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Abstract

Equations for the mean and the turbulent quantities for compressible turbulent flows are derived in this report. Both the conventional Reynolds average and the mass-weighted, Favre average were employed to decompose the flow variable into a mean and a turbulent quantity. These equations are to be used later in developing second-order Reynolds stress models for high-speed compressible flows. A few recent advances in modeling some of the terms in the equations due to compressibility effects are also summarized.

1. Introduction

This report describes the progress that was made during the first phase of an effort to develop new second-order closure models for compressible turbulence.

Compressible turbulence modeling is an essential element of many problems of practical interest, such as external aerodynamic calculations, the design of engine components and noise reduction. Initially, based on Morkovin's hypothesis (1964), the direct extension of incompressible models were used in the calculation of turbulent flows at moderate Mach numbers. This practice has enjoyed a considerable success in the calculations of wall shear layers in the past. However, it failed to predict adequately the reduced growth rate of high speed shear layers, in which the compressibility effects are more prominent. As a result, assuming that the turbulent fluctuations undergo thermodynamic processes such as isentropic or isothermal processes, Oh (1974), Rubesin (1976) and Vandromme (1983), among others, added compressibility corrections to the baseline incompressible models and produced mixed levels of success.

However, with the recent interest and need to develop high-speed flight technologies, models that predict correctly the dynamics of complex compressible turbulent flows are needed to measure up to the advances in other areas such as CFD and aerothermodynamics. To this end, the present task explores second-order models with an explicit account of compressibility effects. With this in mind, we first derive the equations for the mean flow and turbulent quantities within the framework of second-order modeling by using two basic approaches, i.e., the Reynolds average and the Favre average. These equations and some related matters that were observed along the way are described in this report. Two recently proposed arguments about the influence of compressibility are also described. These include the concept of dilatation dissipation and the description of the changes of turbulence structures by flow instabilities. It is concluded that the instability wave description of turbulent large structures complements the second-order modeling methods and, together, the two approaches may be able to satisfy a broad range of needs in engineering calculations involving compressible turbulent flows.

2. Basic Compressible Equations

The equations that govern the flows of compressible fluids are the Navier-Stokes equations. The equations are
Equation of conservation of mass

$$\rho_{,t} + (\rho u_i)_{,i} = 0 \tag{1}$$

Equations of conservation of momentum

$$(\rho u_i)_{,t} \; + \; (\rho u_i u_j)_{,j} \; = \; \sigma_{ij,j} \tag{2}$$

Equation of conservation of energy: in total energy

$$(\rho E)_{,t} + (\rho E u_i)_{,i} = (\sigma_{ij} u_j)_{,i} - q_{i,i} \tag{3}$$

or

$$(\rho H)_{,t} + (\rho H u_i)_{,i} = p_{,t} + (\tau_{ij} u_j)_{,i} - q_{i,i} \tag{4}$$

Equation of state

$$p = \rho R\, T \tag{5}$$

where

$$E = e + \frac{1}{2} u_i u_i \quad , \quad H = e + \frac{p}{\rho} + \frac{1}{2} u_i u_i \quad , \quad e = C_v\, T$$

$$\sigma_{ij} = -p\, \delta_{ij} + \tau_{ij} \quad , \quad \tau_{ij} = 2\mu s_{ij} - \mu^* u_{k,k} \delta_{ij} \ , \quad s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

$$\mu^* = \frac{2}{3}\mu - \mu_{bulk} \ , \quad q_i = -\,\mathrm{k}\, T_{,i}$$

With the expressions for the transport coefficients $\{\mu\ ,\ \mu^*\ ,\ \mathrm{k}\}$ obtained experimentally, the above equations form a closed system with appropriate boundary and initial conditions. Note that according to the second law of thermodynamics, the transport coefficients have to be greater than zero.

3. Equations for Compressible Turbulent Flows

Statistically, the random fluctuation of flow properties in turbulent flows can be decomposed into an average value and a fluctuation. Two commonly used averaging techniques are the conventional and mass-weighted averages, which are often referred to as the Reynolds average and the Favre average, respectively. In the Reynolds average, the flow property, ϕ, is decomposed into a ensemble mean value, $\bar{\phi}$, and a fluctuation, ϕ'', i.e.,

$$\begin{aligned}
\rho &= \overline{\rho} + \rho'' \\
u_i &= \overline{u}_i + u_i'' \\
p &= \overline{p} + p'' \\
T &= \overline{T} + T'' \\
E &= \overline{E} + E'' \\
H &= \overline{H} + H''
\end{aligned}$$

$$\mu = \overline{\mu} + \mu'' \quad , \quad \mu^* = \overline{\mu^*} + \mu^{*''}$$

$$C_v = \overline{C_v} + C_v'' \quad , \quad C_p = \overline{C_p} + C_p'' \tag{6}$$

A Favre-averaged or mass-weighted-averaged mean value, $\tilde{\phi}$ is defined as

$$\tilde{\phi} = \frac{\overline{\rho\phi}}{\bar{\rho}} \tag{7}$$

A mixed average that uses the Reynolds average for the density, pressure, transport coefficients and specific heats and Favre average for the other quantities is often applied in dealing with compressible turbulence. Flow properties can then be written as

$$\begin{aligned}
\rho &= \bar{\rho} + \rho'' \\
u_i &= \tilde{u}_i + u_i' \\
p &= \bar{p} + p'' \\
T &= \tilde{T} + T' \\
E &= \tilde{E} + E' \\
H &= \tilde{H} + H' \\
\mu = \overline{\mu} + \mu'' \quad &, \quad \mu^* = \overline{\mu^*} + \mu^{*''} \\
C_v = \overline{C_v} + C_v'' \quad &, \quad C_p = \overline{C_p} + C_p''
\end{aligned} \tag{8}$$

Note that both decompositions are defined in term of the ensemble average and

$$\overline{u_i''} = 0 \quad , \quad \widetilde{u_i''} \neq 0 \tag{9}$$

$$\widetilde{u_i'} = 0 \quad , \quad \overline{u_i'} \neq 0 \tag{10}$$

This is one of the most important differences between the Reynolds and the Favre averages. The relations between these two averages are

$$\bar{\phi} - \tilde{\phi} = \phi' - \phi'' = \overline{\phi'} = -\overline{\rho''\phi'}\,/\,\bar{\rho} = -\overline{\rho''\phi''}\,/\,\bar{\rho} \tag{11}$$

3.1 Mean Flow Equations

3.1.1 Reynolds-averaged Equations

The Reynolds-averaged mean equation for the conservation of mass can be obtained by taking the Reynolds average of the instantaneous equation of the conservation of mass. The mean equations for the conservation of momentum and energy can be obtained by the same procedure.

Mean Continuity

$$\bar{\rho}_{,t} + (\bar{\rho}\,\bar{u}_i + \overline{\rho'' u_i''})_{,i} = 0 \tag{12}$$

Mean Momentum

$$\overline{\rho}\,\overline{u}_{i,t} \;+\; (\overline{\rho'' u_i''})_{,t} + \;(\overline{\rho}\,\overline{u}_j + \overline{\rho'' u_j''})\,\overline{u}_{i,j} \;+\; (\overline{\rho}\overline{u_i'' u_j''} + \overline{\rho'' u_i'' u_j''} + \overline{u}_j \overline{\rho'' u_i''})_{,j}$$

$$= \;\{-(\overline{p} + \overline{\mu^*}\overline{u}_{k,k} + \overline{\mu^{*''} u_{k,k}''})\delta_{ij} \;+\; 2(\overline{\mu}\overline{s}_{ij} + \overline{\mu'' s_{ij}''})\}_{,j} \tag{13}$$

Mean Energy

$$(\overline{\rho}\overline{E} + \overline{\rho'' E''})_{,t} \;+\; (\overline{\rho}\overline{E}\overline{u}_i + \overline{\rho}\overline{E'' u_i''} + \overline{u}_i \overline{\rho'' E''} + \overline{E}\;\overline{\rho'' u_i''} + \overline{\rho'' E'' u_i''})_{,i}$$

$$= \{\; -(\overline{q_i} + \overline{p}\;\overline{u}_i + \overline{p'' u_i''} + \overline{\mu^*}\overline{u}_{k,k}\;\overline{u}_i + \overline{\mu^*}\overline{u_{k,k}'' u_i''} + \overline{u}_i \overline{\mu^{*''} u_{k,k}''} + \overline{u}_{k,k}\;\overline{\mu^{*''} u_i''}$$

$$+ \overline{\mu^{*''} u_{k,k}'' u_i''}) \;+\; 2\;(\overline{\mu}\;\overline{s}_{ij}\overline{u}_j + \overline{\mu}\overline{s_{ij}'' u_j''} + \overline{u}_j \overline{\mu'' s_{ij}''} + \overline{s}_{ij}\overline{\mu'' u_j''} + \overline{\mu'' u_j'' s_{ij}''})\;\}_{,i} \tag{14}$$

or

$$(\overline{\rho}\overline{H} + \overline{\rho'' H''} - \overline{p})_{,t} \;+\; (\overline{\rho}\overline{H}\overline{u}_i + \overline{\rho}\overline{H'' u_i''} + \overline{u}_i \overline{\rho'' H''} + \overline{H}\;\overline{\rho'' u_i''} + \overline{\rho'' H'' u_i''})_{,i}$$

$$= \;\{\; -\;(\overline{q}_i + \overline{\mu^*}\;\overline{u}_{k,k}\overline{u}_i + \overline{\mu^*}\;\overline{u_{k,k}'' u_i''} + \overline{u}_i \overline{\mu^{*''} u_{k,k}''} + \overline{u}_{k,k}\overline{\mu^{*''} u_i''} + \overline{\mu^{*''} u_{k,k}'' u_i''})$$

$$+\; 2(\overline{\mu}\;\overline{s}_{ij}\overline{u}_j + \overline{\mu}\overline{s_{ij}'' u_j''} + \overline{u}_j \overline{\mu'' s_{ij}''} + \overline{s}_{ij}\overline{\mu'' u_j''} + \overline{\mu'' s_{ij}'' u_j''})\;\}_{,i} \tag{15}$$

where

$$\overline{E} \;=\; \overline{C_v}\;\overline{T} \;+\; \overline{C_v'' T''} \;+\; \frac{1}{2}\,(\overline{u}_i\overline{u}_i + \overline{u_i'' u_i''})$$

and

$$\overline{H} \;=\; \overline{C_p}\;\overline{T} \;+\; \overline{C_p'' T''} \;+\; \frac{1}{2}\,(\overline{u}_i\overline{u}_i + \overline{u_i'' u_i''})$$

