequence of (2.1–2.3),

$$\operatorname{sgn} M = \operatorname{sgn} \Lambda_1 + \operatorname{sgn} \Lambda_0 \quad \text{and} \operatorname{sgn} A = \operatorname{sgn} \Lambda_1. \tag{3.13}$$

Furthermore, with (2.1), (2.2), (3.1), and (3.12), it follows that

$$R = V \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda_0^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} V^H.$$

This shows that $\operatorname{sgn} A = \operatorname{sgn} \Lambda_0$, and, in view of (3.13), the proof is complete. □

Remark 3.4. For the case of nonsingular M, $M^\sharp$ is the usual inverse of M. In particular, for nonsingular real symmetric matrices M, Theorem 3.3 reduces to the recent result [13] of Lazutkin.

4. INERTIA THEOREMS

Numerous authors investigated the connections between the inertias of Hermitian matrices and the inertias of their principal submatrices (see e.g. [9–12,14]). The most general results of this type are due to Maddocks [14]. In this section, we present a different approach, based on the resricted signature normal form, to the main results in [14]. In particular, this will lead to shorter and more elementary proofs.

In [14], Maddocks considered only real symmetric matrices. Here, we will deal with general complex Hermitian $n \times n$ matrices M. Moreover, let F be any $n \times p$ matrix and set $Y = \text{range}(F)$. Generalizing the corresponding notion [14, Lemma 2.2, Corollary 2.3, and Definition 2.2] for real matrices, we introduce

$$\text{in}^*(Y; M) := \text{in}^*(F^H M F) := \text{in}(F^H M F) - (0, 0, \delta(F)). \tag{4.1}$$

Next, we reformulate and prove the main results in [14] for the general complex case.

Theorem C (Maddocks [14, Corollary 4.1]). *Let $M = M^H$ be $n \times n$, F $n \times p$, and G any $n \times q$ matrix whose range is $\ker(F^H M)$. Then:*

$$\text{in}(M) = \text{in}^*(F^H M F) + \text{in}^*(G^H M G) + (d, d, -d - f), \tag{4.2}$$

where

$$d := \dim\left(\text{range}(MF) \cap \ker(F^H)\right) \quad \text{and} \quad f := \dim\left(\text{range}(F) \cap \ker(F^H M)\right).$$

Proof. A straightforward computation, using the singular value decomposition of F, shows that it suffices to consider the case $F = (I_m \quad 0)^T$. Furthermore, let M be partitioned as in (1.2) with leading $m \times m$ principal submatrix A. Note that $m = \operatorname{rank}(F)$, $A = F^H M F$, and $F^H M = (A \quad B)$. Next, we apply Theorem B and reduce M to the restricted signature normal form $T^H M T = \Sigma$. By grouping the columns of T which correspond to the zero columns in (1.4), one obtains the partition

$$T = (T_1 \quad T_2 \quad | \quad T_3 \quad T_4) \quad \text{with } T_1 \; n \times \operatorname{rank}(A),\; T_2 \; n \times \delta(A),\; T_3 \; n \times k.$$

Since

$$\ker(F^H M) = \ker(A \quad B) = \operatorname{range}(T_2 \quad | \quad T_4),$$

we can choose $G = (T_2 \quad T_4)$. It follows that

$$G^H M G = \left(\begin{array}{cc|cc} 0_k & & & \\ & 0_{d_1} & & \\ \hline & & \Lambda_0 & \\ & & & 0_{d_0} \end{array}\right). \tag{4.3}$$

Furthermore, we obtain

$$f = \dim\left(\operatorname{range}\begin{pmatrix} I_m \\ 0 \end{pmatrix} \cap \ker(A \quad B)\right) = \dim\left(\operatorname{range}(T_2)\right) = \delta(A), \tag{4.4}$$

and

$$\begin{aligned} d = \dim\left(\operatorname{range}\begin{pmatrix} A \\ B^H \end{pmatrix} \cap \ker(I_m \quad 0)\right) &= \dim\left(\operatorname{range}(\begin{pmatrix} A \\ B^H \end{pmatrix} U_s)\right) \\ &= \dim\left(\operatorname{range}(V \begin{pmatrix} D_k & 0 \\ 0 & 0 \end{pmatrix})\right) = k, \end{aligned} \tag{4.5}$$

with U_s, V, and D_k as defined in Theorem 2.1. Finally, combining (4.3), (4.4), and (4.5) yields (4.2). □

