NASA Technical Memorandum 105315
ICOMP-91-21

IN-39
53174
P.47

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November 1991

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INSTITUTE FOR COMPUTATIONAL MECHANICS IN PROPULSION

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SUMMARY

In various problems in structural dynamics, the eigenvalues of a linear system depend on a characteristic parameter of the system. Under certain conditions, two eigenvalues of the system approach each other as the characteristic parameter is varied, leading to modal interaction. In a system with "conservative coupling", the two eigenvalues eventually repel each other, leading to the curve veering effect. In a system with "non-conservative coupling", the eigenvalues continue to attract each other, eventually colliding, leading to eigenvalue degeneracy. We study modal interaction in linear systems with conservative and non-conservative coupling using singularity theory, sometimes known as catastrophe theory. Our main result is this: eigenvalue degeneracy is a cause of instability; in systems with conservative coupling it induces only *geometric* instability, whereas in systems with non-conservative coupling eigenvalue degeneracy induces both *geometric* and *elastic* instability. Illustrative examples of mechanical systems are given.

1. INTRODUCTION

This is an expository paper on the application of catastrophe theory, sometimes known as singularity theory, to the stability analyses of linear vibrating systems. One hopes that experts in vibration theory may see from here how the relatively new mathematical ideas in catastrophe theory provide insight into the qualitative, dynamical behavior of engineering structures, while experts in catastrophe theory may learn of yet another way in which their mathematical theories are applicable in engineering analysis. The principal objective in this effort is to investigate in a qualitative sense the stability of linear vibrating systems near degenerate modes.

In vibration analysis, "degenerate modes" means the coincidence of two or more eigenvalues. The most common kind of degeneracy encountered is called "double modes", when only two eigenvalues are equal. The problem of modal interaction considered in this paper is different from problems on "parametric resonance", or matters related to "Arnol'd tongues" which have been addressed in the *nonlinear* dynamics literature. In this paper, we are essentially concerned with the stability properties of *linear* or linearizable vibrating systems. Linear models are often applied in engineering analysis. Experience has shown that the results obtained from such linear models generally compare well with experimental observations, when motion takes place in a small neighborhood of the equilibrium position.

A broad range of concepts and terminology is required in this exposition. Since many of these may be unfamiliar to several readers, an introductory discussion is presented in the remainder of this Section, and also in Section 2. A selection of pertinent literature is given in alphabetic author sequence, [1-48]. In carrying out our principal objective—namely, an investigation of the stability of linear vibrating systems near degenerate modes—we shall combine two important ideas: the first, concerning the nature of coupling in vibrating systems, is due to Crandall; see Crandall and Mroszczyk [23] for a description. The second idea, arising from catastrophe theory, concerns the lack of "structural stability" of degenerate objects, and is due to Thom [42].

1.1 Conservative and Non-conservative Coupling

Suppose a vibrating system has two eigenvalues which depend on a system parameter α. As this parameter is varied, the two eigenvalues approach each other. When they get very close, they could

(a) *repel* each other, leading to curve veering, Figure 1(a); or,

(b) *attract* each other, eventually collide, as in Figure 1 (b), leading to degenerate eigenvalues at the moment of collision.

In the terminology of Crandall, vibrating systems exhibiting the loci repulsion of Figure

1 (a) are said to have "conservative coupling", whereas those exhibiting the attraction of Figure 1 (b) are said to have "non-conservative coupling".

1.2 Structural Stability, Elastic Stability, Geometric Stability

By "structural stability" in the foregoing paragraphs, we mean the concept developed by Thom [42] to describe the behavior of various objects under small perturbations of the objects, or the environment in which the objects are situated. Here, the term "object" may mean a variety of mathematical entities: curves, surfaces, or manifolds; functions; vector fields; etc. Since such objects are widely encountered in engineering analysis, an understanding of their structural stability is important. According to Thom [42, p. 14]

> "The concept of structural stability seems to me to be a key idea in the interpretation of phenomena in all branches of science (except, perhaps, quantum mechanics) ... ".

In the terminology of Thom, an object is said to have "structural stability" if a small perturbation leads to only a small *qualitative* change in the behavior of the object. Conversely, an object lacks structural stability if a very small perturbation changes its qualitative behavior in a very drastic manner.

In structural dynamics, the word "stability" without a qualifier usually means *elastic stability*, i.e. a special kind of structural stability where only the *eigenvalues* play a decisive role. Thus, an engineering structure is said to be "stable" if all of its eigenvalues lie in a certain region of the Nyquist plane. But there is another kind of instability which is not as well-known in engineering dynamics. Although this instability does not lead to the collapse of a structure, it is nevertheless a structural instability in the sense of Thom. Here, and in other recent works by the present author, it is called *geometric instability*; it is a "structural stability" where the *eigenvectors* play the decisive role. The idea of differentiating between elastic and geometric stability in engineering dynamics was introduced by Afolabi [6], and is discussed in detail elsewhere (Afolabi [5]).

1.3 Dependence of a Vibrating System on Parameters

It is well-known that the eigenvalues of many engineering systems have a dependence on one characteristic parameter. We shall refer to such systems as *one-parameter* systems.

Systems depending on more than one parameter are also encountered, but such multi-parameter systems are not discussed here.

In a one-parameter system, it is often easy to identify the "parameter". We cite a few examples.

(a) In the case of a rotating mechanical system, the eigenvalues may vary with the speed of rotation, where *rotation speed* is the parameter (e.g. Crandall and Dugundji, [21]).

(b) In aeroelastic systems, the eigenvalues typically are dependent on the relative speed of air flow as measured by the mach number; one may thus take *mach number* as the parameter (e.g. Dowell, [25]).

(c) In nominally periodic systems, small imperfections known as mistuning are unavoidable in practice; in such systems, a measure of *mistuning* may be used as the system parameter (e.g. Ewins, [26]).

(d) In a mechanical system consisting of a pipe conveying a fluid, its eigenvalues may vary with the speed of the flowing fluid; in this case the *fluid velocity* may be chosen as the parameter (e.g. Benjamin, [16]).

Several other examples of parametric dependence may be cited in the engineering literature.

In a one-parameter system, certain values of the parameter are called "critical", in the sense that some unusual (and sometimes undesirable) dynamics take place in the neighborhood of those critical values of the parameter. Thus, at the *critical speed* of a rotating machine, or the *critical velocity* of a fluid in a flexible pipe, complicated dynamical effects are usually encountered. In this paper, it is shown that such unusual dynamics is often induced by eigenvalue degeneracy, and modal interaction near the degeneracy. It is also shown that systems with conservative coupling have modal interaction leading to geometric instability only, whereas systems with non-conservative coupling have modal interaction leading to both elastic and geometric instabilities.

In arriving at the above result, we applied Thom's transversality theorem to show that when two systems depending on a common parameter are coupled, their "bifurcation diagrams" may take one of the types shown in Figures 1 (a) and (b), which are associated with Crandall's conservative and non-conservative coupling. Thom's transversality theorem (see the Appendix) is one of the foundations of catastrophe theory. Because some readers of this paper are not likely to be familiar with details of catastrophe theory, we present in the next section a review of those aspects of the theory which are necessary for explaining our results. Readers who are familiar with catastrophe theory, or those who are not interested in details of it, may proceed to section 3.

2. CATASTROPHE THEORY

One of the most interesting and relatively recent developments in mathematics is that due to Thom [42], and named by Zeeman [47] as *catastrophe theory*. Certain closely related mathematical concepts, which have been developed mainly by Arnol'd *et al* [8], are known as *singularity theory*. In its early days, catastrophe theory met with some criticism; see, for instance, Sussman and Zahler [41], and the exchange of correspondence in the journal *Nature*, [46]. The criticism, and the resulting controversy, centered on the application of catastrophe theory in the social sciences. Over the years, however, catastrophe theory has matured into a sound mathematical theory. A discussion of some of its applications in the physical sciences and mathematics has been carried out by Zeeman [47], Poston and Stewart [39], Gilmore [27], and Thompson and Hunt [44], among others. Although there are many applications in physics (see, for instance, Berry [15]) there are surprisingly only a few in engineering. Those few applications in engineering are mentioned below.

2.1 Applications of Catastrophe Theory in Engineering

One of the earliest applications of catastrophe theory in engineering is the paper by Holmes and Rand [31]. They investigated the phase portraits and bifurcation characteristics of a single degree of freedom nonlinear oscillator, based on Duffing's equation. Subsequently, this application to *nonlinear* single degree of freedom system was extended and forms, along with other ideas from dynamical systems theory by various workers, the basis of what is now known as "chaos", or chaotic dynamics; see Guckenheimer and Holmes [29] for more details.

Thompson and Hunt [44] have written extensively on the application of catastrophe theory in the analysis of civil engineering structures. Again, their treatment is based on the existence of *nonlinearity* in the system. They treated static, gradient systems predominantly.

With respect to vibration analysis, only one degree of freedom nonlinear systems are usually treated in catastrophe theory and chaos. The implicit assumption has been that catastrophe theory is applicable only to nonlinear systems. It is only in recent years that Afolabi [4-6] has shown that catastrophe theory is also applicable to linear vibrating systems. In this paper, we apply catastrophe theory to *linear* vibrating systems with two degrees of freedom. This application to linear systems is admissible if we go to the fundamentals, and apply the transversality theorem directly. The present work is, therefore, different from previous applications of catastrophe theory in engineering mechanics by other investigators, where only nonlinear systems have been treated.

2.2 Degenerate and Generic Objects

In catastrophe theory, many concepts are formulated in terms of "objects" which, in applications, may represent a variety of things: smooth curves or manifolds, matrices, vector fields, etc. This is common in the branch of mathematics known as category theory. It is then important to distinguish between two types of objects: *degenerate* and *generic*. Generic objects are those that are encountered most frequently, and are said to be in "general position"; they have structural stability in the sense of Thom. On the other hand, degenerate objects are in exceptional position, and lack structural stability.

A degenerate object lacks structural stability because, under an arbitrarily small change in its parameter, the object exhibits a qualitatively different behavior. The object is then said to undergo a bifurcation, since the simplest type of degeneracy induces a breaking into *two* qualitatively different types of behavior after a perturbation.

2.3 Singularity Theory, Catastrophe Theory, Bifurcation Theory

Often, the characteristics of an object are investigated in a "parameter space", where the various parameters on which the object depends are the bases. Any point in the parameter space at which the object shows degenerate characteristics is called a *bifurcation point*; the coordinates of such a point are called *bifurcation parameters*. The set of all such points in the parameter space is known as the *bifurcation set.* Thus exists a close relationship among bifurcation theory, singularity theory, and catastrophe theory. The unifying concept is *degeneracy*.

According to Arnol'd [9],

> "The mathematical description of the world depends on a delicate interplay between continuity and discontinuous, discrete phenomena. ... Singularities, bifurcations and catastrophes are different terms for describing the emergence of discrete structures from smooth, continuous ones. ... Some people consider catastrophe theory as part of the theory of singularities, while others, conversely, include singularity theory in catastrophe theory."

An important deduction from catastrophe theory is the importance of paying special attention to objects exhibiting degenerate characteristics, because structural stability will be lost at or near the degeneracy.