Equation of State

$$\overline{p} \;=\; \overline{\rho} R\overline{T} \;+\; R\overline{\rho'' T''} \tag{16}$$

These mean equations may be used to solve for the mean quantities:

$$\{\overline{\rho} \;,\; \overline{u}_i \;,\; \overline{p} \;,\; \overline{T}\} \tag{17}$$

The set of mean flow equations, however, are not closed. Quantities that need to be known to solve these equations are

$$\overline{\rho'' u_i''} \;,\; \overline{u_i'' u_j''} \;,\; \overline{\rho'' u_i'' u_j''} \;,\; \overline{\mu^{*''} u_{k,k}''} \;,\; \overline{\mu'' s_{ij}''} \;,\; \overline{\rho'' E''} \;,\; \overline{E'' u_i''} \;,\; \overline{\rho'' E'' u_i''} \;,\; \overline{p'' u_i''} \;,$$

$$\overline{u_{k,k}'' u_i''} \;,\; \overline{\mu^{*''} u_i''} \;,\; \overline{\mu^{*''} u_{k,k}''} \;,\; \overline{\mu^{*''} u_{k,k}'' u_i''} \;,\; \overline{u_j'' s_{ij}''} \;,\; \overline{\mu'' s_{ij}''} \;,\; \overline{\mu'' u_j''} \;,\; \overline{\mu'' u_j'' s_{ij}''}$$

$$\overline{\rho'' T''} \;,\; \overline{C_v'' T''} \;,\; \overline{\mu^*} \;,\; \overline{\mu} \;,\; \overline{C_v} \tag{18}$$

3.1.2 Favre-averaged Equations

The equations governing the Favre-averaged mean variables can be obtained by the same process that was used to derive the Reynolds-averaged mean equations. Since $\overline{\rho\phi'} = 0$, the density fluctuation does not appear explicitly in the resulting equations. The equations are listed below.

Mean Continuity

$$\overline{\rho}_{,t} \; + \; (\overline{\rho}\widetilde{u}_i)_{,i} \; = \; 0 \tag{19}$$

Mean Momentum

$$(\overline{\rho}\widetilde{u}_i)_{,t} \; + \; (\overline{\rho}\widetilde{u}_i\widetilde{u}_j + \overline{\rho}\widetilde{u'_i u'_j})_{,j} \; =$$

$$\{-(\overline{p} + \overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^{*''} u'_{k,k}} + \overline{\mu^*}\overline{u'_{k,k}})\delta_{ij} + \; 2(\overline{\mu}\widetilde{s}_{ij} + \overline{\mu'' s'_{ij}} + \overline{\mu}\ \overline{s'_{ij}})\}_{,j} \tag{20}$$

Mean Energy

$$(\overline{\rho}\widetilde{E})_{,t} \; + \; (\overline{\rho}\widetilde{E}\widetilde{u}_i + \overline{\rho}\widetilde{E' u'_i})_{,i} \; = \; \{-(\overline{q_i} + \overline{p}\widetilde{u}_i + \overline{p}\overline{u'_i} + \overline{p'' u'_i}$$

$$+ \overline{\mu^*}\widetilde{u}_{k,k}\widetilde{u}_i + \overline{\mu^*}\overline{u'_{k,k}u'_i} + \widetilde{u}_i\overline{\mu^{*''} u'_{k,k}} + \widetilde{u}_{k,k}\ \overline{\mu^{*''} u'_i} + \overline{\mu^{*''} u'_i u'_{k,k}} + \overline{\mu^*}\widetilde{u}_i\ \overline{u'_{k,k}} + \overline{\mu^*}\ \widetilde{u}_{k,k}\overline{u'_i})$$

$$+ \; 2\ (\overline{\mu}\widetilde{s}_{ij}\widetilde{u}_j + \overline{\mu}\overline{s'_{ij}u'_j} + \widetilde{u}_j\overline{\mu'' s'_{ij}} + \widetilde{s}_{ij}\overline{\mu'' u'_j} + \overline{\mu'' u'_j s'_{ij}} + \overline{\mu}\widetilde{u}_j\overline{s'_{ij}} + \overline{\mu}\widetilde{s}_{ij}\overline{u'_j})\}_{,i} \tag{21}$$

or

$$(\overline{\rho}\widetilde{H} - \overline{p})_{,t} \; + \; (\overline{\rho}\widetilde{H}\widetilde{u}_i + \overline{\rho}\widetilde{H' u'_i})_{,i}$$

$$= \; \{-\ (\overline{q}_i + \overline{\mu^*}\widetilde{u}_{k,k}\widetilde{u}_i + \overline{\mu^*}\overline{u'_{k,k}u'_i} + \widetilde{u}_i\overline{\mu^{*''} u'_{k,k}} \; + \; \widetilde{u}_{k,k}\ \overline{\mu^{*''} u'_i} + \overline{\mu^{*''} u'_i u'_{k,k}} + \overline{\mu^*}\widetilde{u}_i\overline{u'_{k,k}})$$

$$+ \; 2(\overline{\mu}\widetilde{s}_{ij}\widetilde{u}_j + \overline{\mu}\overline{s'_{ij}u'_j} + \widetilde{u}_j\overline{\mu'' s'_{ij}} + \widetilde{s}_{ij}\overline{\mu'' u'_j} + \overline{\mu'' u'_j s'_{ij}} + \overline{\mu}\ \overline{u}_j\overline{s'_{ij}} + \overline{\mu}\widetilde{s}_{ij}\overline{u'_j})\ \}_{,i} \tag{22}$$

where

$$\widetilde{E} \; = \; \overline{C_v}\widetilde{T} + \overline{C''_v T'} + \frac{1}{\overline{\rho}}(\widetilde{T}\overline{C''_v \rho''} + \overline{C''_v \rho'' T'}) \; + \; \frac{1}{2}\ (\widetilde{u}_i\widetilde{u}_i + \widetilde{u'_i u'_i})$$

and

$$\widetilde{H} \; = \; \overline{C_p}\widetilde{T} + \overline{C''_p T'} + \frac{1}{\overline{\rho}}(\widetilde{T}\overline{C''_p \rho''} + \overline{C''_p \rho'' T'}) \; + \; \frac{1}{2}\ (\widetilde{u}_i\widetilde{u}_i + \widetilde{u'_i u'_i})$$

Equation of State

$$\overline{p} \; = \; \overline{\rho}R\widetilde{T} \tag{23}$$

These equations may be used to solve for the mean quantities:

$$\{\ \overline{\rho}\ ,\ \ \widetilde{u}_i\ ,\ \ \overline{p}\ ,\ \ \widetilde{T}\} \tag{24}$$

Not surprisingly, the set of mean flow equations are not closed. Quantities that need to be known to solve these equations are

$$\widetilde{u'_i u'_j}\ ,\ \ \overline{\mu^{*''} u''_{k,k}}\ ,\ \ \overline{\mu'' s'_{ij}}\ ,\ \ \overline{s'_{ij}}\ ,\ \ \widetilde{E' u'_i}\ ,\ \ \overline{u'_i}\ ,$$

$$\overline{p''u_i'}\ ,\ \overline{u_{k,k}'u_i'}\ ,\ \overline{\mu^{*''}u_{k,k}'}\ ,\ \overline{\mu^{*''}u_i'}\ ,\ \overline{s_{ij}'u_j'}\ ,\ \overline{\mu''u_j'}\ ,\ \overline{\mu''u_j's_{ij}'},$$

$$\overline{C_v''T'}\ ,\ \overline{C_v''\rho''}\ ,\ \overline{C_v''\rho''T'}\ ,\ \overline{\mu^*}\ ,\ \overline{\mu}\ ,\ \overline{C_v} \tag{25}$$

It was shown in this section that, in order to solve the mean flow equations, the turbulent moments of order up to three must somehow be modeled. The main objective of this study is to develop rational second-order modeling procedures for compressible turbulence. The transport equations for the second-order turbulent moments are derived in the following sections. Higher order moments are represented as functionals of the mean flow and the second-order moments in the second-order models that will be developed later in the second phase of this project.

3.2 Equations for Turbulent Fluctuations

3.2.1 Reynolds-averaged Equations

The equations for the turbulent fluctuations are needed in the derivation of the equations for moments of various order. The fluctuating parts of the equations of conservation of mass, momentum and energy were obtained by subtracting the mean equations from their corresponding instantaneous equations. The same procedure is also applicable to the equation of state and the thermodynamic relations.
Fluctuating Continuity

$$\rho''_{,t}\ +\ (\overline{\rho}u_i'' + \rho''\overline{u}_i + \rho''u_i'' - \overline{\rho''u_i''})_{,i}\ =\ 0 \tag{26}$$

Fluctuating Momentum

$$\begin{aligned}
&(\rho''u_i'' - \overline{\rho''u_i''} + \rho''\overline{u}_i + \overline{\rho}u_i'')_{,t}\ +\ \{\overline{\rho}(u_i''u_j'' - \overline{u_i''u_j''}) + \overline{u}_i(\rho''u_j'' - \overline{\rho''u_j'})\\
&+ \overline{u}_j(\rho''u_i'' - \overline{\rho''u_i''}) + \rho''u_i''u_j'' - \overline{\rho''u_i''u_j''} + \overline{\rho}\ \overline{u}_iu_j'' + \overline{\rho}\ \overline{u}_ju_i'' + \overline{u}_i\overline{u}_j\rho''\}_{,j}\\
&\quad = \{-(p'' + \mu^{*''}u_{k,k}'' - \overline{\mu^{*''}u_{k,k}''} + \overline{\mu^*}u_{k,k}'' + \mu^{*''}\overline{u}_{k,k})\delta_{ij}\\
&\qquad + 2\ (\mu''s_{ij}'' - \overline{\mu''s_{ij}''} + \overline{\mu}s_{ij}'' + \mu''\overline{s}_{ij}\}_{,j}
\end{aligned} \tag{27}$$