Following [14], we set, for any subspace $Y \subset \mathbf{C}^n$,

$$Y^M := (MY)^\perp \quad \text{and} \quad d^0(Y) := \dim(M(Y \cap Y^M)).$$

Here, $\perp$ indicates the orthogonal complement in $\mathbf{C}^n$. Using these notations, Theorem C can be rewritten as follows.

Corollary D ([14, Corollary 2.7]). *Let M be a Hermitian $n \times n$ matrix and Y a subspace of C^n. Then:*

$$\mathrm{in}(M) = \mathrm{in}^*(Y; M) + \mathrm{in}^*(Y^M; M) + \left(d^0(Y), d^0(Y), -d^0(Y) - \dim(Y \cap Y^M)\right).$$

We remark that the results of Han and Fujiwara ([10, Theorem 2.3] and [9, Theorem 4.1]), and Jongen et al. [12, Theorem 2.1] are only special cases of Corollary D.

As in [14], the formula

$$\ker(F^H M) = M^\dagger[\ker(F^H) \cap \mathrm{range}(M)] \oplus \ker(M)$$

can be used to derive from Theorem C the following result.

Corollary E (cf. [14, Corollary 4.3 and Theorem 3.1]). *Let M and F be as in Theorem C, and let E be any $n \times q$ matrix with* $\mathrm{range}(E) = \ker(F^H) \cap \mathrm{range}(M)$. *Then:*

$$\mathrm{in}(M) = \mathrm{in}^*(F^H M F) + \mathrm{in}^*(E^H M^\dagger E) + (d, d, e - 2d),$$

where

$$d := \dim\left(\mathrm{range}(MF) \cap \ker(F^H)\right) \quad \text{and} \quad e := \delta(M) - \dim\left(\ker(M) \cap \mathrm{range}(F)\right).$$

Remark 4.1. For special cases, the result of Corollary E was also derived by Han [10, Theorem 4.3] and Lazutkin [13] (e.g. Theorem 3.3 and Remark 3.4).

We conclude this section with a result on the relationship of the inertias of M, its submatrix A, and the generalized Schur complement M/A of A in M.

Theorem 4.2. *Let M be a Hermitian matrix of the form (1.2). Then:*

$$\mathrm{in}(M) = \mathrm{in}(A) + \mathrm{in}^*\left(\ker(U_s^H B); M/A\right) + (k, k, -k)$$

with

$$\mathrm{range}(U_s) = \ker(A) \quad \textit{and} \quad k = \mathrm{rank}\begin{pmatrix} A & B \end{pmatrix} - \mathrm{rank}(A).$$

Proof. This result is an immediate consequence of Theorem 2.1. Also, recall Remark 2.3 for the definition of k. □

Remark 4.3. For the special case that the submatrix A in (1.2) is nonsingular Theorem 4.2 reduces to

$$\mathrm{in}(M) = \mathrm{in}(A) + \mathrm{in}(C - B^H A^{-1} B). \tag{4.6}$$

This result is due to Haynsworth [11]. It seems that (4.6) is one of the earliest inertia formulas for partitioned Hermitian matrices.

5. APPLICATIONS TO HERMITIAN MATRIX PENCILS

In this section, we are concerned with Hermitian matrix pencils (see e.g. [16, Chapter 15])

$$\mu M - \lambda N, \quad \mu, \lambda \in \mathbb{R}, \quad (\mu, \lambda) \neq (0,0), \tag{5.1}$$

where $M = M^H$ and $N = N^H \geq 0$ are $n \times n$ matrices. Moreover, without loss of generality, it is always assumed that N is of the form

$$N = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m} \end{pmatrix}, \quad m = \delta(N). \tag{5.2}$$