The foregoing may be illustrated by taking as an object, a *matrix*. If a given matrix depends on parameters, say mistuning parameters, there may be certain critical parameters at which the matrix has degenerate eigenvalues. Then, we deduce immediately that such critical values are bifurcation points in a parameter space, and the matrix having the critical parameter is in a bifurcation set. We would then expect *some* structural instability in the behavior of the matrix, if it is given an arbitrarily small perturbation from the degeneracy. In certain cases, the structural instability may be related to an already known type of instability in engineering, e.g. elastic instability; in other cases, such instability may not be well-known, e.g. *geometric instability*, (Afolabi, [4-6]).

2.4 Transversal Intersections of Manifolds

Another central idea in catastrophe theory, in addition to Thom's transversality theorem, is its extension—the isotopy theorem. Both theorems concern the intersections of manifolds, and their structural stability. One of the routes to the application of catastrophe theory is by a direct application of these theorems. Both are long and highly technical; simplified statements of the transversality theorem are given in the Appendix without proofs. For proofs, the interested reader is referred to Abraham and Robbin [1, ch. 4], Arnol'd [10], Wasserman [45], Poston and Stewart [39], or Thom and Levin [43]. Before stating the basic ideas of the transversality theorem in an intuitive form, it is necessary to define smooth manifolds heuristically, which is sufficient for engineering applications. Throughout this paper, when "manifold" is used without any qualifier, "smooth manifold" is intended.

A smooth manifold is a space that everywhere *locally* seems flat. This is simply an extension of the fact that a smooth curve in a plane locally looks like a straight line—if we consider a small enough segment of the curve. Also, in 3-space the curved surface of a round object like the earth locally appears to be flat, although it is actually curved on a global scale. Locally approximating curved surfaces by their tangent planes, as in the above examples, is very common in infinitesimal calculus and differential topology. In this way, a manifold may be visualized as an nth dimension smooth surface, which locally looks like an nth dimension tangent plane. Manifolds which are smoothly embedded in a larger manifold are known as submanifolds.

2.4.1 Stability of Intersections

When two manifolds intersect, their intersection is either structurally stable or unstable. This concept is best illustrated by examples, as shown in Figures 2. In each figure, two plane curves (i.e. smooth submanifolds of $\mathbb{R}^2$) intersect. The intersection X at Figure 2 (a) is unstable, in the sense that if we perturb M_1 or M_2 ever so slightly, we either have *two* points of intersections as in Figure 2(b), or *none* at all as in Figure 2 (c). The fact that the *number* of times the two manifolds intersect changes abruptly, into two or zero, under the slightest perturbation is a "catastrophe". It is qualitatively significant, and indicates structural instability. On the other hand, the curves shown in Figure 2 (d) intersect in a stable way. No matter what kind of small perturbation we impose on the two curves M_1 and M_2, they always intersect at exactly two points—no more, no less; see Figure 2 (e). Moreover, the coordinates of the points of intersection after perturbation (the solid curves in Figure 2 (e)) are also close to the coordinates before perturbation (the dotted curves). This robustness is a consequence of the isotopy theorem.

Stable intersections as in Figure 2 (d) are called *transversal* intersections in catastrophe theory, and are said to be in "general position". Unstable intersections as in Figure 2 (a) are in exceptional position, and are called degenerate, or non-transversal.

2.4.2 Genericity and Uniqueness of Tangents

The loss of uniqueness of tangent spaces at a non-generic intersection of two manifolds is a degeneracy, consequent on the transversality theorem; it is thus a generator of instability. For example, at the intersection in Figure 2 (a), the intersecting manifolds M_1 and M_2 have the same tangent, therefore *the tangents are not unique*, and are called degenerate; the intersection is hence unstable. Degeneracy always implies instability of some sort. On the other hand, the intersections of Figure 2 (d) are stable because each of the manifolds M_1 and M_2 at each intersection point has its distinct and unique tangent.

2.5 Perestroikas, Versal Unfoldings, and their Codimension

The concept of "unfolding" is very important in singularity theory and catastrophe theory.

Any object at a bifurcation point must evolve along one of at least two different paths after leaving the bifurcation point. A bifurcation can only take place at a *degeneracy*. The number of qualitatively different paths along which a degenerate object evolves after a bifurcation is related to the *codimension* of the degeneracy. The process of undergoing the change is known as an unfolding, or a metamorphosis, or a perestroika. In the Russian text—for instance, by Arnol'd—unfoldings are called *perestroikas*. (In recent years, the word perestroika has been made famous by Mr. Gorbachev, President of the Soviet Union, in the political context). Thom uses biological metaphors, such as a bud *unfolding* to reveal a flower of many colors, or an egg undergoing a *metamorphosis* to become a butterfly, in his mathematical writings.

A versal unfolding is a special unfolding, in the sense that it contains all possible forms that could result from a perturbation. Just as one speaks of a versal unfolding, one also has a versal perturbation. Because a perturbation is sometimes called a "deformation", one also has a *versal deformation*. Thus, a versal deformation is the most general perturbation, from which all other perturbations may be obtained as special cases. The original idea now called versal deformation was introduced by Poincaré in Lemma IV of his thesis†, according to Arnol'd.

Consider, for an example of versal deformation, an "object" which is a smooth curve: $y = x^3$. This curve has a degenerate critical point at $x = 0$. Its versal deformation is $y = x^3 + \alpha_1 x$, where α_1 is an unfolding parameter. We get qualitatively different curves, with respect to the number of distinct critical points, depending on whether α_1 is positive, negative, or zero. Similarly, the curve $y = x^4$ has two degenerate critical points at $x = 0$, and two parameters are needed for a versal unfolding: $y = x^4 + \alpha_2 x^2 + \alpha_1 x$. We get qualitatively different curves for various positive, negative, and zero combinations of α_1 and α_2. All such curves constitute a *versal family*, [10].

Not all unfoldings are versal. For example, $y = x^4 + \alpha_2 x^2$ is an unfolding of the degenerate curve $y = x^4$, but it is not a versal unfolding, as explained below. Here, 'degenerate curve' means a curve with degenerate critical point.

Also, versal unfoldings are not always unique. In the case when a versal unfolding also happens to be *uni*quely determined, it is known as a *uni*versal unfolding. A versal unfolding with the *mini*mum number of unfolding parameters is called a *mini*versal unfolding. (Here, we use the terminology of Arnol'd, rather than that of Thom. What Arnol'd calls a versal unfolding, Thom calls a universal unfolding. Thom does not seem to distinguish between versal and universal unfoldings, whereas Arnol'd does).

An important point regarding the structural stability of the intersection of manifolds is the *codimension* of the submanifold of intersection. This is related to the dimension of the ambient space in which the intersecting manifolds are smoothly immersed. The codimension of a submanifold is an integer, just as its dimension is also an integer. (In recent years, it has been discovered that non-integer, i.e. fractional, dimensions are encountered in *fractal* objects. Fractal catastrophes have been investigated by Professor M. V. Berry in connection with caustics). The integer value of the codimension of an object is very important, for it tells us how many parameters are required for a versal unfolding of that object under arbitrary perturbation. For example, the curve $y = x^4$ discussed above has a codimension 2. The unfolding: $y = x^4 + \alpha_2 x^2$ is non-versal, since

† H. Poincaré, *Sur les propriétés des fonctions définies par des équations aux différences partielles.* (in *Thèse*, 1879, of *Oeuvres Complètes*, vol. I) Paris: Gauthier-Villars, 1928.

it has codimension 1, and one additional unfolding parameter is still required. A versal unfolding must have codimension zero, since only then is it stable.

3. CATASTROPHE THEORY AND BIFURCATION DIAGRAMS

Catastrophes and bifurcations are related via *degeneracy*. A useful device in the analysis of an evolutionary process through a degeneracy is the *bifurcation diagram.*

3.1 Symmetric Pitchfork Bifurcation (Nonlinear)

As an example of bifurcation diagrams, we shall first mention the "pitchfork bifurcation", which has been much studied; see, for instance, Golubitsky and Schaeffer [28] for theoretical comments, and Thompson and Hunt [44] for applications in civil engineering.

A symmetric pitchfork is obtained when a curve intersects a line, as shown in Figure 3 (a). This case may represent, for example, two uncoupled vibrating systems depending *nonlinearly* on one parameter. When the two previously isolated systems are coupled, there are only two ways in which the composite bifurcation diagram may evolve due to the interaction. Both of them are shown in Figures 3 (b) and 3 (c).

3.2 Local Bifurcation (Linear)

When parameter ranges are small, we get local versions of the globally non-linear cases above. Thus, the resulting bifurcation diagrams in Figures 4 are local, *linearized* versions of the global, nonlinear curves of Figure 3.

One notices immediately a resemblance of Figure 4 to Crandall's conservative and non-conservative coupling diagrams, shown earlier as Figures 1 (a) and (b) respectively.

4. SYSTEM MATRICES AND BIFURCATION DIAGRAMS

As stated earlier, the basic theorems of catastrophe theory (such as the transversality, isotopy and versal deformation theorems) pertain to mathematical "objects" which, in applications, may be smooth curves or manifolds, matrices, etc.

In this section, we shall consider matrices; 2×2 matrices specifically. This is relevant in applications, when two undamped single degree of freedom systems interact. Without loss of generality, it will be assumed that the two uncoupled systems are not identical; their difference being measured by a "mistuning" or imperfection parameter. In the 2×2 matrices to be considered here, the imperfection parameters appear on the diagonals, while the off-diagonal terms represent "coupling". The dynamics of such a coupled system may be described by matrices depending on parameters (Arnol'd, [11]). Alternatively, one may also study the dynamics in terms of smooth curves depending on parameters. Such curves often give rise to bifurcation diagrams.

4.1 Matrix Structure in Vibration Analysis

The equations of motion of a linear vibrating system in matrix form are

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{1a}$$

The matrices in equation (1a) above are square, of size $n \times n$, and may have a variety of structures. Take K for instance. K may be diagonal ($k_{ij} = k_{ji} = 0$ for $i \neq j$), symmetric ($k_{ji} = k_{ij}$), or skew symmetric ($k_{ji} = -k_{ij}$). The diagonal elements of a skew-symmetric matrix are all zero, i.e. $k_{ii} = 0$ for all i. Sometimes, the matrices we encounter will be an additive composition of a *dia*gonal and a *skew* symmetric matrix. For this purpose, we shall coin the phrase "*diaskew* symmetric matrix" when $a_{ji} = -a_{ij}$ and $a_{ii} \neq 0$.

4.2 Gyroscopic and Circulatory Matrices

By a well-known theorem in linear algebra, any non-symmetric square matrix may be decomposed into a symmetric part and a skew symmetric (sometimes called anti-symmetric) part.

In equation (1a), M is the mass matrix, which is symmetric, or can usually be made to be purely symmetric. In general, the "damping" C and "stiffness" K matrices are not symmetric. Equation (1a) may therefore be rewritten as

$$M\ddot{x} + \left[C_s + C_a\right]\dot{x} + \left[K_s + K_a\right]x = 0 \tag{1b}$$

where the subscripts s and a, respectively, refer to symmetric and anti-symmetric parts.