Fluctuating Energy

$$\begin{aligned}
&(\overline{\rho}E'' + \overline{E}\rho'' + \rho''E'' - \overline{\rho''E''})_{,t}\ +\ \{\overline{\rho}(E''u_i'' - \overline{E''u_i''}) + \overline{u}_i(\rho''E'' - \overline{\rho''E''})\\
&+ \overline{E}(\rho''u_i'' - \overline{\rho''u_i''}) + \rho''E''u_i'' - \overline{\rho''E''u_i''} + \overline{\rho}\ \overline{u}_iE'' + \overline{\rho}\overline{E}u_i'' + \overline{u}_i\overline{E}\rho''\}_{,i}\\
&= \{ -[\ q_i'' + \overline{p}u_i'' + p''\overline{u}_i + p''u_i'' - \overline{p''u_i''} + \overline{\mu^*}(u_{k,k}''u_i'' - \overline{u_{k,k}''u_i''}) + \overline{u}_i(\mu^{*''}u_{k,k}'' - \overline{\mu^{*''}u_{k,k}''})\\
&+ \overline{u}_{k,k}(\mu^{*''}u_i'' - \overline{\mu^{*''}u_i''}) + \mu^{*''}u_{k,k}''u_i'' - \overline{\mu^{*''}u_{k,k}''u_i''} + \overline{\mu^*}\overline{u}_iu_{k,k}'' + \overline{\mu^*}\overline{u}_{k,k}u_i'' + \mu^{*''}\overline{u}_i\overline{u}_{k,k}]\\
&\qquad + 2\ [\overline{\mu}(s_{ij}''u_j'' - \overline{s_{ij}''u_j''}) + \overline{u}_j(\mu''s_{ij}'' - \overline{\mu''s_{ij}''}) + \overline{s}_{ij}(\mu''u_j'' - \overline{\mu''u_j''})
\end{aligned}$$

$$+\mu'' u_j'' s_{ij}'' - \overline{\mu'' u_j'' s_{ij}''} \ + \ \overline{\mu}\ \overline{u}_j s_{ij}'' + \overline{\mu s}_{ij} u_j'' + \overline{u}_j \overline{s}_{ij} \mu'']\}_{,i} \tag{28}$$

or

$$(\overline{\rho} H'' + \overline{H}\rho'' + \rho'' H'' - \overline{\rho'' H''})_{,t} \ + \ \{\overline{\rho}(H'' u_i'' - \overline{H'' u_i''}) + \overline{u}_i(\rho'' H'' - \overline{\rho'' H''})$$

$$+ \ \overline{H}(\rho'' u_i'' - \overline{\rho'' u_i''}) + \rho'' H'' u_i'' - \overline{\rho'' H'' u_i''} + \overline{\rho}\ \overline{u}_i H'' + \overline{\rho H} u_i'' + \overline{u}_i \overline{H} \rho''\}_{,i}$$

$$= \ \{ - [p_{,t}'' \ + \ q_i'' + \overline{\mu^*}(u_{k,k}'' u_i'' - \overline{u_{k,k}'' u_i''}) + \overline{u}_i(\mu^{*''} u_{k,k}'' - \overline{\mu^{*''} u_{k,k}''})$$

$$+ \ \overline{u}_{k,k}(\mu^{*''} u_i'' - \overline{\mu^{*''} u_i''}) + \mu^{*''} u_{k,k}'' u_i'' - \overline{\mu^{*''} u_{k,k}'' u_i''} + \overline{\mu^*}\overline{u}_i u_{k,k}'' + \overline{\mu^*}\overline{u}_{k,k} u_i'' + \mu^{*''}\overline{u}_i \overline{u}_{k,k}]$$

$$+ \ 2\ [\overline{\mu}(s_{ij}'' u_j'' - \overline{s_{ij}'' u_j''}) + \overline{u}_j(\mu'' s_{ij}'' - \overline{\mu'' s_{ij}''}) + \overline{s}_{ij}(\mu'' u_j'' - \overline{\mu'' u_j''}) + \mu'' u_j'' s_{ij}''$$

$$- \ \overline{\mu'' u_j'' s_{ij}''} \ + \ \overline{\mu}\ \overline{u}_j s_{ij}'' + \overline{\mu s}_{ij} u_j'' + \overline{u}_j \overline{s}_{ij} \mu'']\}_{,i} \tag{29}$$

Fluctuating Equation of State

$$p'' \ = \ R\ (\overline{\rho} T'' + \rho'' \overline{T} + \rho'' T'' - \overline{\rho'' T''}) \tag{30}$$

Fluctuating Thermodynamic Relations

$$E'' \ = \ C_v'' \overline{T} + \overline{C_v} T'' + C_v'' T'' - \overline{C_v'' T''} + \overline{u}_i u_i'' + \frac{1}{2}(u_i'' u_i'' - \overline{u_i'' u_i''}) \tag{31}$$

$$H'' \ = \ C_p'' \overline{T} + \overline{C_p} T'' + C_p'' T'' - \overline{C_p'' T''} + \overline{u}_i u_i'' + \frac{1}{2}(u_i'' u_i'' - \overline{u_i'' u_i''}) \tag{32}$$

3.2.2 Favre-averaged Equations

Fluctuating Continuity

$$\rho_{,t}'' \ + \ \{\rho'' \widetilde{u}_i \ + \ (\overline{\rho} + \rho'') u_i'\}_{,i} \ = \ 0 \tag{33}$$

Fluctuating Momentum

$$\{\overline{\rho} u_i' + \rho''(\widetilde{u}_i + u_i')\}_{,t} \ + \ \{\rho'' \widetilde{u}_i \widetilde{u}_j + (\overline{\rho} + \rho'') u_i' u_j' - \overline{\rho}\widetilde{u_i' u_j'} + (\overline{\rho} + \rho'')\widetilde{u}_i u_j' + (\overline{\rho} + \rho'') u_i' \widetilde{u}_j\}_{,j}$$

$$= \ \{-[p'' + \mu^{*''} u_{k,k}' - \overline{\mu^{*''} u_{k,k}'} + \overline{\mu^*}(u_{k,k}' - \overline{u_{k,k}'}) + \mu^{*''}\widetilde{u}_{k,k}]\delta_{ij}$$

$$+ \ 2\ [\mu'' s_{ij}' - \overline{\mu'' s_{ij}'} + \overline{\mu}(s_{ij}' - \overline{s_{ij}'}) + \mu'' \widetilde{s}_{ij}]\}_{,j} \tag{34}$$

Fluctuation Energy

$$\{\rho'' \widetilde{E} + (\overline{\rho} + \rho'') E'\}_{,t} \ + \ \{\rho'' \widetilde{E} \widetilde{u}_i + (\overline{\rho} + \rho'') E' u_i' - \overline{\rho}\widetilde{E' u_i'} + (\overline{\rho} + \rho'')\widetilde{u}_i E' + (\overline{\rho} + \rho'') u_i' \widetilde{E}\}_{,i}$$

$$= \ \{-[\widetilde{p}(u_i' - \overline{u_i'}) + p'' \widetilde{u}_i + p'' u_i' - \overline{p'' u_i'} + \overline{\mu^*}(u_{k,k}' u_i' - \overline{u_{k,k}' u_i'}) + \widetilde{u}_i(\mu^{*''} u_{k,k}' - \overline{\mu^{*''} u_{k,k}''})$$

$$+ \widetilde{u}_{k,k}(\mu^{*''} u'_i - \overline{\mu^{*''} u'_i}) + \mu^{*''} u'_i u'_{k,k} - \overline{\mu^{*''} u'_i u'_{k,k}} + \overline{\mu^*}\widetilde{u}_i(u'_{k,k} - \overline{u'_{k,k}}) + \overline{\mu^*}\widetilde{u}_{k,k}(u'_i - \overline{u'_i})$$

$$+\mu^{*''}\widetilde{u}_i\widetilde{u}_{k,k}] + 2[\overline{\mu}(s'_{ij}u'_j - \overline{s'_{ij}u'_j}) + \widetilde{u}_j(\mu'' s'_{ij} - \overline{\mu'' s'_{ij}}) + \widetilde{s}_{ij}(\mu'' u'_j - \overline{\mu'' u'_j}) + \mu'' u'_j s'_{ij}$$

$$-\overline{\mu'' u'_j s'_{ij}} + \overline{\mu}\,\widetilde{u}_j(s'_{ij} - \overline{s'_{ij}}) + \overline{\mu}\widetilde{s}_{ij}(u'_j - \overline{u'_j}) + \widetilde{u}_j\widetilde{s}_{ij}\mu'']\}_{,i} \qquad (35)$$

Fluctuating Equation of State

$$p'' = R\,[\rho''\widetilde{T} + (\overline{\rho} + \rho'')T'] \qquad (36)$$

Fluctuating Thermodynamic

$$E' = C''_v\widetilde{T} + \overline{C_v}T' + C''_vT' - \overline{C''_vT'} - \frac{1}{\overline{\rho}}(\widetilde{T}\overline{C''_v\rho''} + \overline{C''_v\rho''T'}) + \widetilde{u}_i u'_i + \frac{1}{2}(u'_iu'_i - \widetilde{u'_iu'_i}) \qquad (37)$$

$$H' = C''_p\widetilde{T} + \overline{C_p}T' + C''_pT' - \overline{C''_pT'} - \frac{1}{\overline{\rho}}(\widetilde{T}\overline{C''_p\rho''} + \overline{C''_p\rho''T'}) + \widetilde{u}_i u'_i + \frac{1}{2}(u'_iu'_i - \widetilde{u'_iu'_i}) \qquad (38)$$

3.3 Turbulent Moment Equations

3.3.1 Reynolds-averaged Equations

Since $\overline{\phi''} = 0$, the first-order moments of all the Reynolds-averaged turbulence fluctuations are zero. Equations for the second-order moments: $\overline{\rho u''_i u''_j}$, $\overline{\rho E'' u''_i}$, $\overline{\rho'' u''_i}$ and turbulent kinetic energy, $\overline{k}$, are derived below.