Let $T^H M T = \Lambda$ of M be a quasi-spectral decomposition (2.3) of M with matrices T and Λ of the type (2.1) and (2.2), respectively. Then, by (2.1) and (5.2), we have $T^H N T = N$. Together with (2.3) it follows that

$$T^H(\mu M - \lambda N)T = \left(\begin{array}{ccc|ccc} \mu\Lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu D_k & 0 & 0 \\ 0 & 0 & 0_{d_1} & 0 & 0 & 0 \\ \hline 0 & \mu D_k & 0 & -\lambda I_k & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu\Lambda_0 - \lambda I_l & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda I_{d_0} \end{array}\right) \tag{5.3}$$

Next, we show that the essential properties of pencil (5.1) can be deduced from its normal form (5.3). First, recall that a matrix pencil (5.1) is said to be singular, if $\det(\mu M - \lambda N) = 0$ for all $\mu, \lambda \in \mathbb{R}$, and it is called regular otherwise. From (5.3), we immediately obtain

Theorem 5.1. *The matrix pencil (5.1) is regular if, and only if, $d_1 = 0$ in (5.3).*

In the following it is always assumed that (5.1) is a regular matrix pencil. Then, by (5.3) and since $\det(T^H T) = 1$ (cf. (2.8)), we get

$$\begin{aligned} \det(\mu M - \lambda N) &\equiv \det(\mu\Lambda_1)\ \det\begin{pmatrix} 0 & \mu D_k \\ \mu D_k & -\lambda I \end{pmatrix}\ \det(\mu\Lambda_0 - \lambda I)\ \det(-\lambda I_{d_0}) \\ &\equiv \det(\mu\Lambda_1)\ (-\lambda)^{d_0} \prod_{j=1}^{l}(\mu\lambda_j - \lambda) \prod_{j=1}^{k}(-\mu^2\sigma_j^2), \end{aligned} \tag{5.4}$$

where $\Lambda_0 = \operatorname{diag}(\lambda_1, \ldots, \lambda_l)$ and $D_k = \operatorname{diag}(\sigma_1, \ldots, \sigma_k)$. The next theorem readily follows from (5.4).

Theorem 5.2. *All solutions $(\mu, \lambda) \neq 0$ of $\det(\mu M - \lambda N) = 0$ are given by:*

(i) $\lambda = 0$, μ *arbitrary, if* $d_0 > 0$;

(ii) $\lambda = \mu\lambda_j$, μ *arbitrary, for all* $j = 1, \ldots, l$.

(iii) $\mu = 0$, λ *arbitrary, if* $m + k > 0$.

The solutions of $\det(\mu M - \lambda N) = 0$ with $\mu = 1$ resp. $\mu = 0, \lambda \neq 0$ are just the eigenvalues of the generalized eigenvalue problem

$$Mx = \lambda Nx. \tag{5.5}$$

For this special case, Theorem 5.2 together with (5.4) leads to the following

Corollary 5.3. *The eigenvalues λ of (5.5) are given by:*

(i) $\lambda = 0$ *with multiplicity* d_0, *if* $d_0 > 0$;

(ii) $\lambda = \lambda_j$, $j = 1, \ldots, l$.

(iii) $\lambda = \infty$ *with multiplicity* $m + k$, *if* $m + k > 0$.

As a further application, by means of (5.3), one can easily characterize all cases for which $\mu M - \lambda N > 0$.

Theorem 5.4. *Let $\mu, \lambda \in \mathbb{R}$ and $(\mu, \lambda) \neq 0$. Then, the matrix (5.1) $\mu M - \lambda N$ is positive definite if, and only if, the following four conditions are satisfied:*

(i) $d_1 = k = 0$;

(ii) $m = 0$ *or* $\mu\Lambda_1 > 0$;

(iii) $l = 0$ *or* $\mu\Lambda_0 - \lambda I_l > 0$;

(iv) $d_0 = 0$ *or* $\lambda < 0$.

In particular, we obtain the following

Corollary 5.5. *There exist $\mu, \lambda \in \mathbb{R}$ such that the matrix (5.1) $\mu M - \lambda N$ is positive definite if, and only if, the submatrix A in the partition (1.2) of M is positive or negative definite.*

Finally, we conclude this section with an inertia formula which again immediately follows from (5.3).