Following Ziegler [48, p. 29], the symmetric part K_s of the matrix pre-multiplying the displacement vector is called the *stiffness* matrix, whereas the skew symmetric part K_a is called the *circulatory* matrix. This is to be distinguished from a circulant matrix, which is a matrix in which each row is a circular shift of the previous row by a fixed number of columns; a circulant matrix may be symmetric or skew symmetric. Also, the symmetric part C_s of the matrix pre-multiplying the velocity vector is called the *damping* matrix, whereas the skew symmetric part C_a is called the *gyroscopic* matrix. Although C_a is called a gyroscopic matrix, it does not always arise from gyroscopic effects. For example, it may be due to magnetic forces, Coriolis forces, etc.

4.3 Versal Deformation of Matrices

In this section, we consider undamped vibrating systems with two degrees of freedom, i.e. those for which equation (1a) applies for $n = 2$. This is just for convenience, and leads to no loss of generality for $n > 2$. Assume $M = I$ where I is the identity matrix, and that elements of K are real. Let k be the nominal stiffness of the reference isolated system; ε a mistuning parameter; and s is the coupling strength. Consider the following stiffness matrix of the tuned or perfect, uncoupled system

$$K_0 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \tag{2a}$$

Such a matrix has degenerate eigenvalues, and is "structurally unstable" by virtue of the degeneracy. By the versal deformation theorem of Arnol'd [11] we know that a versal deformation of equation (2a) may be written as

$$K_v = \begin{bmatrix} k + \varepsilon_1 & \varepsilon_3 \\ \varepsilon_2 & k + \varepsilon_4 \end{bmatrix} \tag{2b}$$

By translating the origin, one may set $\varepsilon_1 = 0$, without affecting the qualitative nature of the system dynamics. There are exactly *four* qualitatively different ways in which the coupling and mistuning parameters ε_i may be varied in the matrix K_v to give qualitatively different dynamical systems. The resulting four matrices from the versal family are enumerated below, in sections 4.3.1-4.3.4. Although all four are listed, only two of them are considered in detail here; see section 5. The other cases, as well as those involving damping and gyroscopic matrices, will be returned to in subsequent work.

In each subsection below, we give :

(a) first, the stiffness matrix K_i in the conventional form;

(b) then, the eigenvalue matrix A_i in the Crandall canonical form;

(c) and, finally, the bifurcation diagram.

In the canonical form, α is the variable imperfection parameter and β, the fixed coupling term. If desired, β may be made the variable parameter with α being constant. Of course, in a two parameter system, both α and β would be variable, with neither being held constant.

4.3.1 Case I: Eigenvalue Veering

The eigenvalues repel each other at the critical parameter, and never collide.
The stiffness matrix of the coupled system is symmetric with real elements. Setting $\varepsilon_1 = 0,\ \varepsilon_2 = \varepsilon_3 = -s,\ \varepsilon_4 = \varepsilon$, in equation (2b), one gets

$$K_1 = \begin{bmatrix} k & -s \\ -s & k+\varepsilon \end{bmatrix}; \quad A_1 = \begin{bmatrix} 1-\alpha & -\beta \\ -\beta & 1+\alpha \end{bmatrix} \tag{3a,b}$$

$$\alpha = \frac{\varepsilon}{2k}; \quad \beta = \frac{s}{k}. \tag{3c,3d}$$

The bifurcation diagram for this case is depicted in Figure 5. It usually leads to *geometric* instability, when the eigenvalues get too close, in the neighborhood of $\alpha = 0$. This bifurcation diagram has been reported upon extensively in the structural dynamics literature, and is usually known as "curve veering"; see Leissa [34], Perkins and Mote [37], Pierre [38], Bajaj *et al* [13], etc. In physics, the eigenvalue repulsion is known as "avoided crossing", while the attraction discussed below is known as "frequency coalescence".

4.3.2 Case II: Eigenvalue Collision — Degeneracy of the First Kind

The eigenvalues attract each other, and collide destructively at the critical parameter.
The system matrices for this version may be put in the form of the following dia-skew matrices

$$K_2 = \begin{bmatrix} k & -s \\ s & k+\varepsilon \end{bmatrix}; \quad A_2 = \begin{bmatrix} 1-\alpha & -\beta \\ \beta & 1+\alpha \end{bmatrix} \tag{4a,b}$$

When $\alpha = \beta$ or $s = \varepsilon/2$, the paths of two approaching eigenvalues intersect *non transversely*, and we have eigenvalue degeneracy, as mentioned earlier. The eigenvectors are also degenerate. The bifurcation diagram for this case is in Figure 6. It may be observed that the two eigenvalues collide ("destructively") at a critical value of the mistuning parameter, when $\alpha = -\beta$. Then, as the parameter continues to vary in the same direction (in this case, the mistuning continues to increase in the positive direction) a second critical parameter value is reached $\alpha = +\beta$, at which two distinct eigenvalues suddenly emerge. The separation between the eigenvalues continues to widen as the parameter continues along its path of variation.

Between the two critical parameter values lies a zone of elastic instability or *flutter*. The width of the zone is 2β. The critical parameter is sometimes called the flutter boundary. In the canonical form, negative values of the parameter are encountered, placing half of the inadmissive parameter range, $-\beta \leq \alpha \leq \beta$, in the negative sector of the parameter plane. In practical cases, the inadmissive range may completely lie in the positive half of the parameter plane. This may be achieved by translating the origin, e.g. by replacing α with α' where $\alpha' = \alpha + \mu, \ \mu > |\beta| > 0$.

4.3.3 Case III: Degeneracy of the Second Kind

The eigenvalues pass through each other at the critical parameter; the eigenvalues, as well as the eigenvectors, are degenerate at the bifurcation point.

Coupling in this case gives rise to a matrix which is not diagonalizable. Its physical effect is to act as a *diode* does in an electronic circuit, or as a *one-way valve* in a hydraulic circuit. It may be written in the form

$$K_3 = \begin{bmatrix} k & -s \\ 0 & k+\varepsilon \end{bmatrix}; \quad A_3 = \begin{bmatrix} 1-\alpha & -\beta \\ 0 & 1+\alpha \end{bmatrix} \qquad \text{(5a,b)}$$

Case II discussed above may, at first glance, seem to be identical to this present case, and also with Case IV discussed below, in the sense that they both have degenerate eigenvalues at the critical parameter. However, the degenerate eigenvalues occur under very different circumstances. Here, $\alpha = 0$ is the condition at which eigenvalues are degenerate, compared with $\alpha = |\beta|$ in Case II. Also, at the critical value of the parameter, the two eigenvalues meet but do not collide destructively as in Case II; they simply pass through each other, as depicted in Figure 7. Moreover, only one critical parameter value is encountered here, unlike in Case II when two different critical values exist. An important fact for system dynamics in this case is this: at the critical parameter, *the system matrix cannot be diagonalized*; it can only be reduced to the Jordan normal form. Further, there is no zone of inadmissive parameter values as in Case II. It is important to remark here that this case can give rise to extremely complicated dynamics in the neighborhood of degeneracy, and the subject is currently being developed in another paper.

4.3.4 Case IV: Degeneracy of the Third Kind

The eigenvalues pass through each other at the critical parameter; the degenerate eigenvalues have independent eigenvectors at the bifurcation point, but the eigenvectors are dense in the unit disk.

The system matrix in this case is rather simple; it is diagonal, representing uncoupled non-interacting systems.

$$K_4 = \begin{bmatrix} k & 0 \\ 0 & k+\varepsilon \end{bmatrix}; \quad A_1 = \begin{bmatrix} 1-\alpha & 0 \\ 0 & 1+\alpha \end{bmatrix} \tag{6a,b}$$

The bifurcation diagram is shown in Figure 8. It is identical to that in Figure 7. However, the dynamics represented in the present case is very different, due to the fact the eigenvectors here are linearly independent at the degenerate eigenvalue, and the system matrix is diagonalizable for all values of α. This kind of degeneracy occurs in a *decoupled* two degree of freedom system, or in cyclic symmetric systems with purely conservative coupling where double modes are frequently encountered.

5. EXAMPLES OF MODAL INTERACTION IN MECHANICAL SYSTEMS

In section 4.3, we enumerated the four qualitatively different matrices obtainable from a versal family of a degenerate 2×2 matrix. In this section, we are basically interested in the first two cases, and now examine examples of mechanical systems exhibiting characteristics of their bifurcation diagrams.

Suppose a mechanical system P has an eigenvalue depending on a parameter α as shown in Figure 9 (a). Then, another system Q which, not coupled to P, depends on the same parameter, but in a different fashion as shown in Figure 9 (b). By coupling the two systems together, the composite structure will have an eigenvalue loci diagram which has as *asymptotes*, Figure 9 (c). If the coupling is conservative, then the bifurcation diagram is as shown in Figure 9 (d). For non-conservative coupling, we have Figure 9 (e). These "unfoldings" of the degenerate bifurcation diagram can be deduced from Thom's transversality theorem.

In the foregoing, we have assumed that the subsystems in their uncoupled state both depend on the same parameter. In some cases, only one subsystem P depends on the parameter, α, as in Figure 10 (a), while Q is independent of α, Figure 10 (b). For such systems, we have the versal family of the bifurcation diagrams shown in Figures 10 (d) and (e).

5.1 Conservative Coupling

5.1.1 One-to-One Interaction

A *one* degree of freedom subsystem coupled to another *one* degree of freedom subsystem is the standard, simplest case. The system shown in Figure 11 has been treated by Arnol'd [7, pp. 105*ff*]. Curve veering and localization in the system of two coupled pendula have been studied by several authors; see, for instance, Pierre [38] and Bajaj *et al* [13]. It is easy to write down the system equations; see, for instance, Arnol'd [7, p. 106], or Bajaj *et al* [13]. The eigenvalue matrix may be placed in the form

$$\begin{bmatrix} 1+\beta & -\beta \\ -\beta & \dfrac{1}{1+\alpha}+\beta \end{bmatrix} \tag{7}$$

By a suitable change of basis, the above equations may be converted to the Crandall canonical form, Equation (3b). A variety of undamped coupled mechanical systems having a *real, symmetric* stiffness matrix will show the same kind of curve veering associated with conservative coupling. They are far too numerous to be listed here.

5.1.2 Two-to-One Interaction

By using the concept of binary composition, it is a routine matter to predict the bifurcation diagrams of a *two* degree of freedom system coupled to a *one* degree of freedom. For instance, suppose a two degree of freedom subsystem P is invariant with system parameter α, while the second one degree of freedom subsystem Q depends on the parameter in the manner of Figure 12. Upon imposing conservative coupling at the applicable coordinates, one obtains the bifurcation diagram shown in Figure 12 (c). This kind of bifurcation diagram is very common in many vibrating systems depending on one parameter.

5.1.3 Multi-to-One Interaction

As another example, consider a *multi* degree of freedom subsystem P with the parameter dependence of Figure 13 (a), and a *one* degree of freedom subsystem Q varying as shown in Figure 13 (b). The asymptotes or framework of the composite bifurcation diagram will look like Figure 13 (c), upon conservative coupling of P and Q. A mechanical example has been studied by Su [40], among others; see Figures 5 and 6 of Su.