$\overline{\rho u''_i u''_j}$

An equation for the total momentum exchanges, $\rho u_i u_j$ can be derived by multiplying the momentum equations in u_i by u_j, exchanging the indices and adding the resulting equations together. An equation for $\overline{\rho}\ \overline{u}_i\overline{u}_j$ can be obtained in a similar way.

$$(\overline{\rho}\,\overline{u}_i\overline{u}_j)_{,t} + \overline{u}_i(\overline{\rho'' u''_j})_{,t} + \overline{u}_j(\overline{\rho'' u''_i})_{,t} + (\overline{\rho}\,\overline{u}_i\overline{u}_j\overline{u}_m)_{,m} + (\overline{u}_i\overline{u}_j\overline{\rho'' u''_m})_{,m} + \overline{u}_i(\overline{\rho u''_j u''_m})_{,m}$$

$$+ \overline{u}_j(\overline{\rho u''_i u''_m})_{,m} + \overline{u}_j(\overline{u}_m\overline{\rho'' u''_i})_{,m} + \overline{u}_i(\overline{u}_m\overline{\rho'' u''_j})_{,m} + \overline{u}_i(\overline{\rho'' u''_j u''_m})_{,m} + \overline{u}_j(\overline{\rho'' u''_i u''_m})_{,m}$$

$$= -\overline{u}_j\overline{p}_{,i} - \overline{u}_i\overline{p}_{,j} - \overline{u}_i(\overline{\mu^*}\,\overline{u}_{k,k} + \overline{\mu^{*''} u''_{k,k}})_{,j} - \overline{u}_j(\overline{\mu^*}\,\overline{u}_{k,k} + \overline{\mu^{*''} u''_{k,k}})_{,i}$$

$$+ 2\,\overline{u}_i(\overline{\mu s}_{jm} + \overline{\mu'' s''_{jm}})_{,m} + 2\,\overline{u}_j(\overline{\mu s}_{im} + \overline{\mu'' s''_{im}})_{,m} \qquad (39)$$

An equation for $\overline{\rho}\ \overline{u''_i u''_j}$, the turbulent Reynolds stresses, may be obtained by subtracting the equation for $\overline{\rho}\ \overline{u}_i\overline{u}_j$ from the one for $\rho u_i u_j$. The resulting equation is

$$
\begin{aligned}
&(\overline{\bar{\rho} u_i'' u_j''} + \overline{\rho'' u_i'' u_j''})_{,t} + \overline{\rho'' u_j''}\,\overline{u}_{i,t} + \overline{\rho'' u_i''}\,\overline{u}_{j,t} \\
&+ \overline{u}_{i,m}(\overline{\bar{\rho} u_j'' u_m''} + \overline{u}_m \overline{\rho'' u_j''} + \overline{\rho'' u_j'' u_m''}) + \overline{u}_{j,m}(\overline{\bar{\rho} u_i'' u_m''} + \overline{u}_m \overline{\rho'' u_i''} + \overline{\rho'' u_i'' u_m''}) \\
&+ (\bar{\rho}\, \overline{u}_m \overline{u_i'' u_j''} + \overline{\bar{\rho} u_i'' u_j'' u_m''} + \overline{u}_m \overline{\rho'' u_i'' u_j''} + \overline{\rho'' u_i'' u_j'' u_m''})_{,m} \\
&= -\, \overline{u_j'' p_{,i}''} - \overline{u_i'' p_{,j}''} - \overline{u_j''(\overline{\mu^*} u_{k,k}'' + \mu^{*''} \overline{u}_{k,k} + \mu^{*''} u_{k,k}'')_{,i}} \\
&\quad - \overline{u_i''(\overline{\mu^*} u_{k,k}'' + \mu^{*''} \overline{u}_{k,k} + \mu^{*''} u_{k,k}'')_{,j}} \\
&+ 2\, \{\overline{u_j''(\bar{\mu} s_{im}'' + \mu'' \overline{s}_{im} + \mu'' s_{im}'')_{,m} + u_i''(\bar{\mu} s_{jm}'' + \mu'' \overline{s}_{jm} + \mu'' s_{jm}'')_{,m}}\} \qquad (40)
\end{aligned}
$$

The continuity equation and the following relations have been used in the process

$$
a\,(bc)_{,m} \;+\; b\,(ac)_{,m} \;=\; (abc)_{,m} \;+\; b\;a\;c_{,m} \qquad (41)
$$

$$
a\,(bcd)_{,m} \;+\; b\,(acd)_{,m} \;=\; (abcd)_{,m} \;+\; b\;a\;(cd)_{,m} \qquad (42)
$$

$\overline{k}$

An equation for the turbulent kinetic energy $\overline{k}$ can be obtained by letting $i = j$ in the equation for $\overline{\bar{\rho} u_i'' u_j''}$.

$$
\begin{aligned}
&(\bar{\rho}\overline{k} + \overline{\rho'' k})_{,t} \;+\; \overline{\rho'' u_i''}\,\overline{u}_{i,t} \\
&+ (\overline{\bar{\rho} u_i'' u_j''} + \overline{u}_j \overline{\rho'' u_i''} + \overline{\rho'' u_i'' u_j''})\overline{u}_{i,j} \;+\; (\bar{\rho}\, \overline{u}_j \overline{k} + \overline{\bar{\rho} u_j'' k} + \overline{u}_j \overline{\rho'' k} + \overline{\rho'' u_j'' k})_{,j} \\
&= -\overline{u_i'' p_{,i}''} \;-\; \overline{u_i''(\overline{\mu^*} u_{k,k}'' + \mu^{*''} \overline{u}_{k,k} + \mu^{*''} u_{k,k}'')_{,i}} \\
&\quad + 2\, \overline{u_i''(\bar{\mu} s_{ij}'' + \mu'' \overline{s}_{ij} + \mu'' s_{ij}'')_{,j}} \qquad (43)
\end{aligned}
$$

$\overline{\rho'' u_i''}$

An equation for $\overline{\rho'' u_i''}$ can be derived by manipulating the equations for u_i'' and ρ''. An equation for ρ'' can be obtained from the continuity equation. It is

$$
\rho_{,t}'' \;+\; u_j \rho_{,j}'' \;+\; u_j'' \bar{\rho}_{,j} \;+\; \rho u_{j,j}'' \;+\; \rho'' \overline{u}_{j,j} \;=\; 0 \qquad (44)
$$

An equation for u_i'' can be obtained from the momentum equations. Note that

$$
\frac{1}{\rho} = \frac{1}{\bar{\rho} + \rho''} = \frac{1}{\bar{\rho}}\left(\frac{1}{1 + \frac{\rho''}{\bar{\rho}}}\right) = \frac{1}{\bar{\rho}}\left(1 - \frac{\rho''}{\bar{\rho}} + \left(\frac{\rho''}{\bar{\rho}}\right)^2 + \ldots\right) \simeq \frac{1}{\bar{\rho}} - \frac{\rho''}{\bar{\rho}^2} + O\left(\left(\frac{\rho''^2}{\bar{\rho}^3}\right)\right) \qquad (45)
$$

This linearized form for $\frac{1}{\rho}$ is used here to simplify the equation. With this simplification, the equation for u_i'' becomes

$$u_{i,t}'' + \overline{u}_j u_{i,j}'' + u_j''\overline{u}_{i,j}$$

$$= \frac{1}{\overline{\rho}} \{-(p'' + \overline{\mu}u_{k,k}'' + \mu''\overline{u}_{k,k} + \mu'' u_{k,k}'')\delta_{ij} + 2(\overline{\mu}s_{ij}'' + \mu''\overline{s}_{ij} + \mu'' s_{ij}'')\}_{,j}$$

$$- \frac{\rho''}{\overline{\rho}^2}\{-(\overline{p} + p'' + \overline{\mu^*}\overline{u}_{k,k} + \mu^{*''}\overline{u}_{k,k} + \overline{\mu^*}u_{k,k}'' + \mu^{*''} u_{k,k}'')\delta_{ij}$$

$$+ 2(\overline{\mu}\ \overline{s}_{ij} + \mu''\overline{s}_{ij} + \overline{\mu}s_{ij}'' + \mu'' s_{ij}'')\}_{,j} \tag{46}$$

The equation for $\overline{\rho'' u_i''}$ can be obtained by multiplying the equation for ρ'' by u_i'' and the equation for u_i'' by ρ'' and summing up the two. The resulting equation is

$$(\overline{\rho'' u_i''})_{,t} + \overline{u}_j(\overline{\rho'' u_i''})_{,j} + \overline{\rho'' u_j''}\overline{u}_{i,j} + \overline{\rho'' u_i''}\overline{u}_{j,j} + (\overline{\rho'' u_i'' u_j''})_{,j} + \overline{u_i'' u_j''}\overline{\rho}_{,j} + \overline{\rho}\overline{u_i'' u_{j,j}''}$$

$$= - \overline{\frac{\rho''}{\overline{\rho}}\{(p'' + \overline{\mu^*}u_{k,k}'' + \mu^{*''}\overline{u}_{k,k} + \mu^{*''} u_{k,k}'')\delta_{ij}}$$

$$\overline{-2\ (\overline{\mu}s_{ij}'' + \mu''\overline{s}_{ij} + \mu'' s_{ij}'')\}_{,j}}$$

$$- \overline{(\frac{\rho''}{\overline{\rho}})^2\ \{\ (\overline{p} + p'' + \overline{\mu^*}\overline{u}_{k,k} + \overline{\mu^*}u_{k,k}'' + \mu^{*''}\overline{u}_{k,k} + \mu^{*''} u_{k,k}'')\delta_{ij}}$$

$$\overline{+ 2\ (\overline{\mu}\ \overline{s}_{ij} + \overline{\mu}s_{ij}'' + \mu''\overline{s}_{ij} + \mu'' s_{ij}'')\ \}_{,j}} + O((\frac{\rho''}{\overline{\rho}})^3) \tag{47}$$

The linear approximation was used in Shih et al. (1987) in modeling a variable-density mixing layer and may be applicable in the modeling of compressible turbulence as well.

$\overline{\rho}\overline{E'' u_i''}$

One form of the transport equation for $\overline{\rho}\overline{E'' u_i''}$ can be obtained by tracking the following steps: (1) Derive the $\rho E u_i$ equation by multiplying the momentum equations by E and the energy equation by u_i and summing up the two resulting equations (2) Derive the $\overline{\rho}\overline{E}\ \overline{u}_i$ equations by the same procedure (3) Subtract the $\overline{\rho}\overline{E}\ \overline{u}_i$ equation from the $\rho E u_i$ equation. The resulting equations are

$$(\overline{\rho}\overline{E'' u_i''} + \overline{\rho'' E'' u_i''})_{,t} + \overline{\rho'' u_i''}\ \overline{E}_{,t} + \overline{\rho'' E''}\overline{u}_{i,t}$$

$$+ (\ \overline{\rho}\ \overline{u}_j\overline{E'' u_i''} + \overline{\rho}\overline{E'' u_i'' u_j''} + \overline{u}_j\overline{\rho'' E'' u_i''} + \overline{\rho'' E'' u_i'' u_j''}\ )_{,j}$$

$$+ (\overline{\rho}\overline{u_i'' u_j''} + \overline{u}_i\overline{\rho'' u_j''} + \overline{u}_j\overline{\rho'' u_i''} + \overline{\rho'' u_i'' u_j''}\ )\ \overline{E}_{,j}$$