Theorem 5.6. *For the matrix pencil (5.1) with $\mu, \lambda \in \mathbb{R}$, $\mu \neq 0$, it holds*

$$\begin{aligned} \mathrm{in}(\mu M - \lambda N) &= \mathrm{in}(\mu\Lambda_1) + \mathrm{in}(\mu\Lambda_0 - \lambda I) + \mathrm{in}(-\lambda I_{d_0}) + (k, k, \delta_1 - k) \\ &= \mathrm{in}(\Lambda_1) + (k, k, \delta_1 - k) + \mathrm{in}^*(\ker(U_s^H B); M/A + \lambda I). \end{aligned}$$

Here, U_s is defined in (2.4), (2.5) and the notation in* *in (4.1).*

Remark 5.7. Different inertia formulas for matrix pencils (5.1) can be found in [12, Section 4].

6. POSSIBLE INERTIAS FOR A HERMITIAN MATRIX AND ITS PRINCIPAL SUBMATRICES

In this section, we are concerned with the following problem: For given Hermitian $n_i \times n_i$ matrices, $i = 1, 2$, characterize the possible inertias of Hermitian $n \times n$ matrices

$$M = \begin{pmatrix} M_1 & B \\ B^H & M_2 \end{pmatrix} \tag{6.1}$$

in terms of the inertias of M_1 and M_2.

Along these lines, a main result is the following theorem of Dancis [6, Theorem 1.2].

Theorem F. *Let M be a Hermitian matrix of the form (6.1). Set $\delta_1 := \delta(M_1)$, $d :=$ $\dim(\ker M \cap \ker M_1)$, $\Delta := \delta_1 - d$, and $\Delta^* := \delta(M) - d$. Then:*

$$n_2 + \pi_1 - \Delta^* \geq \pi \geq \pi_1 + \Delta, \tag{6.2}$$

$$\pi_1 + \delta_1 + n_2 - \Delta \geq \pi + \delta(M) \geq \pi_1 + \delta_1 + \Delta^*, \tag{6.3}$$

$$\delta_1 + n_2 - 2\Delta \geq \delta(M) \geq \delta_1 - n_2 + 2\Delta^*. \tag{6.4}$$

Proof. Using Theorem B on the restricted signature normal form of M, we obtain

$$k = \delta_1 - d = \Delta \text{ and } \Delta^* = \delta(M) - d = k + \delta(M) - \delta_1.$$

Note that (6.2) has already been proven in Section 1, Theorem A. To show (6.3), we remark that in view of (1.4)

$$n_2 + \pi_1 + (\delta_1 - k) \geq \pi + \delta(M) \geq \pi_1 + k + \pi_0 + \delta(M) \geq \pi_1 + k + \delta(M).$$

Finally, since $n_2 \geq \Delta + \Delta^* = k + d_0$, we have

$$(\delta_1 - k) + (n_2 - k) \geq \delta(M) = \Delta^* + (\delta_1 - k) \geq 2\Delta^* + \delta_1 - n_2$$

and this implies (6.4). □

Remark 6.1. In addition, for $\pi_0 = \nu_0 = 0$ or, equivalently, $\delta(M) = n_2 + 2d - \delta_1$, by (1.4) we get $\pi = \pi_1 + k$.

Next, let us consider some inertia properties of $X^H M X$.

Theorem G (Dancis [4, Theorem 3.1] and de Sá [15, Theorem 1]). *Let M be an $n \times n$ Hermitian matrix with $\mathrm{in}(M) = (\pi, \nu, \delta)$ and m, s integers such that $1 \leq m \leq \min(s, n)$. Then, there exists an $n \times s$ matrix X of rank m with $\mathrm{in}(X^H M X) = (\pi_1, \nu_1, \delta_1)$ if, and only if, the following inequalities are satisfied:*

$$\pi + m - n \leq \pi_1 \leq \pi\,, \quad \nu + m - n \leq \nu_1 \leq \nu\,, \quad \pi_1 + \nu_1 \leq m. \tag{6.5}$$