5.1.4 Multi-to-Multi Interaction

As an example of a *multi* degree of freedom coupled to another *multi* degree of freedom, one may cite the studies of Cheng and Perkins [19]. By inspection, one notes that all the intersections in their Figure 5 correspond to conservative coupling.

5.2 Non-Conservative Coupling

Non-conservative coupling frequently arises when we have a circulatory matrix. Often, this arises from taking a vector cross product. As is well-known, there are mathematical anomalies associated with the vector cross product. For instance, the vector cross product—unlike the scalar or dot product—exists only in a three dimensional real vector space. To extend the operation to $\mathbb{R}^n$, $n > 3$, is not possible; the best we can do is use Lie algebra.

In a forthcoming paper Afolabi [3], it is shown that the vector cross product arises in systems with *direction sensitivity*, thereby leading to skew-symmetric matrices. Direction sensitivity occurs in problems with convective forces (including moving loads, translating cables, pipes conveying fluid, aeroelasticity, etc) rotating systems, systems with follower forces, etc. When such matrices premultiply the mass or stiffness matrix, elastic instability (flutter) is induced for certain parameter values.

Thompson and Hunt [44] have conducted a comprehensive review of many elastic instability phenomena from the static and nonlinear point of view, using catastrophe theory. Our approach here is from the dynamic and linear view point. The following are representative samples of a few well-known problems which are now re-examined from our present viewpoint: namely, that *eigenvalue degeneracy is a generator of "structural instability"*, which leads to elastic instability in certain cases when the eigenvectors are also degenerate, and to geometric instability in others where the eigenvectors may be resolved into linearly independent directions. It must be emphasized that there are various routes to elastic instability, and the route of eigenvalue degeneracy is only one of them.

All the problems described in this section have 1:1 interaction. These are the most common forms of nonconservative coupling. Once a *range* of parameters is encountered where instability holds, it is often difficult to operate the machinery beyond the critical zone.

5.2.1 One-to-One Interaction: Rotating Systems

The problem of "ground resonance" of helicopters is a classic example of this one. This is discussed in many texts; see, for instance, Den Hartog [24, p249]. Other problems related to this have been treated by, for instance, Crandall [20], and Crandall and Dugundji [21-22] who reported on the aircraft engine/propeller interaction problem from linear and nonlinear analyses.

5.2.2 One-to-One Interaction: Pipe conveying fluid

Linear and nonlinear versions of the problem of a pipe conveying fluid have been studied by several authors. For, instance Benjamin [16], Bishop and Fawzy [17], Paidoussis and co-workers [35, 36], Bajaj and Sethna [12] and Holmes [30]; among others.

Elastic instability of such systems are principally due to two causes, the presence of "circulatory" and "centrifugal" terms in the stiffness matrix. The circulatory terms lead to flutter, while the centrifugal terms lead to divergence. I conjecture that flutter in this case may also be explained in terms of the collision of two eigenvalues of opposite "Krein signatures", see Krein [33]. At any rate, structural instability induced by eigenvalue degeneracy is observed, as predicted by catastrophe theory. When nonlinear as well as gyroscopic terms are present, they tend to stabilize the system, in general.

5.2.3 One-to-One Interaction: Linear Flutter

The problem of panel flutter has been studied extensively by several authors. See, for instance, Dowell [25]. He details two kinds of linear flutter: 'single mode flutter due to negative damping', and coupled mode flutter due to 'merging or coalescence of frequencies'. The latter case which arises at supersonic Mach numbers is discussed here, from the point of view of eigenvalue degeneracy.

For Mach number $M > \sqrt{2}$, Dowell [25, p. 20, eq. 27] gives the following matrix for a two mode approximation of the dynamics of a flat plate; see [25] for nomenclature:

$$\begin{bmatrix} M_1(\omega_1^2 - \omega^2) & \rho_\infty U_\infty^2 \bar{Q}_{21}^M \\ -\rho_\infty U_\infty^2 \bar{Q}_{21}^M & M_2(\omega_2^2 - \omega^2) \end{bmatrix} \tag{8a}$$

One may cast the above in Crandall canonical form, by using the following substitutions:

$$\begin{aligned} \omega_1^2 &= 1 - \alpha; \\ \omega_2^2 &= 1 + \alpha; \\ \alpha &= \tfrac{1}{2}(\omega_2^2 - \omega_1^2); \\ \beta &= \frac{\rho_\infty U_\infty^2 \bar{Q}_{21}^M}{\sqrt{M_1 M_2}} \\ \lambda &= \omega^2 \end{aligned} \tag{8b}$$

leading to the eigenvalue matrix

$$\begin{bmatrix} 1 - \alpha - \lambda & \beta \\ -\beta & 1 + \alpha - \lambda \end{bmatrix} \tag{8c}$$

The condition for degenerate eigenvalues are obtained when the "mistuning" or imperfection parameter α cancels out the "coupling" term β, i.e. when $\alpha = \beta$, or

$$\omega_2^2 - \omega_1^2 = \frac{2\rho_\infty U_\infty^2 \bar{Q}_{21}^M}{\sqrt{M_1 M_2}}. \tag{8d}$$

Again, the flutter instability here is due to the collision of two eigenvalues. The collision originates from the circulatory matrix, due to aeroelastic forces. Thus, structural instability induced by degeneracy as predicted from catastrophe theory is also in evidence here.

Another flutter problem with two degrees of freedom has been treated by Bakhle, Reddy, and Keith [14]. The non-symmetric nature of their stiffness matrix means it may be resolved to a symmetric part, and a skew-symmetric part—the circulatory matrix. Again, flutter occurs under the appropriate condition demanded by eigenvalue degeneracy of catastrophe theory.

5.2.4 One-to-One Interaction: Follower Force

Several mechanical systems incorporating a "follower force", have been described in detail by Bolotin [18]. In Bolotin [18, p. 90, section 2.2], the matrix equations given are of the non-conservative coupling type. Thus, flutter occurs due to eigenvalue degeneracy. Flutter induced by eigenvalue collision is also evident in an earlier section of the book: Figures 21-22 on pages 56-57. Again, all this is placed in clearer perspective when viewed from the position that *eigenvalue degeneracy* is a generator of structural instability, in the sense of Thom.

5.3 Mixed Conservative and Non-conservative Coupling

A system may have conservative coupling at a certain range of the system parameter, $\alpha_1 < \alpha < \alpha_2$ only to have non-conservative coupling at a different (usually, *higher*) value of the parameter $\alpha_3 < \alpha < \alpha_4$. We give some examples.

5.3.1 Rotating Systems

The stability of an engine/propeller systems has been studied by Crandall and Mroszczyk [23]. The parameter in this system is the rotation speed of the propeller. The engine, as a two degree of freedom system, has two resonant frequencies that do not vary with rotation speed; Figure 14 (a). The propeller's frequency, however, varies as with rotation speed as in Fig 14 (b), due to centrifugal effects. The first coupled resonance is conservative, while the second is nonconservative; Figure 14 (c).

5.3.2 Convective Problems

By convective problems we mean problems with spatial or temporal direction sensitivity.

Among these are problems of moving load, which have been treated by many investigators in a variety of contexts: Alabi and Afolabi [2], Cheng and Perkins [19], Crandall [20], Kortum and Wormley [32], Perkins and Mote [37], etc. Mixed coupling has been reported in connection with this type of problem by Cheng and Perkins [19]. Their "subcritical" speed corresponds to conservative coupling, while the postcritical regime corresponds to nonconservative coupling.

6. NUMERICAL EXPERIMENTS

It is generally a useful strategy in engineering analysis to invent canonical forms for various problems. The advantage is that once a solution is obtained for the canonical form, we can solve a variety of similar problems simply by casting any such problem in the standard form.

Using the canonical forms of coupling due to Crandall, as discussed in Section 4, one may obtain the time domain response of both coupling types in closed form, following standard procedures in the theory of ordinary differential equations. This requires the prior calculation of eigenvalues and eigenvectors. Numerical methods may also be used to integrate the required ordinary differential equations with prescribed initial conditions.

6.1 Conservative Coupling

When formulated as a second order differential equation problem, the system with conservative coupling in the Crandall canonical form (equation 3b) has the following eigenvalues and eigenvectors.

$$\lambda_1 = 1 - \sqrt{\alpha^2 + \beta^2} \, ; \qquad \lambda_2 = 1 + \sqrt{\alpha^2 + \beta^2} \, . \tag{9a}$$

$$u_1 = \begin{bmatrix} -\beta \\ \alpha - \sqrt{\alpha^2 + \beta^2} \end{bmatrix} ; \quad u_2 = \begin{bmatrix} -\beta \\ \alpha + \sqrt{\alpha^2 + \beta^2} \end{bmatrix} . \tag{9b}$$

From (9a), one notices that eigenvalues here *cannot* collide. There is a minimum distance $\Delta\lambda = |\lambda_1 - \lambda_2| = 2\sqrt{\alpha^2 + \beta^2}$ between the two eigenvalues. If $\alpha^2 + \beta^2 \geq 1$, then buckling instability (i.e. divergence) takes place, arising from the fact that the first eigenvalue ceases to be positive. Else, the system always has elastic stability.

6.1.1. Vibration Localization and Geometric Stability

Figure 15 (a) shows the eigenvalue loci, plotted from equation (9a) for both weak coupling ($\beta = .01$) and strong coupling ($\beta = 0.5$). Curve veering is evident in the neighborhood of $\alpha = 0$ in both cases. The radius of curvature of the curve for weak coupling is much smaller than that for strong coupling. Thus, the veering is more pronounced for weak coupling.

In discussing vibration localization, it is important to consider two limiting parameter cases: $\alpha = 0$ and $\beta = 0$. In the first case, when $\alpha = 0$, an examination of the eigenvector equation (9b) shows that localization *cannot* take place, irrespective of the strength of coupling, β, provided $\beta \neq 0$. Thus, a perfect, tuned, system does not experience localization, since its mode shape elements are all constants, invariant with coupling strength. For all other values of α, varying degrees of localization occur, depending on how large α is relative to β. In the second limit when $\beta = 0$, perfect localization takes place, irrespective of the value of mistuning, α, even when $\alpha = 0$. Thus, a completely decoupled system always exhibits localization, since there is no mechanism for energy interchange between the subsystems.

The case when both $\alpha = 0$ and $\beta = 0$ simultaneously is a degenerate condition. In the neighborhood of the degeneracy, we have geometric instability. This means that a very small change in either α or β changes the eigenvectors considerably. Vibration localization is an extreme case of this instability, and occurs when β/α is very close to zero. Since $\beta/\alpha = 0$ represents a decoupled system, vibration localization is an indication that the system is close to being decoupled.

By plotting the *eigenvector* parameters, $u_1 = \alpha - \sqrt{\alpha^2 + \beta^2}$ versus α, and $u_2 = \alpha + \sqrt{\alpha^2 + \beta^2}$ versus α, one also obtains veering in the eigenvector curves, Figure 15 (b). This type of veering may be contrasted with that exhibited by the eigenvalues.