$$
\begin{aligned}
&+ \left(\overline{\rho E'' u_j''} + \overline{u}_j \overline{\rho'' E''} + \overline{\rho'' E'' u_j''} \right) \overline{u}_{i,j} \\
&= - \overline{E''(p'' + \overline{\mu^*} u''_{k,k} + \mu^{*''} \overline{u}_{k,k} + \mu^{*''} u''_{k,k})_{,j}} \\
&\quad - \overline{u}_j \overline{u_i''(p'' + \overline{\mu} u'_{k,k} + \mu'' \overline{u}_{k,k} + \mu'' u''_{k,k})_{,j}} \\
&\quad - \overline{u_i'' u_j'' (\overline{p} + p'' + \overline{\mu^*} \overline{u}_{k,k} + \overline{\mu^*} u''_{k,k} + \mu^{*''} \overline{u}_{k,k} + \mu^{*''} u''_{k,k})_{,j}} \\
&\quad - \overline{u}_{j,j} \overline{u_i''(p'' + \overline{\mu^*} u''_{k,k} + \mu^{*''} \overline{u}_{k,k} + \mu^{*''} u''_{k,k})} \\
&\quad - \overline{u_j'' u''_{j,j} (\overline{p} + p'' + \overline{\mu^*} \overline{u}_{k,k} + \overline{\mu^*} u''_{k,k} + \mu^{*''} \overline{u}_{k,k} + \mu^{*''} u''_{k,k})} \\
&+ 2\ [\overline{u_i''(\overline{\mu s}_{ij} u_i'' + \overline{\mu} s''_{ij} \overline{u}_i + \overline{\mu} s''_{ij} u_i'' + \mu'' \overline{s}_{ij} \overline{u}_i + \mu'' \overline{s}_{ij} u_i'' + \mu'' \overline{u}_i s''_{ij} + \mu'' u_i'' s''_{ij})_{,j}} \\
&\quad + \overline{E''\ (\overline{\mu} s''_{ij} + \mu'' \overline{s}_{ij} + \mu'' s''_{ij})_{,j}}] \ - \ \overline{q''_{j,j} u_i''}
\end{aligned}
\tag{48}
$$

3.3.2 Favre-averaged Equations

In order to close the Favre-averaged mean equations, the quantity $\overline{u_i'}$ has to be known. An equation for $\overline{u_i'}$ can be obtained by taking the Reynolds average of the equation for u_i'. The Favre-averaged mean momentum equations can be written as, by using the Favre-averaged mean continuity equation,

$$
\begin{aligned}
(\widetilde{u}_i)_{,t} \ + \ \widetilde{u}_j \widetilde{u}_{i,j} \ &= \ \frac{1}{\overline{\rho}} \{ -(\overline{p} + \overline{\mu^*} \widetilde{u}_{k,k} + \overline{\mu^{*''} u'_{k,k}} + \overline{\mu^*}\, \overline{u'_{k,k}}) \delta_{ij} \\
&\quad + 2(\overline{\mu} \widetilde{s}_{ij} + \overline{\mu'' s'_{ij}} + \overline{\mu}\ \overline{s'_{ij}}) - \overline{\rho} \widetilde{u_i' u_j'} \}_{,j}
\end{aligned}
\tag{49}
$$

An equation for u_i' can be obtained by subtracting this equation from the total momentum equations. The equation for $\overline{u_i'}$ becomes

$$
\begin{aligned}
&\overline{u'}_{i,t} \ + \ \widetilde{u}_j \overline{u'}_{i,j} \ + \ \overline{u'}_j \widetilde{u}_{i,j} \ + \ \overline{u_j' u'_{i,j}} \\
&= \overline{\frac{1}{\rho} \{ -(\overline{p} + p'' + \overline{\mu^*} \widetilde{u}_{k,k} + \overline{\mu^*} u'_{k,k} + \mu^{*''} \widetilde{u}_{k,k} + \mu^{*''} u'_{k,k}) \delta_{ij} \}} \\
&\quad + \overline{\frac{2}{\rho} (\overline{\mu} \widetilde{s}_{ij} + \overline{\mu} s'_{ij} + \mu'' \widetilde{s}_{ij} + \mu'' s'_{ij})_{,j}} \\
&= \frac{1}{\overline{\rho}} \{ -(\overline{p} + \overline{\mu^*} \widetilde{u}_{k,k} + \overline{\mu^{*''} u'_{k,k}} + \overline{\mu^*}\, \overline{u'_{k,k}}) \delta_{ij} \ + \ 2(\overline{\mu} \widetilde{s}_{ij} + \overline{\mu'' s''_{ij}} + \overline{\mu}\, \overline{s'_{ij}}) - \widetilde{u_i' u_j'} \}_{,j}
\end{aligned}
\tag{50}
$$

Since,

$$
\overline{u_i'} \ = \ - \ \overline{\rho'' u_i''} \ / \ \overline{\rho}
\tag{51}
$$

an equation for $\overline{u_i'}$ can be converted to an equation for the mass flux $\overline{\rho'' u_i''}$.

Similarly, an equation for $\overline{E'}$ can be obtained from the mean continuity, mean energy and the total energy equations. The resulting equation is

$$\overline{E'}_{,t} + \widetilde{u}_j\overline{E'}_{,j} + \overline{u'_j}\widetilde{E}_{,j} + \overline{u'_j E'_{,j}}$$

$$= -\overline{\frac{1}{\rho}\{\bar{q}_j + q''_j + (\bar{p} + p'' + \overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}u'_{k,k})\delta_{ij}\}_{,j}}$$

$$+ \overline{\frac{2}{\rho}(\overline{\mu}\widetilde{s}_{ij} + \overline{\mu}s'_{ij} + \mu''\widetilde{s}_{ij} + \mu''s'_{ij})_{,j}} + \frac{1}{\bar{\rho}}\ \{\overline{q_j} + (\overline{\bar{p}\widetilde{u}_i + \bar{p}u'_i + p''u'_i}$$

$$+ \overline{\overline{\mu^*}\widetilde{u}_{k,k}\widetilde{u}_i + \overline{\mu^*}\widetilde{u}_{k,k}u'_i + \overline{\mu^*}u'_{k,k}\widetilde{u}_i + \overline{\mu^*}u'_{k,k}u'_i}$$

$$+ \overline{\mu^{*''}\widetilde{u}_{k,k}u'_i + \mu^{*''}u'_{k,k}\widetilde{u}_i + \mu^{*''}u'_{k,k}u'_i})\delta_{ij}$$

$$- 2\ (\overline{\overline{\mu}\widetilde{s}_{ij}\widetilde{u}_i + \overline{\mu}\widetilde{s}_{ij}u'_i + \overline{\mu}s'_{ij}\widetilde{u}_i + \overline{\mu}s'_{ij}u'_i + \mu''\widetilde{s}_{ij}u'_i + \mu''s'_{ij}\widetilde{u}_i + \mu''s'_{ij}u'_i})$$

$$- \bar{\rho}\widetilde{E'u'_j}\}_{,j} \tag{52}$$

Note that, unlike $\overline{u'_i}$, $\overline{E'}$ does not appear explicitly in the Favre-averaged mean energy equation. However, this equation can be converted easily to an equation for $\overline{\rho''E''}$ which appears in the Reynolds-averaged mean equations.

The equations for the Favre-averaged second-order moments can be obtained through the same process used in the derivation of the corresponding Reynolds-a averaged second-order moments. Therefore, only the final form of the equations will be given.

$\bar{\rho}\widetilde{u'_i u'_j}$

$$(\bar{\rho}\widetilde{u'_iu'_j})_{,t} + (\bar{\rho}\widetilde{u}_m\widetilde{u'_iu'_j} + \bar{\rho}\widetilde{u'_iu'_ju'_m})_{,m} + \bar{\rho}(\widetilde{u'_iu'_m}\widetilde{u}_{j,m} + \widetilde{u'_ju'_m}\widetilde{u}_{i,m})$$

$$= -[\overline{(\bar{p} + p'')u'_j}]_{,i} - [\overline{(\bar{p} + p'')u'_i}]_{,j} + \overline{2(\bar{p} + p'')s'_{ij}}$$

$$- \overline{u'_j(\overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}\widetilde{u}_{k,k} + \mu^{*''}u'_{k,k})_{,i}}$$

$$- \overline{u'_i(\overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}\widetilde{u}_{k,k} + \mu^{*''}u'_{k,k})_{,j}}$$

$$+ 2\ \{\overline{u'_j(\overline{\mu}\widetilde{s}_{im} + \overline{\mu}s'_{im} + \mu''\widetilde{s}_{im} + \mu''s'_{im})_{,m}}$$

$$+ \overline{u'_i(\overline{\mu}\widetilde{s}_{jm} + \overline{\mu}s'_{jm} + \mu''\widetilde{s}_{jm} + \mu''s'_{jm})_{,m}}\} \tag{53}$$

$\widetilde{k}$

$$(\bar{\rho}\widetilde{k})_{,t} + (\bar{\rho}\ \widetilde{u}_j\widetilde{k})_{,j} + \bar{\rho}\widetilde{u'_iu'_j}\widetilde{u}_{i,j} + (\bar{\rho}\widetilde{u'_jk})_{,j}$$

$$= -\ [\overline{(\bar{p} + p'')u'_i}]_{,i} + \overline{(\bar{p} + p'')u'_{i,i}}$$

$$
\begin{aligned}
&- \overline{u_i'(\overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}\widetilde{u}_{k,k} + \mu^{*''}u'_{k,k})_{,i}} \\
&+ 2\,\overline{u_i'(\overline{\mu}\widetilde{s}_{ij} + \overline{\mu}s'_{ij} + \mu''\widetilde{s}_{ij} + \mu''s'_{ij})_{,j}}
\end{aligned}
\tag{54}
$$

$\underline{\overline{\rho}\widetilde{E'u'_i}}$

$$
\begin{aligned}
&(\overline{\rho}\widetilde{E'u'_i})_{,t} \;+\; (\overline{\rho}\,\widetilde{u}_i\widetilde{E'u'_i} + \overline{\rho}\widetilde{E'u'_iu'_j})_{,j} \;+\; \overline{\rho}\widetilde{u'_iu'_j}\widetilde{E}_{,j} \;-\; \overline{\rho}\widetilde{E'u'_j}\widetilde{u}_{i,j} \\
&= \;-\; \overline{E'\{(\bar{p} + p'' + \overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}\widetilde{u}_{k,k} + \mu^{*''}\widetilde{u}_{k,k})\delta_{ij}} \\
&\qquad + \overline{2(\overline{\mu}\widetilde{s}_{ij} + \overline{\mu}s'_{ij} + \mu''\widetilde{s}_{ij} + \mu''s'_{ij})\}_{,j}} \\
&\quad - \tilde{u}_{j,j}\overline{u'_i(\bar{p} + p'' + \overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}\widetilde{u}_{k,k} + \mu^{*''}\widetilde{u}_{k,k})} \\
&\quad - \widetilde{u}_j\overline{u'_i(\bar{p} + p'' + \overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}\widetilde{u}_{k,k} + \mu^{*''}\widetilde{u}_{k,k})_{,j}} \\
&\quad - \overline{u'_{j,j}u'_i(\bar{p} + p'' + \overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}\widetilde{u}_{k,k} + \mu^{*''}\widetilde{u}_{k,k})} \\
&\quad - \overline{u'_iu'_j(\bar{p} + p'' + \overline{\mu^*}\widetilde{u}_{k,k} + \overline{\mu^*}u'_{k,k} + \mu^{*''}\widetilde{u}_{k,k} + \mu^{*''}\widetilde{u}_{k,k})_{,j}} \\
&+ 2\,\overline{u'_i(\overline{\mu}\widetilde{s}_{ij}\widetilde{u}_i + \overline{\mu}\widetilde{s}_{ij}u'_i + \overline{\mu}s'_{ij}\widetilde{u}_i + \overline{\mu}s'_{ij}u'_i + \mu''\widetilde{s}_{ij}\widetilde{u}_i + \mu''\widetilde{s}_{ij}u'_i + \mu''s'_{ij}\widetilde{u}_i + \mu''s'_{ij}u'_i)_{,j}} \\
&\qquad\qquad - \overline{q}_{j,j}\overline{u'_i} \;-\; \overline{q''_{j,j}u'_i}
\end{aligned}
\tag{55}
$$

Equations for the third-order moments can be similarly derived, if need arises, and may provide some insights into the physical mechanisms involved. In the mean time, however, we derive moment equations only up to the second order.