Proof. In view of the singular value decomposition of X, we can assume that

$$X = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $X^H M X$ is the $m \times m$ leading principal submatrix of M and Theorem A shows that

$$\pi + m - n \leq \pi + m - n + (\delta - d) \leq \pi_1 \leq \pi - (\delta_1 - d) \leq \pi.$$

On the other hand, for given π_1 and ν_1 satisfying (6.5), we can define a solution X via the quasi-spectral decomposition. Set Λ a quasi-diagonal matrix (2.2) with $\mathrm{in}(\Lambda) = \mathrm{in}(M)$ and $\mathrm{in}(\Lambda_1) = (\pi_1, \nu_1, \delta_1)$. Because Λ and M are both Hermitian matrices with the same inertia, there exists a nonsingular matrix S with $S^H M S = \Lambda$. Then for

$$X = S \begin{pmatrix} I_m & 0 \\ 0 & 0_{(n-m)\times(s-m)} \end{pmatrix}$$

we get $\mathrm{rank}(X) = m$ and $\mathrm{in}(X^H M X) = (\pi_1, \nu_1, \delta_1)$. □

Theorem H (de Sá [15, Theorem 5]). *Let M_i be $n_i \times n_i$ Hermitian matrices with inertias (π_i, ν_i, δ_i), $i = 1, 2$. Let π and ν be nonnegative integers. Then there exists an $n_1 \times n_2$ matrix X such that*

$$\mathrm{in}(M_1 + X M_2 X^H) = (\pi, \nu, n_1 - \pi - \nu)$$

if, and only if, the following inequalities hold:

$$\pi_1 - \nu_2 \leq \pi \leq \pi_1 + \pi_2, \tag{6.6}$$

$$\nu_1 - \pi_2 \leq \nu \leq \nu_1 + \nu_2, \tag{6.7}$$

$$\pi + \nu \leq n_1. \tag{6.8}$$

Proof. Obviously, we can assume that M_1 and M_2 are diagonal matrices with diagonal elements ± 1 and 0, and that, in addition, M_2 is nonsingular. Thus, we have $M_1 = \operatorname{diag}(l_1, \ldots, l_{n_1})$ and $M_2 = \operatorname{diag}(r_1, \ldots, r_{n_2})$ with

$$l_i = \begin{cases} 1 & \text{for } i = 1, \ldots, \pi_1 \\ -1 & \text{for } i = \pi_1 + 1, \ldots, \pi_1 + \nu_1 \\ 0 & \text{for } i = \pi_1 + \nu_1 + 1, \ldots, n_1 \end{cases} \quad \text{and} \quad r_i = \begin{cases} 1 & \text{for } i = 1, \ldots, \pi_2 \\ -1 & \text{for } i = \pi_2 + 1, \ldots, \pi_2 + \nu_2 \end{cases}.$$

First, let π and ν be given nonnegative integers which satisfy (6.6)–(6.8). It suffices to consider matrices $X := \sigma \operatorname{diag}(x_1, \ldots, x_{n_2})$ with real x_i and σ a permutation matrix. Applying σ on diagonal matrices induces a permutation of the diagonal elements. Then $M_1 + XM_2X^H = \operatorname{diag}(d_1, \ldots, d_{n_1})$ is also a diagonal matrix with

$$d_i = \begin{cases} 1 + x_i^2 r_{\sigma(i)} & \text{for } i = 1, 2, \ldots, \pi_1, \\ -1 + x_i^2 r_{\sigma(i)} & \text{for } i = \pi_1 + 1, \ldots, \pi_1 + \nu_1, \\ x_i^2 r_{\sigma(i)} & \text{for } i = \pi_1 + \nu_1 + 1, \ldots, n_1. \end{cases}$$