6.1.2. Vibration Localization and Beats

Since the time of Huygens‡ in the 17th century, the phenomenon of "beats" has been

‡ C. Huygens, *Holorgium Oscillatorium.* Paris: 1673. (English Translation by R. J. Blackwell, Iowa State University Press, 1986).

known in relation to pendulum clocks. In discussing the relationship between beats and vibration localization, it is important to consider two limiting cases $\alpha = 0$ and $\beta = 0$, in the coupling of two identical systems P and Q, as before.

In the first case when $\alpha = 0$, there is a perfectly periodic and *complete* exchange of energy between the two systems, irrespective of the value of β, provided it does not actually vanish. The only effect of the coupling strength β is to regulate the period of beating. Weak coupling leads to longer periods, while strong coupling leads to shorter periods. A time domain analysis illustrates the foregoing very well. If we set system P in motion while leaving Q at rest, then initially all the kinetic energy is in P while Q is stationary. After a while energy is transferred from P to Q, so that Q has all the kinetic energy while P is stationary. Then, after another while, the energy is transferred back to P, and so forth. The rate of energy transfer is governed primarily by the magnitude of coupling, since there is no mistuning.

In the second limit when $\beta = 0$, there is no energy exchange whatsoever, irrespective of the value of α, including $\alpha = 0$. The complete decoupling, therefore, manifests itself as perfect localization for all levels of mistuning.

In the neighborhood of degeneracy (i.e. $\alpha \simeq \beta \simeq 0$ simultaneously), more complicated beating and varying degrees of localization take place. For instance, a very weak coupling $\beta = 0.001$ and relatively moderate mistuning $\alpha = 0.1$, lead to the response curves in Figure 16. The initial conditions correspond to $x_0 = 1, y_0 = 0, \dot{x}_0 = 0$, and $\dot{y}_0 = 0$. Here, x represents the displacement of subsystem P, while y denotes that of Q. Notice that, because of the extremely weak coupling, the amplitude of y is almost zero, while that of x is pretty much close to the initial condition. The beating is not very evident from a short term trace as in Figure 16 (a). Figures 16 (b) and 16 (c) show long term trends for both $x(t)$ and $y(t)$ respectively.

As the coupling strength is increased while keeping the mistuning constant at $\alpha = 0.1$, we obtain the traces in Figures 17 (a) - (e). Respectively, the coupling strengths are $\beta = 0$, 0.01, 0.1, 0.9, 1.0. The initial conditions here are the same as in Figures 16, and through out similar figures in the remainder of this paper. Figure 17 (a), $\beta = 0$, is the reference case representing the decoupled system. In Figure 17 (b), $\beta = 0.01$, the beating frequency is higher than before, and the amplitudes of both systems are almost equal. Because the system is undamped and is stable in the elastic sense, the long term trend is an infinite sequence of energy exchange between the interacting subsystems P and Q. Figure 17 (d), $\beta = 0.9$, represents a system that is almost close to buckling. Therefore, rather than beating, i.e. energy exchange, there is energy modulation, with both subsystems almost out of phase at every time. The critical value of $\beta = 1$ in Figure 17 (e) corresponds to the buckling limit. Since the system is elastically unstable at the buckled state, the amplitudes of both subsystems increase without bound, while undergoing harmonic oscillation in the process. This kind of motion is not the same as the classical *static divergence*, and may be termed *dynamic divergence,* in contrast to static divergence where no oscillation takes place, or flutter, which is sometimes known as *oscillatory divergence*; see the illustrations in Figures 18 (a) - (c).

6.2 Non-Conservative Coupling

The above procedures are now repeated for the system with non-conservative coupling. The eigenvalues and eigenvectors of the system in canonical form are

$$\lambda_1 = 1 - \sqrt{\alpha^2 - \beta^2}\ ; \qquad \lambda_2 = 1 + \sqrt{\alpha^2 - \beta^2}\ . \tag{10a}$$

$$u_1 = \begin{bmatrix} -\beta \\ \alpha - \sqrt{\alpha^2 - \beta^2} \end{bmatrix}; \quad u_2 = \begin{bmatrix} -\beta \\ \alpha + \sqrt{\alpha^2 - \beta^2} \end{bmatrix}. \tag{10b}$$

We notice that at $\alpha = |\beta|$, there is eigenvalue degeneracy of the first kind, section 4.3.2. The eigenvectors are also degenerate. There exists a parameter range, $-\beta \le \alpha \le \beta$, at which flutter occurs; its width is 2β, and is governed solely by the coupling, β.

Figure 19 (a) shows the eigenvalue loci for both weak and strong non-conservative coupling. For weak coupling, the unstable region is very small, whereas it is large for strong coupling. In this sense, weak coupling $\beta \simeq 0$ is advantageous while strong coupling $\beta >> 0$ leads to a wide range of α where the system flutters. The eigenvector loci are also shown in Figure 19b. Here, instead of curve veering, we have curve collision.

A time domain solution reveals that *the system with purely non conservative coupling inherently lacks elastic stability, whenever there is no mistuning; therefore, the system flutters*. Mistuning or imperfection is thus a stabilizing factor in these kinds of system.

But this is conditional stabilization, since the stability offered by mistuning depends also on the coupling strength, β. In fact, the ratio of mistuning to coupling strength is a very important parameter in this regard. For a large ratio of mistuning to coupling strength as in Figure 20 (a) ($\alpha = 0.1, \beta = .001$), the system behaves as the corresponding system with non-conservative coupling. As the magnitude of coupling is increased while keeping mistuning fixed at $\alpha = 0.1$, we get Figures 20 (b) and (c). At Figure 20 (b), $\beta = 0.01$, and both systems are stable, in the long term. Beating also takes place as energy is passed back and forth between the two subsystems. But as coupling increases to, for instance $\beta = 0.1$ as in Figure 20 (c), we have elastic instability. From here on, any further increase in coupling introduces large amplitudes to *both* systems. The stability boundary in the parameter space is at the point where mistuning balances coupling, $\alpha = \beta$.

7. CONCLUSIONS

It is instructive to distinguish between conservative and non conservative coupling in vibrating systems. This idea is due to Professor Crandall. It is also instructive to pay attention to degenerate conditions, which are generators of "structural instability". This other idea is due to Professor Thom. By combining Crandall's classification of conservative and non conservative coupling from vibration theory with Thom's transversality theorem from catastrophe theory, certain deductions concerning the elastic stability and geometric stability of *linear* vibrating systems have been outlined. The following is a summary.

(a) Tuned systems with purely non conservative coupling (equation 4b, $\alpha = 0$) are always elastically unstable. This is often called "flutter instability". Stabilization of such systems is possible if (a) a certain amount of conservative coupling is also present, or (b) a certain amount of mistuning is present, $\alpha \neq 0$.

(b) When mistuning or imperfection is used to stabilize the system with purely non conservative coupling, the amount of mistuning must be carefully regulated. The flutter boundary is at the parameter value where the ratio of mistuning to coupling equals unity. A system with purely non conservative coupling and too little a ratio of mistuning to coupling strength, or no mistuning at all, will flutter.

(c) The range of parameter values at which the system with non conservative coupling is elastically unstable depends on the magnitude of the coupling strength, β. Weak coupling gives a larger range of stability, whereas strong coupling gives a wider range of instability. In this sense, weak nonconservative coupling is more advantageous than strong coupling.

(d) In the neighborhood of degenerate modes, systems with either purely conservative coupling or purely non conservative coupling both have geometric instability. This means that the eigenvectors rotate their direction of alignment much more rapidly here than in any other region of the parameter space.

(e) At the exact moment of eigenvalue degeneracy, i.e. at the bifurcation point, the *eigenvectors* of the system with non-conservative coupling are also degenerate; whereas in a system with conservative coupling the eigenvectors are dense in the unit disk, from which one may *arbitrarily* select any mutually orthogonal pair of vectors.

(f) Undamped systems with purely conservative coupling cannot flutter. But elastic instability can arise when buckling takes place, leading to "dynamic divergence". This is not the same as "static divergence", or "oscillatory divergence".

Examples of mechanical systems are provided to illustrate the foregoing.

ACKNOWLEDGEMENT.

Part of this work was carried out while I was a Visiting Scientist at the Massachusetts Institute of Technology. I am indebted to Stephen H. Crandall, Ford Professor of Engineering at MIT for his hospitality, encouragement, and several insightful discussions.

APPENDIX: The Transversality Theorems

This exposition arose from my interpretation of the following theorems, with the objective of applying them in *linear* vibration analysis. Neither in the statement of the theorems, nor in their proofs, is it required that their application be limited to nonlinear, gradient dynamic systems as far as I can determine. Theorem 3, for instance, seems applicable for matrices originating from linear or nonlinear systems, while theorem 4 appears to be valid for all C^∞ functions, whether or not such functions originate from nonlinear, gradient dynamic systems. Theorem 2 is a generalization of Theorem 1. Theorem 4 reduces to the Morse Lemma in the case when f is a function of one variable. The statements here are based on those of Arnol'd [10]. For proofs and other variants of the theorems, see Abraham and Robin [1], Arnol'd [10], Poston and Stewart [39], or Thom and Levin [43].

Theorem 1 (The Weak Transversality Theorem)

> *Let $f : B \longrightarrow A$ be a smooth map of a compact manifold to a manifold A containing a compact submanifold $C \subset A$. Then the maps f that are transversal to C form an everywhere dense subset of the function space of all maps $f : B \longrightarrow A$ with the C^r-topology, $s \le r \le \infty$, where* $s > \max(\dim B - \dim A + C,\ 0)$.

Theorem 2 (Thom's Transversality Theorem)

> *Let C be any regular submanifold of the jet space $J^k(B, A)$. Then, the set of maps $f : B \longrightarrow A$ whose k-jet extensions are transversal to C is an everywhere dense intersection of open sets in the space of smooth maps.*

Theorem 3 (Corollary to Theorem 1, for matrices)

> *In the space of smooth families of mappings $f: B \longrightarrow A$ of $n{\times}n$ matrices, those families that are transversal to the stratified variety C form an everywhere dense set. In particular, the values of the parameters corresponding to matrices of rank r form, in general, a smooth submanifold of the parameter space B of codimension $(m-r)(n-r)$.*

Theorem 4 (Corollary to Theorem 2, for smooth functions)

> *In the space of smooth functions $f : B \longrightarrow A$, those functions whose critical points are non degenerate form an everywhere dense intersection of open sets.*

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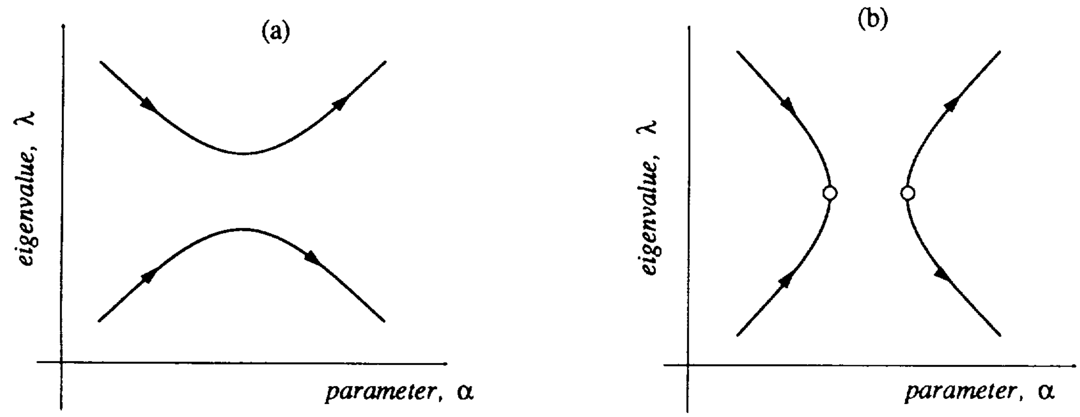


Figure 1: Eigenvalues as functions of the system parameter, α.
(a) conservative coupling; (b) non conservative coupling.