3.4 Simplified Equations

Equations for the mean flow and the second-order moments in Reynolds or Favre decompositions derived above are rather complicated compared to their incompressible counterparts. These equations can be greatly simplified by assuming that the correlations of the transport coefficient fluctuations, μ'' and $\mu^{*''}$, and thermodynamic coefficients, C''_v and C''_p, with other turbulent fluctuations are negligible. That is,

$$
\frac{\overline{\mu''\phi''}}{\overline{\mu}\overline{\phi}} \ll 1 \;, \quad \frac{\overline{\mu^{*''}\phi''}}{\overline{\mu^*}\overline{\phi}} \ll 1 \;, \quad \frac{\overline{C''_v\phi''}}{\overline{C_v}\overline{\phi}} \ll 1 \;, \quad \frac{\overline{C''_p\phi''}}{\overline{C_p}\overline{\phi}} \ll 1 \tag{56}
$$

The resulting simplified equations for the mean flow and turbulence moments are given below.

3.4.1 Reynolds-averaged Equations

The Reynolds-averaged mean flow equations, the turbulent kinetic energy equation and the Reynolds-stress equations become

Mean Continuity

$$\overline{\rho}_{,t} \ + \ (\overline{\rho}\ \overline{u}_i + \overline{\rho'' u''_i})_{,i} \ = \ 0 \tag{57}$$

Mean Momentum

$$(\overline{\rho}\ \overline{u}_{i,t} + \overline{\rho'' u''_i})_{,t} + \ (\overline{\rho}\ \overline{u}_i \overline{u}_j + \overline{u}_i \overline{\rho'' u''_j} + \overline{\rho} \overline{u''_i u''_j} + \overline{\rho'' u''_i u''_j} + \overline{u}_j \overline{\rho'' u''_i})_{,j}$$

$$= \ \{-\overline{p}\ \delta_{ij} \ + \ \overline{\tau}_{ij}\}_{,j} \tag{58}$$

Mean Energy

$$(\overline{\rho}\overline{E} + \overline{\rho'' E''})_{,t} \ + \ (\overline{\rho}\overline{E}\overline{u}_i + \overline{\rho}\overline{E'' u''_i} + \overline{u}_i \overline{\rho'' E''} + \overline{E}\ \overline{\rho'' u''_i} + \overline{\rho'' E'' u''_i})_{,i}$$

$$= \ \{-(\overline{q_i} + \overline{p}\ \overline{u}_i + \overline{p'' u''_i}) \ + \ \overline{\tau}_{ij}\ \overline{u}_j \ + \ \overline{\tau''_{ij} u''_j}\ \}_{,i} \tag{59}$$

where

$$\overline{E} \ = \ \overline{C_v}\ \overline{T} \ + \ \frac{1}{2}\ (\overline{u}_i \overline{u}_i + \overline{u''_i u''_i}),$$

and

$$\overline{\tau}_{ij} \ = \ 2\ \overline{\mu}\ \overline{s}_{ij} \ - \ \overline{\mu^*}\ \overline{u}_{k,k}\delta_{ij}, \qquad \tau''_{ij} \ = \ 2\ \overline{\mu}\ s''_{ij} \ - \ \overline{\mu^*}\ u''_{k,k}\delta_{ij}$$

Equation of State

$$\overline{p} \ = \ \overline{\rho} R \overline{T} \ + \ R\overline{\rho'' T''} \tag{60}$$

$\overline{\rho} \overline{u''_i u''_j}$

$$(\overline{\rho}\overline{u''_i u''_j})_{,t} \ + \ (\overline{\rho}\ \overline{u}_m \overline{u''_i u''_j})_{,m} + (\overline{\rho'' u''_i u''_j})_{,t} \ + \ \overline{\rho'' u''_j}\overline{u}_{i,t} \ + \ \overline{\rho'' u''_i}\overline{u}_{j,t}$$

$$= \ \mathbf{P}^R_{ij} \ + \ \mathbf{T}^R_{ijm,m} \ + \ \mathbf{\Pi}^R_{ij} \ + \ \epsilon^R_{ij} \tag{61}$$

where

$$\begin{aligned}\mathbf{P}^R_{ij} \ &= \ - \ \{\ (\overline{\rho}\overline{u''_j u''_m} \ + \ \overline{u}_m \overline{\rho'' u''_j} \ + \ \overline{\rho'' u''_j u''_m})\ \overline{u}_{i,m} \\ &\qquad + \ (\overline{\rho}\overline{u''_i u''_m} \ + \ \overline{u}_m \overline{\rho'' u''_i} \ + \ \overline{\rho'' u''_i u''_m})\ \overline{u}_{j,m}\ \}\end{aligned}$$

$$\begin{aligned}\mathbf{T}^R_{ijm} \ &= \ - \ \{\ \overline{\rho}\overline{u''_i u''_j u''_m} \ + \ \overline{u}_m \overline{\rho'' u''_i u''_j} \ + \ \overline{\rho'' u''_i u''_j u''_m} \ + \ \overline{p'' u''_j}\delta_{im} \ + \ \overline{p'' u''_i}\delta_{jm} \\ &\qquad - \ \overline{\tau''_{im} u''_j} \ - \ \overline{\tau''_{jm} u''_i}\ \}\end{aligned}$$

$$\Pi^R_{ij} \ = \ 2\ \overline{p'' s''_{ij}}$$

$$\epsilon^R_{ij} \ = \ - \ \overline{\tau''_{im} u''_{j,m}} \ - \ \overline{\tau''_{jm} u''_{i,m}}$$

$\underline{\overline{k}}$

$$(\overline{\rho k})_{,t} + (\overline{\rho}\,\overline{u}_j \overline{k})_{,j} + (\overline{\rho'' k})_{,t} + \overline{\rho'' u_i''}\,\overline{u}_{i,t} = \mathbf{P}^R + \mathbf{T}^R_{j,j} + \mathbf{\Pi}^R + \epsilon^R \quad (62)$$

where

$$\mathbf{P}^R = (- \overline{\rho}\overline{u_i'' u_j''} - \overline{u}_j \overline{\rho'' u_i''} - \overline{\rho'' u_i'' u_j''})\, \overline{u}_{i,j}$$

$$\mathbf{T}^R_j = - (\overline{\rho}\overline{u_j'' k} + \overline{u}_j \overline{\rho'' k} + \overline{\rho'' u_j'' k} + \overline{p'' u_i''}\delta_{ij} - \overline{\tau_{ij}'' u_i''})$$

$$\mathbf{\Pi}^R = \overline{p'' u_{k,k}''}$$

$$\epsilon^R = - \overline{\tau_{ij}'' u_{i,j}''}$$

The mean flow is described by the following quantities :

$$\{\overline{\rho} \ , \ \overline{u}_i \ , \ \overline{p} \ , \ \overline{T}\} \quad (63)$$

Quantities that need to be known to solve these equations are

$$\overline{\rho'' u_i''} \ , \ \overline{u_i'' u_j''} \ , \ \overline{\rho'' u_i'' u_j''} \ , \ \overline{\rho'' E''} \ , \ \overline{E'' u_i''} \ , \ \overline{\rho'' E'' u_i''} \ , \ \overline{p'' u_i''} \ ,$$

$$\overline{u_{k,k}'' u_i''} \ , \ \overline{u_j'' s_{ij}''} \ , \ \overline{\rho'' T''} \ , \ \overline{\mu^*} \ , \ \overline{\mu} \ , \ \overline{C_v} \quad (64)$$

3.4.2 Favre-averaged Equations

The simplified equations for Favre-averaged quantities are

Mean Continuity

$$\overline{\rho}_{,t} + (\overline{\rho}\widetilde{u}_i)_{,i} = 0 \quad (65)$$

Mean Momentum

$$(\overline{\rho}\widetilde{u}_i)_{,t} + (\overline{\rho}\widetilde{u}_i\widetilde{u}_j + \overline{\rho}\widetilde{u_i' u_j'})_{,j} = \{- \overline{p}\,\delta_{ij} + \widetilde{\tau}_{ij} + \overline{\tau_{ij}'}\}_{,j} \quad (66)$$

Mean Energy

$$(\overline{\rho}\widetilde{E})_{,t} + (\overline{\rho}\widetilde{E}\widetilde{u}_i + \overline{\rho}\widetilde{E' u_i'})_{,i}$$

$$= \{-(\overline{q_i} + \overline{p}\widetilde{u}_i + \overline{p}\overline{u_i'} + \overline{p'' u_i'}) + \widetilde{\tau}_{ij}\widetilde{u}_j + \overline{\tau_{ij}'}\widetilde{u}_j + \widetilde{\tau}_{ij}\overline{u_j'} + \overline{\tau_{ij}' u_j'} \}_{,i} \quad (67)$$

where

$$\widetilde{E} = \overline{C_v}\widetilde{T} + \frac{1}{2} (\widetilde{u}_i\widetilde{u}_i + \widetilde{u_i' u_i'})$$

and

$$\widetilde{\tau}_{ij} = 2\,\overline{\mu}\,\widetilde{s}_{ij} - \overline{\mu^*}\,\widetilde{u}_{k,k}\,\delta_{ij}, \quad \tau_{ij}' = 2\,\overline{\mu}\,s_{ij}' - \overline{\mu^*}\,u_{k,k}'\,\delta_{ij}$$