Setting $x_i := 0$ for $i = 1, 2, \ldots, \min(\pi_1, \pi)$ and $i = \pi_1 + 1, \ldots, \pi_1 + \min(\nu_1, \nu)$, yields $\min(\pi_1, \pi)$ positive and $\min(\nu_1, \nu)$ negative d_i.
If $\pi_1 \le \pi \le \pi_1 + \pi_2$ and $\nu_1 \le \nu \le \nu_1 + \nu$, we set:

$$x_i := \begin{cases} 1 & \text{for } i = \pi_1 + \nu_1 + 1, \ldots, \pi_1 + \nu_1 + (\pi - \pi_1) + (\nu - \nu_1), \\ 0 & \text{otherwise}, \end{cases}$$
$$r_{\sigma(i)} := \begin{cases} 1 & \text{for } i = \pi_1 + \nu_1 + 1, \ldots, \pi_1 + \nu_1 + (\pi - \pi_1), \\ -1 & \text{for } i = \pi + \nu_1 + 1, \ldots, \pi + \nu_1 + (\nu - \nu_1). \end{cases}$$

If $\pi_1 \le \pi \le \pi_1 + \pi_2$ and $\nu_1 - \pi_2 \le \nu < \nu_1$, we set:

$$x_i := \begin{cases} \sqrt{2} & \text{for } i = \pi_1 + \nu + 1, \ldots, \pi_1 + \nu_1, \\ 1 & \text{for } i = \pi_1 + \nu_1 + 1, \ldots, \pi + \nu, \\ 0 & \text{otherwise}, \end{cases}$$
$$r_{\sigma(i)} := 1 \text{ for } i = \pi_1 + \nu + 1, \ldots, \pi_1 + \nu + (\pi - \pi_1).$$

If $\pi_1 - \nu_1 \le \pi < \pi_1$ and $\nu_1 - \pi_2 \le \nu < \nu_1$, we set:

$$x_i := \begin{cases} 1 & \text{for } i = \pi + 1, \ldots, \pi_1 \text{ and } i = \pi_1 + \nu + 1, \ldots, \pi_1 + \nu_1, \\ 0 & \text{otherwise}, \end{cases}$$
$$r_{\sigma(i)} := \begin{cases} -1 & \text{for } i = \pi + 1, \ldots, \pi_1, \\ 1 & \text{for } i = \pi_1 + \nu + 1, \ldots, \pi_1 + \nu_1. \end{cases}$$

For the remaining case $\pi_1 - \nu_2 \le \pi < \pi_1$ and $\nu_1 \le \nu \le \nu_1 + \nu_2$ we can define X and σ analogously. Clearly, the above defined matrices have the prescribed inertia.

Secondly, let us assume that M_i, $i = 1,2$, and X are given matrices, and that M_2 is nonsingular. Then, Theorem F applied to the first matrix of the equation

$$\mathrm{in}\begin{pmatrix} -M_2^{-1} & X^H \\ X & M_1 \end{pmatrix} = \mathrm{in}\begin{pmatrix} -M_2^{-1} & 0 \\ 0 & M_1 + XM_2X^H \end{pmatrix}$$

yields (6.6) and (6.7). □

As a consequence of Theorem H and Theorem B on the restricted signature normal form, we obtain the following

Theorem I (Cain and de Sà [2]). *Let*

$$M = \begin{pmatrix} M_1 & B \\ B^H & M_2 \end{pmatrix}$$

be a Hermitian matrix with $\mathrm{in}(M) = (\pi, \nu, n - \pi - \nu)$ *and* $\mathrm{in}(M_i) = (\pi_i, \nu_i, n_i - \pi_i - \nu_i)$, $i = 1,2$. *Then the following inequalities are satisfied:*

$$\begin{aligned} &\max(\pi_1, \pi_2) \le \pi \le \min(n_1 + \pi_2, n_2 + \pi_1) \\ \text{and} \quad &\max(\nu_1, \nu_2) \le \nu \le \min(n_1 + \nu_2, n_2 + \nu_1), \end{aligned} \tag{6.9}$$

$$\pi - \nu \le \pi_1 + \pi_2 \quad \text{and} \quad \nu - \pi \le \nu_1 + \nu_2, \tag{6.10}$$

$$\pi + \nu \le n_1 + n_2.$$

Proof. By the restricted signature normal form (1.4) of M, the inequalities (6.9) are obvious. Next, note that, with the notation of Theorem 2.1, $\mathrm{diag}(\Lambda_0, 0) = V_s^H(M_2 - B^H M_1^\dagger B)V_s$. Then, Theorem G and H show $\pi(\Lambda_0) \le \pi_2 + \nu_1$. Thus we get

$$\pi - \nu = \pi_1 + \pi(\Lambda_0) - \nu_1 - \nu(\Lambda_0) \le \pi_1 - \nu_1 + \pi(\Lambda_0) \le \pi_1 - \nu_1 + \pi_2 + \nu_1 \le \pi_1 + \pi_2.$$