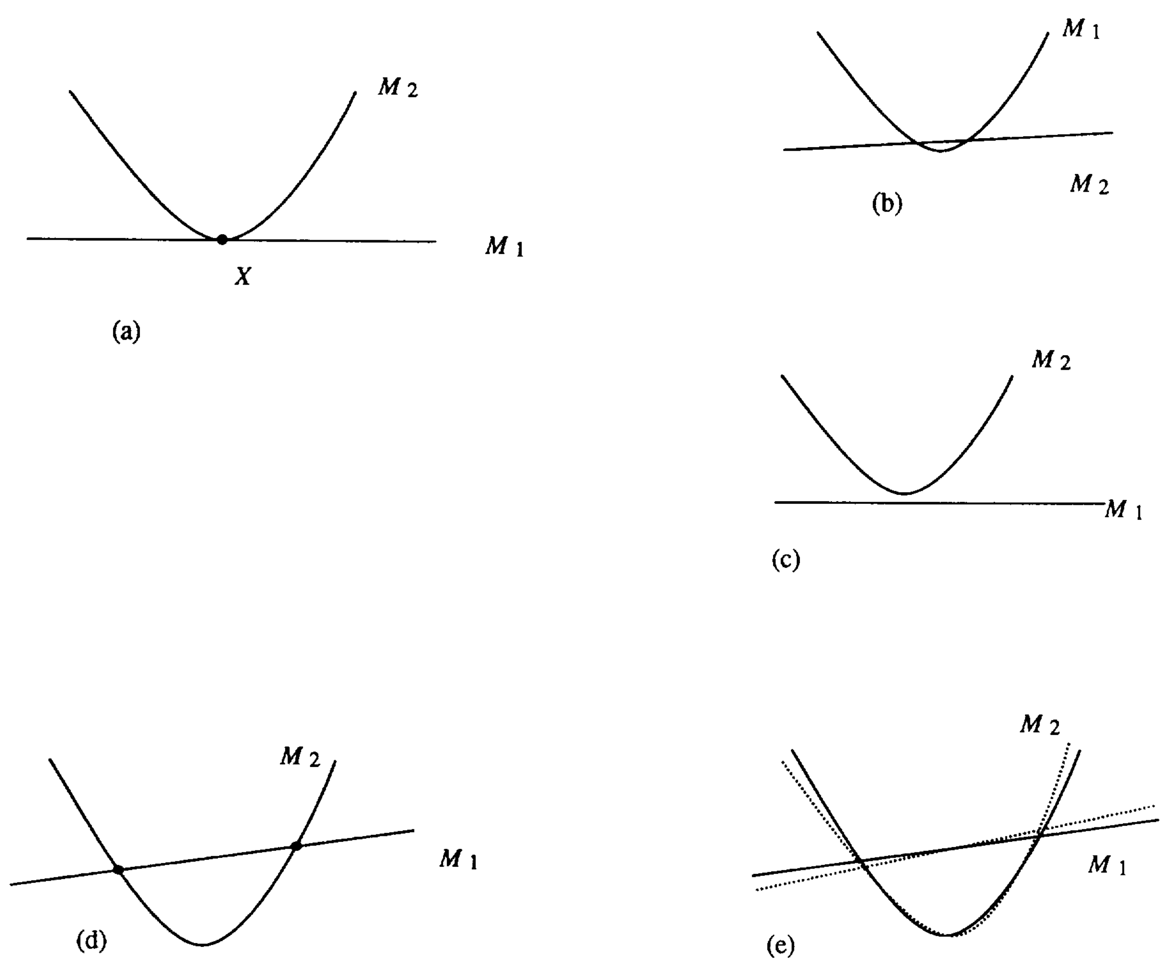


Figure 2: Transversal and non-transversal intersections; (a) the non-transversal intersection at X is unstable, a small perturbation either leads to (b) *two* intersection points, or (c) *none* at all ; (d) the two intersections are transversal, and (e) are stable under small perturbations.

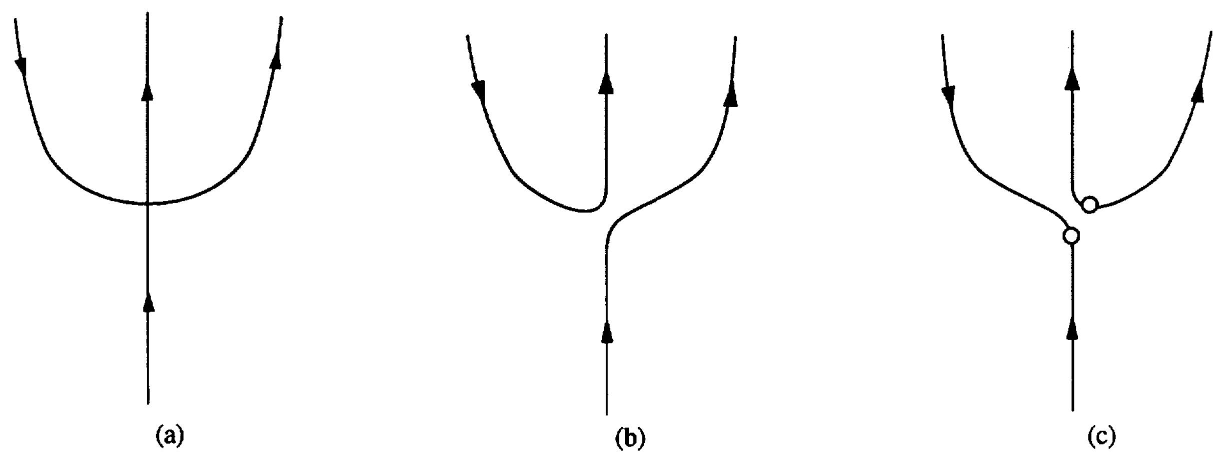


Figure 3: The symmetric pitchfork bifurcation; (a) degenerate case; (b), (c) versal family.

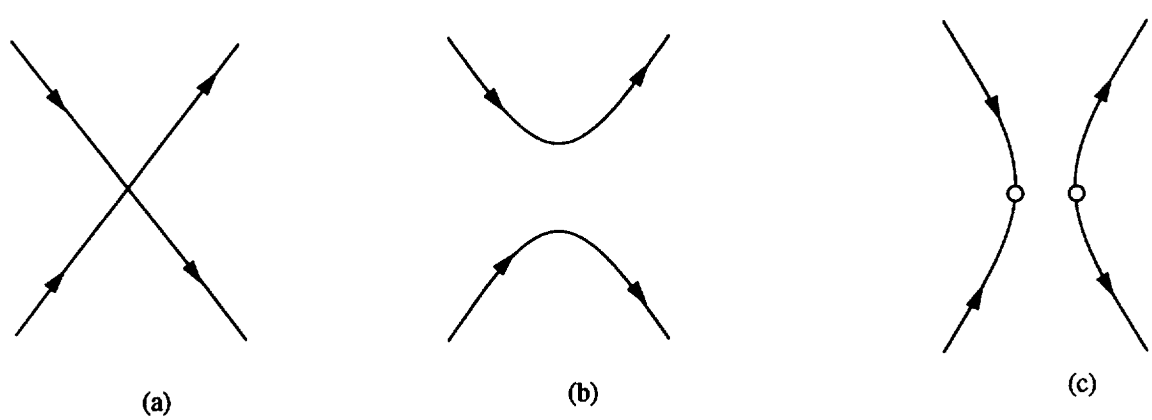


Figure 4: Local bifurcation diagrams are linear versions of the globally nonlinear curves of Figure 3. (a) degenerate case; (b), (c) versal family.

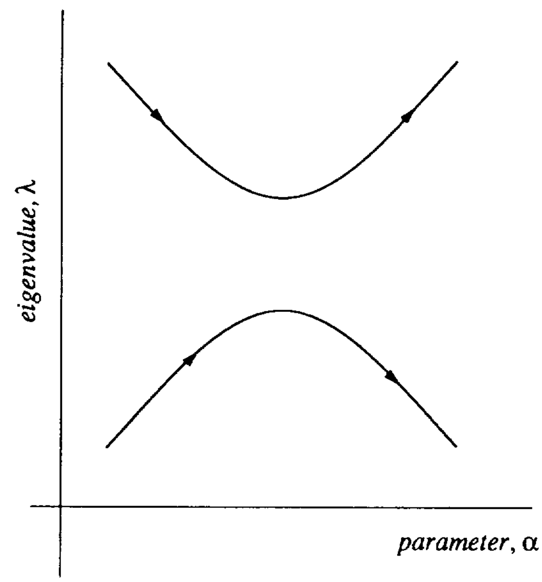


Figure 5: Eigenvalue Veering.

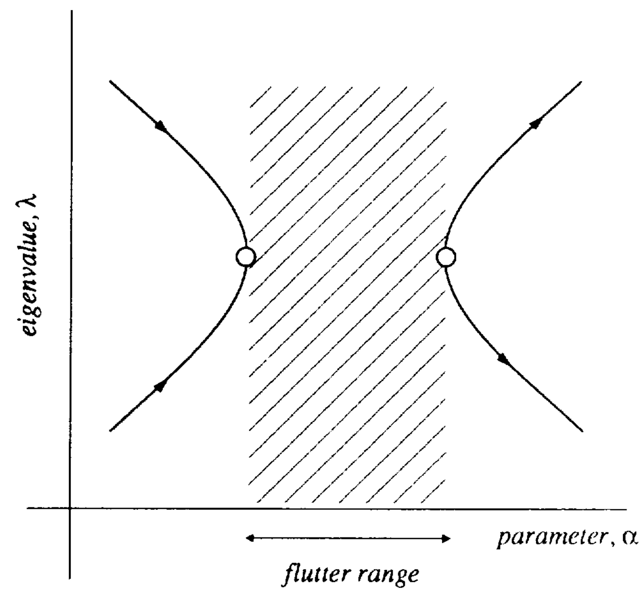


Figure 6: Eigenvalue Collision; a range of parameter exists, at which flutter takes place.

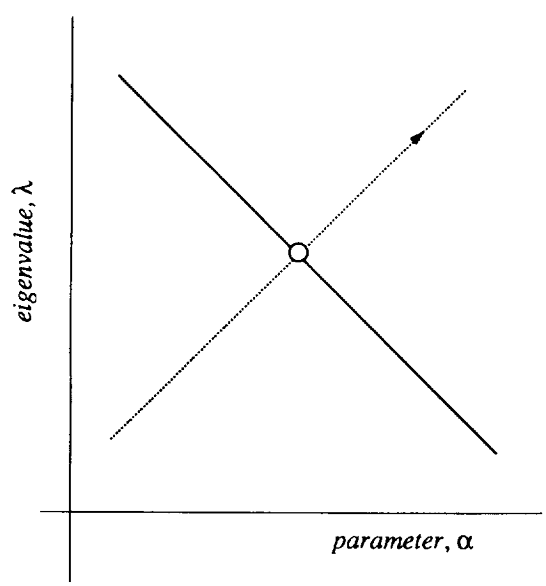


Figure 7: Eigenvalues are degenerate at the bifurcation point; the eigenvectors are also degenerate.