Equation of State

$$\overline{p} = \overline{\rho} \, R \, \widetilde{T} \tag{68}$$

$\overline{\rho}\widetilde{u_i' u_j'}$

$$(\overline{\rho}\widetilde{u_i' u_j'})_{,t} + (\overline{\rho}\widetilde{u}_m \widetilde{u_i' u_j'})_{,m}$$

$$= \mathbf{P}_{ij}^F + \mathbf{T}_{ijm,m}^F + \mathbf{\Pi}_{ij}^F + \epsilon_{ij}^F - \overline{p}_{,j}\overline{u_i'} - \overline{p}_{,i}\overline{u_j'} + \widetilde{\tau}_{im,m}\overline{u_j'} + \widetilde{\tau}_{jm,m}\overline{u_i'} \tag{69}$$

where

$$\mathbf{P}_{ij}^F = -\, \overline{\rho} \, (\, \widetilde{u_i' u_m'}\widetilde{u}_{j,m} + \widetilde{u_j' u_m'}\widetilde{u}_{i,m} \,)$$

$$\mathbf{T}_{ijm}^F = -\, \{ \, \overline{\rho}\widetilde{u_i' u_j' u_m'} + \overline{p'' u_j'}\delta_{im} + \overline{p'' u_i'}\delta_{jm} - \overline{\tau_{im}' u_j'} - \overline{\tau_{jm}' u_i'} \, \}$$

$$\mathbf{\Pi}_{ij}^F = 2 \, \overline{p'' s_{ij}'}$$

$$\epsilon_{ij}^F = -\, \overline{\tau_{im}' u_{j,m}'} - \overline{\tau_{jm}' u_{i,m}'}$$

$\widetilde{k}$

$$(\overline{\rho}\widetilde{k})_{,t} + (\overline{\rho} \, \widetilde{u}_j \widetilde{k})_{,j} = \mathbf{P}^F + \mathbf{T}_{j,j}^F + \mathbf{\Pi}^F + \epsilon^F - \overline{p}_{,i}\overline{u_i'} + \widetilde{\tau}_{ij,j}\overline{u_i'} \tag{70}$$

where

$$\mathbf{P}^F = -\, \overline{\rho} \, \widetilde{u_i' u_j'} \, \widetilde{u}_{i,j}$$

$$\mathbf{T}_j^F = -\, \{ \, \overline{\rho}\widetilde{u_j' k} + \overline{p'' u_j'} - \overline{\tau_{ij}' u_i'} \, \}$$

$$\mathbf{\Pi}^F = \overline{p'' u_{i,i}'}$$

$$\epsilon^F = -\, \overline{\tau_{ij}' u_{i,j}'}$$

Since,

$$\overline{\psi'' \phi'} = \overline{\psi'' \phi''} \tag{11}$$

$$\mathbf{\Pi}_{ij}^F = \mathbf{\Pi}_{ij}^R \quad \text{and} \quad \mathbf{\Pi}^F = \mathbf{\Pi}^R \tag{71}$$

i.e. the pressure-strain terms in the Reynolds and Favre averages are identical. Moreover, if the viscous stress tensors are expressed in the form of Reynolds-averaged rates of deformation tensors, the following expressions can be derived

$$\epsilon_{ij}^F = \epsilon_{ij}^R \quad \text{and} \quad \epsilon^F = \epsilon^R \tag{72}$$

These equations may be used to solve for the mean quantities:

$$\{\, \overline{\rho} \;,\; \widetilde{u}_i \;,\; \overline{p} \;,\; \widetilde{T} \} \tag{73}$$

The unknowns are

$$\widetilde{u_i' u_j'} \;,\; \widetilde{E' u_i'} \;,\; \overline{u_i'} \;,$$
$$\overline{p'' u_i'} \;,\; \overline{u_{k,k}' u_i'} \;,\; \overline{s_{ij}' u_j'} \;,\; \overline{\mu^*} \;,\; \overline{\mu} \;,\; \overline{C_v} \tag{74}$$

4. Some Observations

The equations governing the mean motions of compressible turbulent flows have been derived. The mean motions were defined in two ways. Firstly, the mean is taken as the ensemble-averaged mean of instantaneous quantities. The mean motions are thus described by the mean density, velocity, temperature, etc. This is also called Reynolds average or Reynolds decomposition. The second averaging method, Favre-averaging, is a mass-weighted average. The mean flow is described by the mass-weighted quantities. Regardless of the decomposition that was used, quite a few correlations or moments of turbulent fluctuations have to be known to close the equations governing the mean flow.

Due to the nonlinear terms in the Navier-Stokes equations, such as the convective terms in the momentum equations and the surface-force work terms in the energy equation, an equation for an n^{th}-order moment always contains correlations of order higher than n. All these terms in the turbulent moment equations have to be modeled so that the mean flow and the turbulence equations form a closed system with appropriate boundary and/or initial conditions. Second-order modeling efforts involve solutions of the mean flow and the second order correlations that appear in the mean equations through rational modeling techniques. In the following, the equations for compressible turbulent flows obtained by using the Reynolds average and the Favre average will be examined.

It should be noted that the form of the Reynolds-averaged turbulent mean flow equations is not the same as that of the compressible Navier-Stokes equations. The equations for the turbulent mean flow contain additional turbulent correlations due to the turbulent density fluctuation such as $\overline{\rho''\phi''}$ and $\overline{\rho'' u_i'' \phi''}$ and the turbulent fluctuation of transport coefficients such as $\overline{\mu^{*\prime\prime} u_{k,k}''}$ and $\overline{\mu^{*\prime\prime} S_{ij}''}$. As was mentioned earlier, the correlations of the fluctuating transport coefficients may be negligible in most problems. This, however, is not necessarily the case for the correlations of the density fluctuation. Density fluctuation terms such as $\overline{\rho'' u_i''}$ and $\overline{\rho'' E'' u_i''}$ in the Reynolds-averaged mean equations may be important and have to be modeled, Varma et al. (1974). Since these moments of density fluctuation do not appear in the equations for incompressible turbulence, there are no incompressible baseline models that modeling of these terms can be built upon. Therefore, the Reynolds-averaged equations are seldom used in the calculations of compressible turbulent flows.

Two procedures that can be used to obtain transport equations for the density correlation terms were proposed in the previous sections. These equations may be used to develop modeled transport equations for these moments of density fluctuation. Note that the average turbulent mass flux $\overline{\rho'' u_i''}$ are different from the mass flux fluctuation, $(\rho u_i)''$, since

$$(\rho u_i)'' \;=\; \bar{\rho} u_i'' \;+\; \rho'' \bar{u}_i \;+\; \rho'' u_i'' \;-\; \overline{\rho'' u_i''} \tag{75}$$

On the other hand, Favre averaging eliminates moments of the density fluctuation and the resulting equations bear much more resemblance to those in incompressible flows. This is attributable to the use of mass-weighted quantities : $\tilde{\phi}$, since

$$\overline{\rho \phi'} \;=\; 0 \qquad \text{where} \qquad \phi' \;=\; \phi \;-\; \tilde{\phi} \tag{11}$$

In light of this analogy and Morkovin's hypothesis that the turbulence structure is unaffected by compressibility as long as the fluctuation Mach number is much less than unity, there was common optimism for the extension of incompressible models to high Mach number flows. This practice has enjoyed a considerable success in the calculations of wall shear layers in the past. The success, however, was not shared by the predictions of high speed free shear layers, even when the conditions satisfied Morkovin's hypothesis. The extension would also fail in the presence of shock waves and expansion fans where the dilatation is large. One prominent feature that extension models fail to predict is the reduction of the growth rate of compressible free mixing layers as the Mach number increases. Oh (1974) considered the pressure dilatation terms in the turbulent kinetic energy equation. These terms were modeled using an eddy shock wave hypothesis. Oh argued that the convecting turbulence structures in free shear flows provide the key mechanism to produce pressure-dilatation and predicted the reduced growth rate of high speed mixing layers.

The Favre-averaged mean equations can be reduced to a form similar to that of the Navier-Stokes equations in Favre-averaged mean quantities. With an eddy-viscosity type of model, an operational compressible Navier-Stokes solver can then be extended to include "turbulence" by simply (1) neglecting $\overline{u_i''}$, $\overline{u'_{k,k}}$ and $\overline{s'_{ij}}$ and (2) replacing the molecular viscosity by an equivalent turbulent eddy viscosity without modifying the main structure of the codes. This is generally the case in the implementation of eddy-viscosity turbulence models to existing CFD codes. It should be remembered, however, that the solutions thus obtained are mass-averaged quantities, rather than ensemble-averaged ones, since

$$\tilde{\phi} - \overline{\phi} = \overline{\phi'} = -\frac{\overline{\rho''\phi''}}{\overline{\rho}} \tag{11}$$

$$\overline{\rho}\widetilde{u'_i u'_j} - \overline{\rho}\,\overline{u''_i u''_j} = \overline{\rho'' u''_i u''_j} - \frac{\overline{\rho'' u''_i} \cdot \overline{\rho'' u''_j}}{\overline{\rho}} \tag{76}$$

The differences between Reynolds- and Favre-averaged turbulent quantities may be less than 10% for low supersonic flows. Nevertheless, it may not be negligible for flows of higher speed, say for M $\gg$ 5. Therefore, it is essential to be consistent not only in the equations that are used but also in the comparison of results. For instance, Favre-averaged turbulence equations must be used with the Favre-averaged mean equations and the solutions be converted to Reynolds-averaged quantities to compare with ensemble averaged experimental data.