□

Finally, we will prove the most general result on the connection between the inertia of a partitioned matrix M of the form (6.1) and its submatrix M_1.

Theorem J (Dancis [5, Theorem 1.3 for m=2]). *Let M_1 and M_2 be Hermitian matrices with inertia* $(\pi_i, \nu_i, n_i - \pi_i - \nu_i)$, *and* $V_i \subset \ker(M_i)$, $d_i = \dim(V_i)$, *for* $i = 1,2$. *Then, M_1 and M_2 can be extended to a Hermitian matrix*

$$M = \begin{pmatrix} M_1 & B \\ B^H & M_2 \end{pmatrix}$$

with inertia $(\pi, \nu, n-\pi-\nu)$ *such that* $\ker(B^H) \cap \ker(M_1) = V_1$ *and* $\ker(B) \cap \ker(M_2) = V_2$, *if, and only if,*

$$\begin{aligned} &\delta \geq d_1 + d_2, \quad \pi \geq \max_{i=1,2}(\pi_1 + \delta_i - d_i), \\ &\nu \geq \max_{i=1,2}(\nu_i + \delta_i - d_i) \quad \text{and} \quad \pi + \nu + \delta = n_1 + n_2. \end{aligned} \tag{6.11}$$

Proof. The 'only if' part has already been established in Theorem A. Now let us turn to the proof of the 'if' part of Theorem J. Using orthogonal eigenvectors of M_1 and M_2, we can reduce M to the form

$$\left(\begin{array}{ccc|ccc} 0_{d_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & E_{11} & E_{12} & 0 \\ 0 & 0 & 0_{d_1'} & E_{21} & E_{22} & 0 \\ \hline 0 & E_{11}^H & E_{21}^H & 0_{d_2'} & 0 & 0 \\ 0 & E_{22}^H & E_{12}^H & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{d_2} \end{array}\right), \quad d_i' := \delta_i - d_i, \ i = 1, 2, \tag{6.12}$$

where D_i, $i = 1, 2$ denote the nonzero eigenvalues of M_1 and M_2. By eliminating the zero lines and columns in (6.12) we get the following reduced version of Theorem J:
Let $M_1 = \operatorname{diag}(D_1, 0)$ and $M_2 = \operatorname{diag}(0, D_2)$ be diagonal matrices with inertia (π_i, ν_i, δ_i), $i = 1, 2$. Assume that

$$\pi \geq \max_{i=1,2}(\pi_i + \delta_i) \quad \text{and} \quad \nu \geq \max_{i=1,2}(\nu_i + \delta_i). \tag{6.13}$$

Then there exist matrices $E_{i,j}$, $i, j = 1, 2$, such that

$$\operatorname{rank}(E_{21}\ E_{22}) \quad \text{and} \quad \operatorname{rank}(E_{11}^H\ E_{21}^H) \quad \text{are both maximal,} \tag{6.14}$$

and $\operatorname{in}(M) = (\pi, \nu, \delta)$ with

$$M = \left(\begin{array}{cc|cc} D_1 & 0 & E_{11} & E_{12} \\ 0 & 0 & E_{21} & E_{22} \\ \hline E_{11}^H & E_{21}^H & 0 & 0 \\ E_{22}^H & E_{12}^H & 0 & D_2 \end{array}\right) = \begin{pmatrix} M_1 & B \\ B^H & M_2 \end{pmatrix}. \tag{6.15}$$

Here, the rank conditions (6.14) correspond to $d_i = 0$, $i = 1, 2$, in the reduced version of Theorem J.
Without loss of generality it suffices to consider the case $\delta_1 \geq \delta_2$. Now we have to find