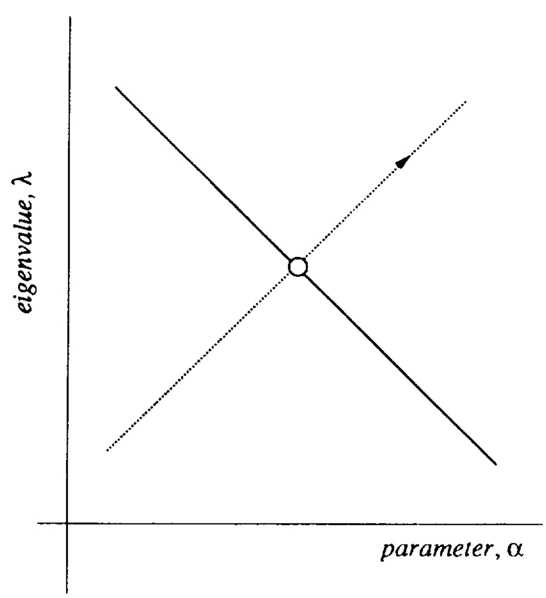


Figure 8: Eigenvalues are degenerate at the bifurcation point; the eigenvectors are not degenerate, but are dense in the unit disk.

(a)

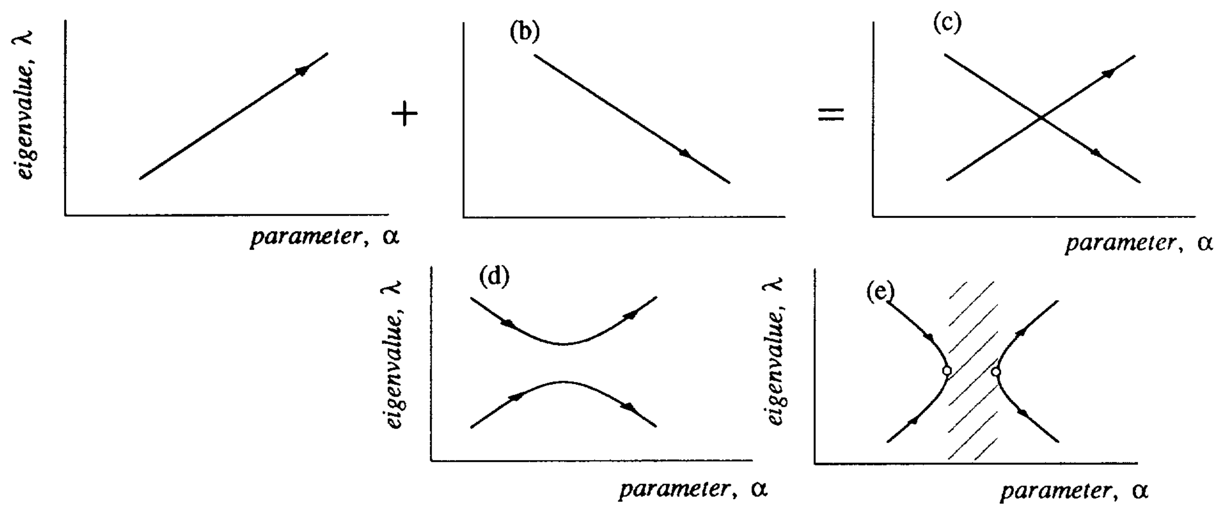


Figure 9: Composition of two one degree of freedom systems.
(a) , (b) Eigenvalue varies with parameter;
(c) composite bifurcation diagram (degenerate);
(d) conservative coupling; (curve veering); (e) non-conservative coupling (eigenvalue collision).

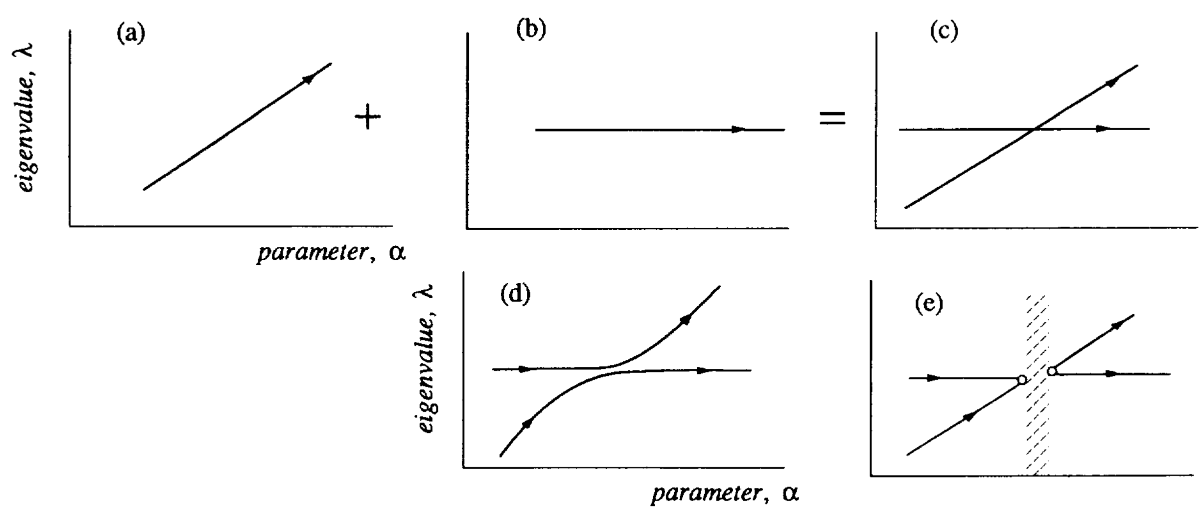


Figure 10: Composition of two one degree of freedom systems. Eigenvalue
(a) varies with parameter in subsystem 1; (b) is invariant in subsystem 2;
(c) composite bifurcation diagram (degenerate);
(d) conservative coupling; (curve veering); (e) non-conservative coupling (eigenvalue collision).

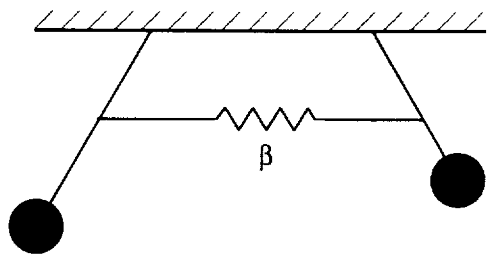


Figure 11: Two pendula with conservative coupling, β .

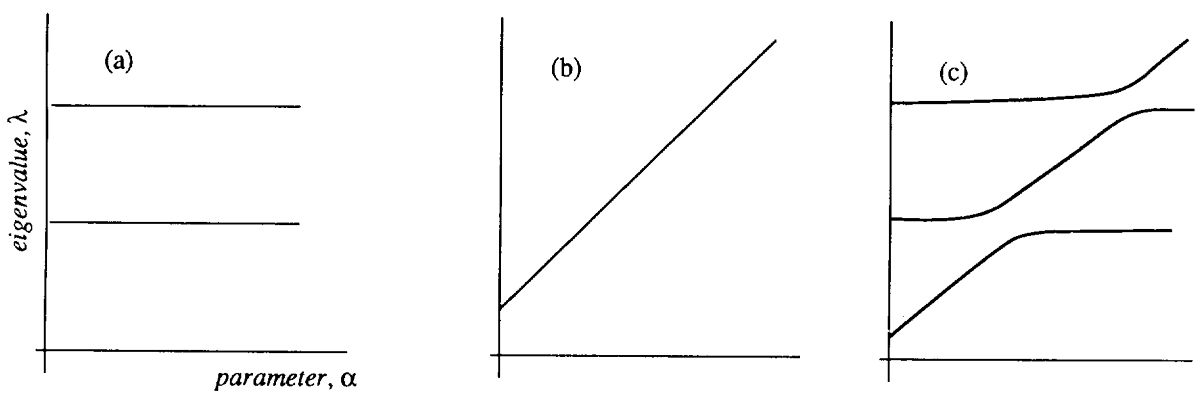


Figure 12: Interaction between a 2-degree-of-freedom system P and a single degree of freedom system Q with conservative coupling; (a) the eigenvalues of system P are invariant with system parameter; (b) that of system Q increases linearly with system parameter; (c) composite system's eigenvalue loci.

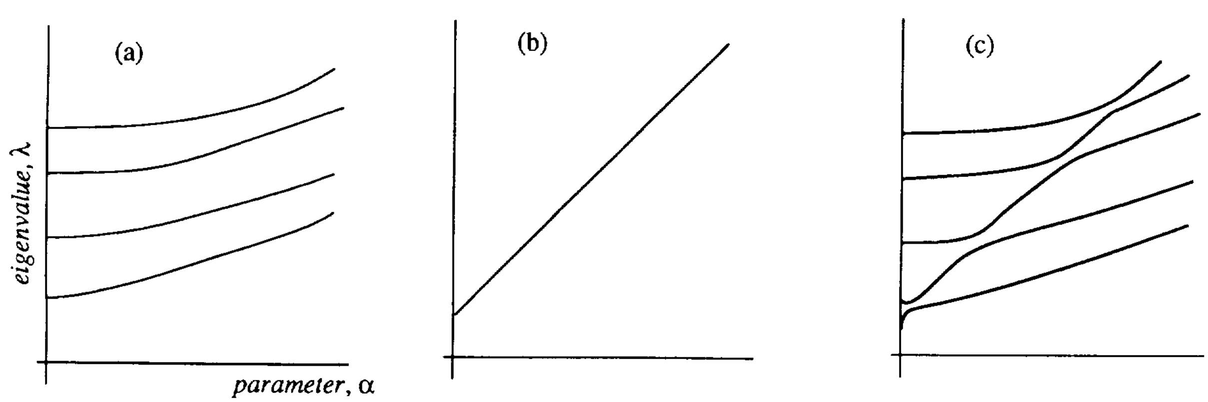


Figure 13: Interaction between a multi degree-of-freedom system P and a single degree of freedom system Q with conservative coupling; (a) the eigenvalues of system P vary *nonlinearly* with system parameter; (b) that of system Q varies linearly with system parameter; (c) composite system's eigenvalue loci.