Note that $\overline{\phi'} \neq 0$ and the Favre-decomposition looses its advantage over the Reynolds-decomposition in cases in which density does not appear, for instance, the surface force terms in the mean momentum and energy equations. There are a variety of ways to handle these moments. Rubesin (1990), Vandromme et al. (1983), Speziale et al. (1991) , among others, simply neglected $\overline{S'_{ij}}$ and $\overline{u'_{k,k}}$ terms in the mean momentum equations. That is,

$$\overline{\tau}_{ij} = 2\overline{\mu}\widetilde{S}_{ij} - \overline{\mu^*}\,\tilde{u}_{k,k}\delta_{ij} \tag{77}$$

Sarkar and Balakrishnan (1990), instead, use Reynolds decomposed velocity in the constitutive relation

$$\overline{\tau}_{ij} = 2\overline{\mu}\overline{S}_{ij} - \overline{\mu^*}\,\overline{u}_{k,k}\delta_{ij} \tag{78}$$

In this case, a model is needed to convert Favre-decomposed velocity to Reynolds-decomposed velocity, since

$$\overline{u}_i - \tilde{u}_i = -\frac{\overline{\rho'' u''_i}}{\overline{\rho}} \tag{79}$$

The surface force terms in the mean energy equation can be similarly simplified. Rubesin (1990) argued that the $\overline{S'_{ij}}, \overline{u'_{k,k}}$ and $\overline{u'_i}$ terms in the equation can simply be omitted as in the mean momentum equations. Sarkar and Balakrishnan (1990) used Reynolds-decomposition in all the surface-force work terms. Speziale and Sarkar (1991) used Reynolds-averaged velocity in the shear stress and Favre-averaged quantities otherwise. All but Rubesin's (1990) proposals used a model for

$\overline{u_i'}$. Sarkar and Balakrishnan (1990) and Speziale and Sarkar (1990) used a gradient transport expansions, i.e.,

$$\overline{u_i'} = - \frac{\overline{\rho'' u_i''}}{\overline{\rho}} \sim \overline{\rho}_{,i} \tag{80}$$

A more refined analysis can be found in Taulbee (1991), in which a modeled transport equation was used for the mass flux, $\overline{u_i'}$.

Note that there has been a lack of consensus in the identification of the compressibility effects on turbulence, quite aside from the modeling of compressible turbulent flows. For example, Nichols (1990) and Grasso and Speziale (1989) calculated separated flows and considered the effects of compressibility in the turbulent kinetic energy equation. They chose completely different terms in the equation to represent the dominant effects of compressibility. Nichols (1990) argued that the "turbulent velocity-density dissipation" terms, $2\bar{u}_i\overline{\rho'' u_j''}\ \overline{S}_{ij}$, are the most significant terms; Grasso and Speziale (1989), on the other hand, proposed that the pressure-dilatation terms, $\overline{p''\frac{\partial u_j''}{\partial x_j}}$, and the product of the pressure-gradient and mass fluctuations, $\frac{\overline{\rho'' u_j''}}{\bar{\rho}}\frac{\partial \overline{p}}{\partial x_j}$, play a strong role in high speed flows with adverse pressure gradients and could not be disregarded. Both Grasso and Speziale (1989) and Nichols (1990) used a 1D isentropic approximation to model the relations between the density and velocity fluctuations. Nichols (1990) applied his model to a variety of flows, which included a supersonic compression ramp flow similar to that in Grasso and Speziale (1989). For the compressible ramp flow, both models obtained some degree of sucess in the prediction of the skin friction coefficients and it is difficult to identify the dominant compressibility effects.

5. Recent Advances

Recently a rather new concept called dilatational dissipation was proposed by Sarkar et al. (1990) and Zeman (1990). For $\mu_{bulk} = 0$ and $\mu = constant$, the dissipation rate

$$\bar{\rho}\,\epsilon = \overline{\tau_{ij}'' u_{i,j}''} \tag{81}$$

can be represented by

$$\bar{\rho}\epsilon = \overline{\mu}\,(\,\overline{\omega_i''\omega_i''} + \frac{4}{3}\,\overline{(u_{k,k}'')^2}\,) = \epsilon_s + \epsilon_c \tag{82}$$

where ω_i is the vorticity vector and ϵ_s and ϵ_c denote the solenoidal and the dilatational part of the turbulent dissipation. The newly identified dilatational dissipation term accounts for the viscous dissipation of turbulent kinetic energy due to volume fluctuations. Both Sarkar et al. (1990) and Zeman (1990) argued that the solenoidal part of the dissipation assumed its standard form as in incompressible cases and that the dilatational dissipation was proportional to the square of the Mach number. Zeman's (1990) analysis was based on the existence of turbulent eddy shocklets in high speed mixing layers. On the other hand, Sarkar et al. (1990) used an asymptotic analysis. Model predictions from both work were very encouraging in the prediction of the reduced growth rates of compressible mixing layers as a function of the convective Mach number. Dilatational dissipation appears to be among the direct consequency of compressibility effects and has to be properly accounted for in modeling compressible turbulence.

Sarkar et al. (1990) indicated that ϵ_s is rather insensitive to the change of compressible indicators in moderate Mach number turbulence and modeled ϵ_s using a traditional incompressible form. For other cases such as for flows of higher Mach number a refined model may be needed. An equation of ϵ_s $(\equiv \overline{\overline{\mu}\omega_i''\omega_i''})$ can be obtained by subtracting the equation for $\overline{\omega}_i\overline{\omega}_i$ from the $\overline{\omega_i\omega_i}$ equation. The resulting equation is

$$
\begin{aligned}
\epsilon_{s,t} + \overline{u}_j\epsilon_{s,j} &= -(\overline{u_j''\epsilon_s})_{,j} - \overline{\epsilon_s u_{j,j}''} - 2\overline{\mu}\,\overline{\omega}_i\overline{\omega_i''u_{j,j}''} \\
&- 2\overline{\mu}\overline{u_j''\omega_i''}\overline{\omega}_{i,j} + 2\overline{\mu}\,\overline{\omega}_j\overline{\omega_i''u_{i,j}''} + 2\overline{\mu}\overline{\omega_i''\omega_j''}\overline{u}_{i,j} - 2\epsilon_s\overline{u}_{j,j} + 2\overline{\mu}\overline{\omega_i''\omega_j''u_{i,j}''} \\
&+ 4\,\overline{\mu}\,\{\,\overline{r_{kj}''T_{,j}''}\,\overline{S}_{,k} + \overline{r_{kj}''S_{,k}''}\,\overline{T}_{,j} + \overline{r_{kj}''T_{,j}''S_{,k}''} \\
&+ \frac{1}{\overline{\rho}}\overline{r_{kj}''\tau_{kp,pj}''} - \frac{\overline{\rho''r_{kj}''}}{\overline{\rho}^2}\overline{\tau}_{kp,pj} - \frac{1}{\overline{\rho}^2}\overline{r_{kj}''\rho''\tau_{kp,pj}''} \\
&+ (\frac{1}{\overline{\rho}})_{,j}\,\overline{r_{kj}''\tau_{kp,p}''} - \overline{r_{kj}''(\frac{\rho''}{\overline{\rho}^2})_{,j}}\,\overline{\tau}_{kp,p} - \overline{r_{kj}''(\frac{\rho''}{\overline{\rho}^2})_{,j}\tau_{kp,p}''}\,\}
\end{aligned}
\qquad (83)
$$

where

$$r_{kj} = \frac{1}{2}\,(\,u_{k,j} - u_{j,k})$$

and S denotes entropy. The approximation (45) was also invoked. The equation suggests that the dynamics of ϵ_s in compressible turbulence may be very different from that of the turbulent dissipation rate in incompressible turbulence. The equation also indicates that in compressible turbulence, the effects of the thermodynamic states of the flow system may be important to the evolution of ϵ_s.

Another school of though on compressibility effects focuses on the changes of turbulence structures at high Mach numbers. It was proposed that turbulent energetics evolve with preferred modes of perturbations as the Mach number, or compressibility effects, increases. These energetic turbulent structures were interpreted as physical manifestation of evolving instability. The selective amplifying effects of compressibility were mentioned in Sandham and Reynolds (1987), Ragab and Wu (1989), Lele (1989) and Morkovin (1990). Due to the communicability problem between interacting elements in supersonic flows, structures that are highly efficient in extracting energy from the mean flow at low Mach numbers no longer prevail as the Mach number increases. They are replaced by structures associated with the modes generated by compressibility effects. Therefore, the quasi-2D vortical structures in low speed mixing layers may be more susceptible to the communicability problem than the highly 3D structures in bounded shear flows. Linear stability analysis also shows that for even higher M_c, the amplified modes travel supersonically and obliquely and thus may generate only limited mixing. Morris et al (1990) applied linear analysis and predicted quantitatively the reduced growth of supersonic free mixing layers without any adjustment of operating conditions. Liou (1990) developed three closure models to predict the evolution of incompressible mixing layers at the large scale using a weakley nonlinear theory. The models predicted the mean flow field as well as the unsteady large-scale turbulent motions. This approach is rather unique and appears to be very promising in predicting the unsteadiness of turbulent structures. Note that a reasonable modeling of the unsteadiness and spatial intermittence of turbulent structures is essential in many areas that are related to high speed turbulent flow calculations. These include, for example, the predictions of high speed jet noise and chemically reacting flows. Therefore, this approach appears to be complementary rather than contradictory to the traditional second-order modeling techniques.

6. Initiative to Develop New Second-Order Compressible Models

To develop second-order models for compressible turbulence, a reasonable first step is probably to distinguish compressibility terms from incompressible background. This is what Zeman (1990) and Sarkar et al. (1990) have performed. The dilatation dissipation they identified appears to be an important mechanism in compressible turbulence. A successful development of a compressible second-order model may inevitably invoke techniques and methodologies that were proved useful in the modeling of incompressible turbulence. The techniques and methodologies, however, have to be tuned in to the experimental observations and the theoretical constrains of compressible turbulence. For example, one of the consequences of compressibility effects is the finite speed of the propagation of information. In supersonic flows, modulation of flow properties occurs only within Mach cones of influence with acoustic time delay. This feature introduces additional scales that may have to be included in the modeling of compressible flows.

It can also be observed from the turbulent equations that there will be exchanges between the turbulent kinetic energy and thermoenergy through rather different routes than in incompressible flows. In compressible turbulent flows of high fluctuating Mach numbers, eddy shocklets may begin to appear. Lumley indicated (private communication) that once the eddy shocklets appear in any significant numbers, a quite different approach, "... one which centers on entropy fluctuations, and which concentrates on the rather large changes that take place in the shocklets..." is needed. Corresponding terms in the equations may need to be identified and properly modeled. These unique features, among others, may have to be considered explicitly in the development of new compressible turbulence models.

Compressible turbulence modeling is still in its infancy. This seems to be true both theoretically and computationally. Therefore, it is prudent to have a open mind in the pursuit of rational, accurate and physically sound models.

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13. ABSTRACT *(Maximum 200 words)*
Equations for the mean and the turbulent quantities for compressible turbulent flows are derived in this report. Both the conventional Reynolds average and the mass-weighted, Favre average were employed to decompose the flow variable into a mean and a turbulent quantity. These equations are to be used later in developing second-order Reynolds stress models for high-speed compressible flows. A few recent advances in modeling some of the terms in the equations due to compressibility effects are also summarized.

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