$E_{i,j}$, $i,j = 1,2$, that fulfill (6.14) and (6.15). To this aim set $E_{11} = 0$. Theorem H shows that, by choosing E_{12} appropriately, we can generate matrices

$$F = M_2 - B^H M_1^\dagger B = \begin{pmatrix} 0_{\delta_2} & 0 \\ 0 & D_2 - E_{12}^H D_1^{-1} E_{12} \end{pmatrix} = \begin{pmatrix} 0_{\delta_2} & 0 \\ 0 & F_1 \end{pmatrix}$$

with any inertia for which

$$\pi_2 - \pi_1 \leq \pi(F) \leq \pi_2 + \nu_1 \quad \text{and} \quad \nu_2 - \nu_1 \leq \nu(F) \leq \nu_2 + \pi_1$$

is fulfilled. For W_s an $(n_2 - \delta_2) \times (n_1 - \delta_1)$ matrix with orthogonal columns define

$$V_s = \begin{pmatrix} 0_{\delta_2 \times (n_2 - \delta_1)} \\ W_s \end{pmatrix}$$

and $G = V_s^H F V_s = W_s^H F_1 W_s$. By Theorem G, for the inertia of G we can reach

$$\max(0, \pi_2 - \pi_1 - (\delta_1 - \delta_2)) \leq \pi(G) \leq \pi_2 + \nu_1 \tag{6.16}$$

and

$$\max(0, \nu_2 - \nu_1 - (\delta_1 - \delta_2)) \leq \nu(G) \leq \nu_2 + \pi_1 \tag{6.17}$$

for different choices of W_s. In addition we can assume that G is a diagonal matrix by multiplying W_s with a suitable unitary matrix.
Now for given π and ν, (6.13) shows that (6.16) and (6.17) are fulfilled for $\pi(G) = \pi - \pi_1 - \delta_1$ and $\nu(G) = \nu - \nu_1 - \delta_1$. Thus, by choosing E_{12} and W_s appropriately, it holds $\pi = \pi(G) + \pi_1 + \delta_1$ and $\nu = \nu(G) + \nu_1 + \delta_1$. Now set $(E_{21} \ E_{22}) = V_r^H = (W_r^H \ *)$ such that $V = (V_r \ V_s)$ is a unitary matrix and W_r is a $\delta_2 \times (n_2 - \delta_1)$ matrix of full rank. All in all we have defined $E_{i,j}$, $i,j = 1,2$, and thus M via (6.15). It remains to show that M has the desired properties. Eliminating E_{12} leads to

$$\left(\begin{array}{cc|cc} D_1 & 0 & 0 & 0 \\ 0 & 0 & E_{21} & E_{22} \\ \hline 0 & E_{21}^H & 0 & 0 \\ 0 & E_{22}^H & 0 & F_1 \end{array}\right),$$

and the congruence transformation with $T = diag\,(I \quad V)$ then gives

$$\left(\begin{array}{cc|cc} D_1 & 0 & 0 & \\ 0 & 0 & I_{\delta_1} & 0 \\ \hline 0 & I_{\delta_1} & 0 & 0 \\ 0 & 0 & 0 & G \end{array}\right).$$

Therefore, $\pi(M) = \pi(G) + \pi_1 + \delta_1 = \pi$ and $\nu(M) = \nu(G) + \nu_1 + \delta_1 = \nu$. Futhermore, it holds $E_{21} = W_r^H$ and thus $\operatorname{rank}(E_{21}\ E_{22}) = \operatorname{rank}(V_r^H) = \delta_1$ and $\operatorname{rank}(E_{11}^H\ E_{21}^H) = \operatorname{rank}(E_{21}^H) = \operatorname{rank}(W_r) = \delta_2$ are both maximal. Hence, (6.14) is also fulfilled. □

Remark 6.2. In Theorem 1.3 of [5] Dancis considers the more general case of the extension of r matrices M_i to a matrix M of prescribed inertia. By a nontrivial induction Theorem J can be extended to this case.

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