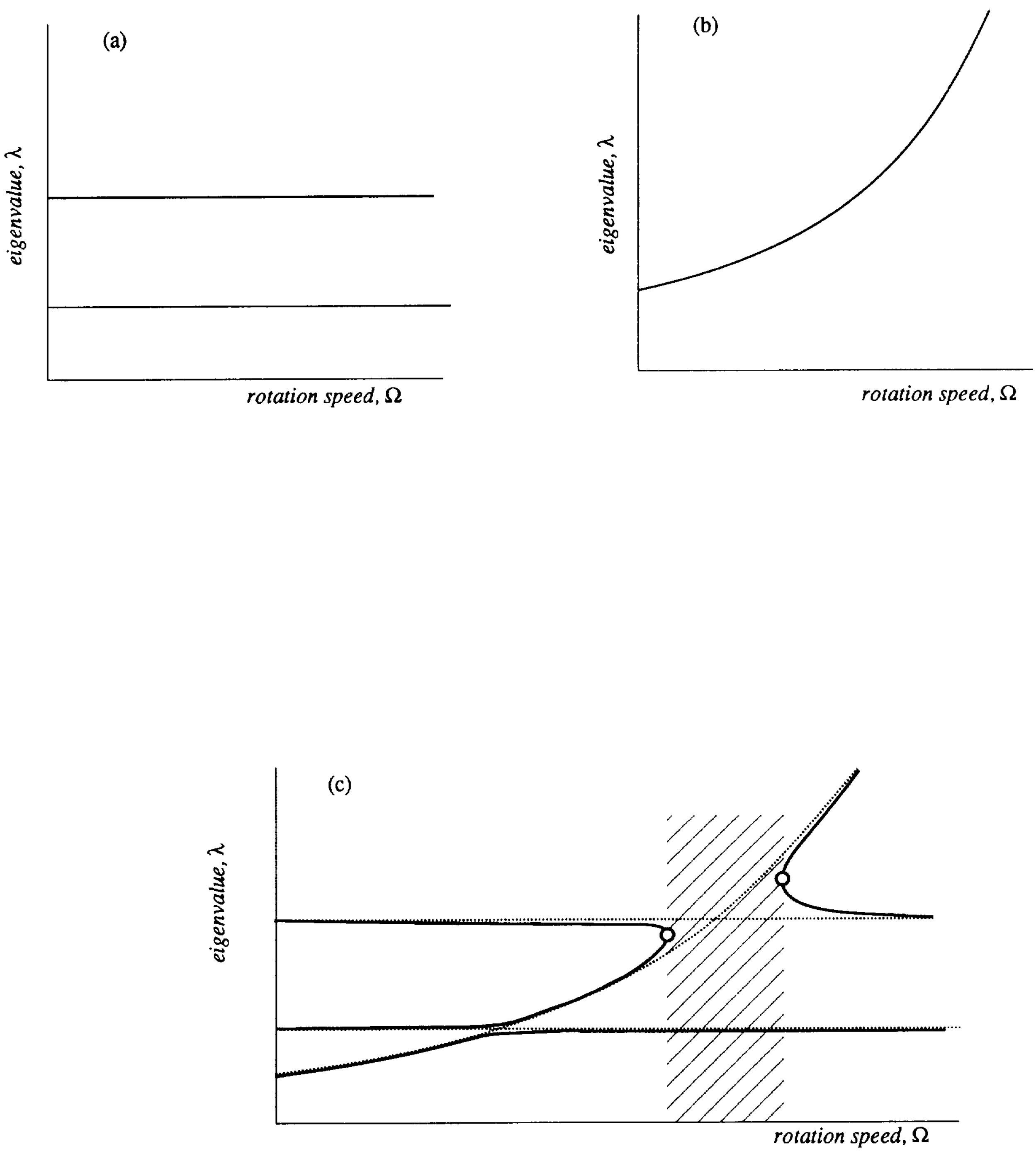


Figure 14: Engine/propeller interaction leading, first to eigenvalue veering, then to eigenvalue collision.
(a) engine frequency is invariant with parameter (rotation speed);
(b) variation of propeller frequency with rotation speeed;
(c) variation of the coupled system's frequencies with rotation speed.

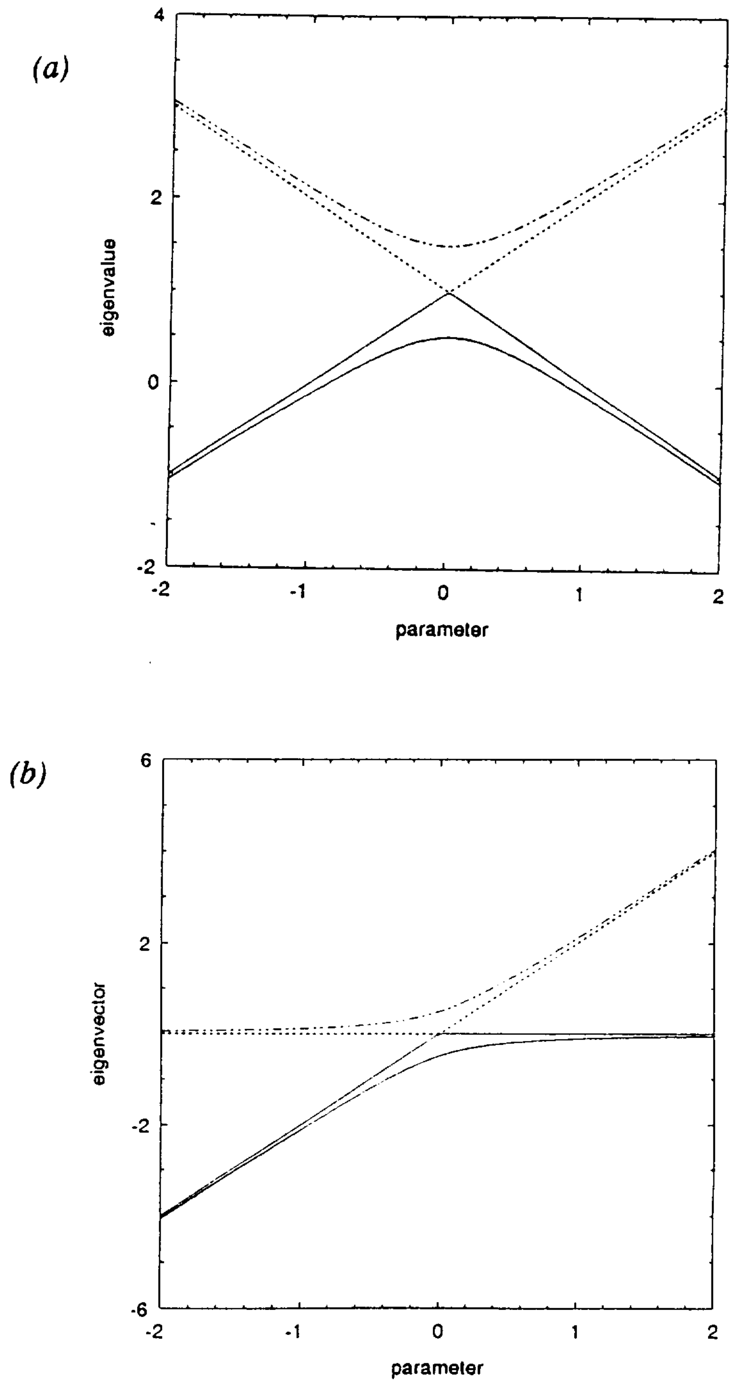


Figure 15: Loci showing veering of the (a) eigenvalues, and (b) eigenvectors in the canonical system with conservative coupling.

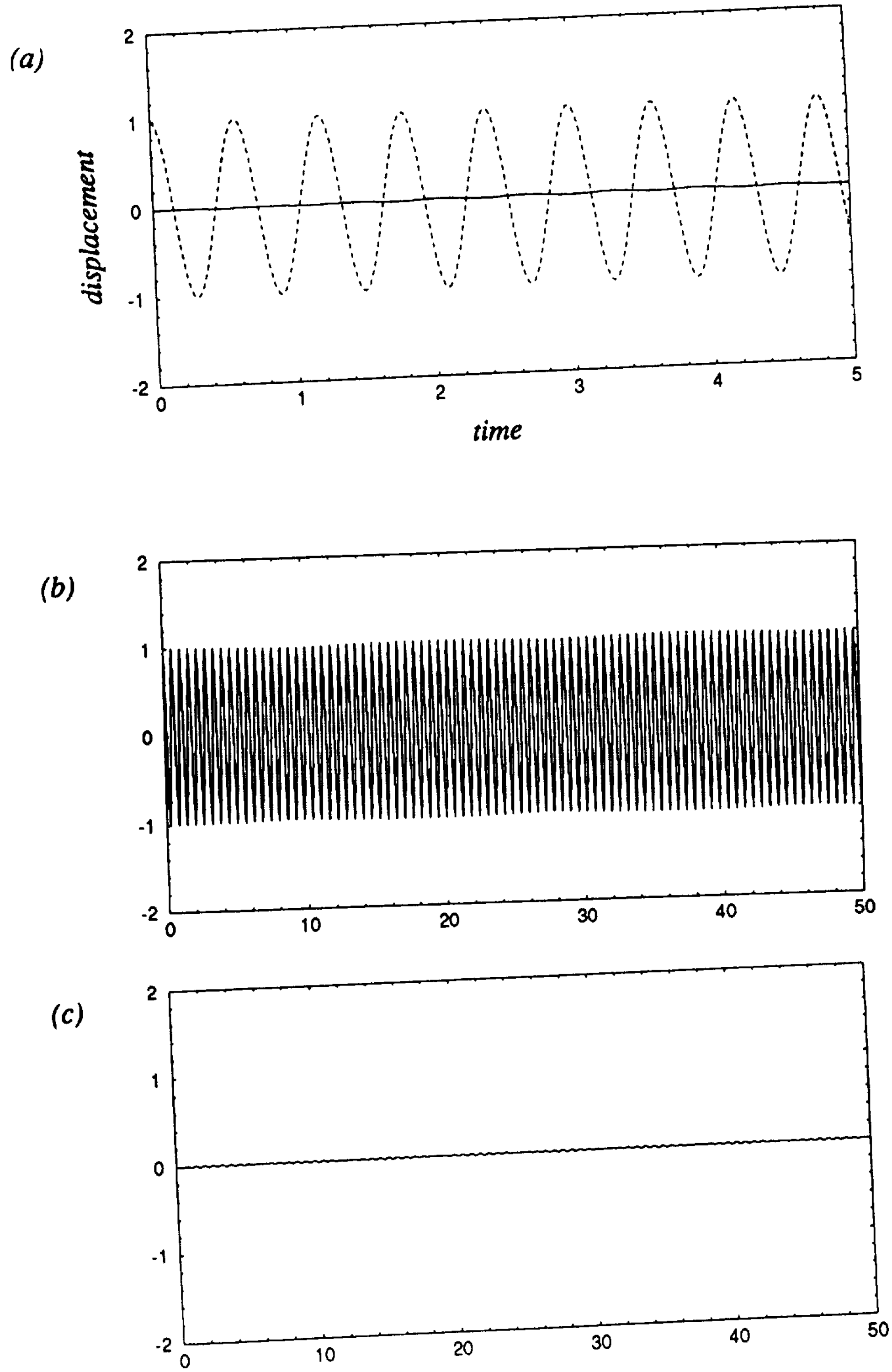


Figure 16: Time domain response of the canonical system with conservative coupling
(a) short term, x and y displacement; (b) long term, x displacement;
(c) long term, y displacement
...... $x(t)$; ____ $y(t)$; $x(0) = 1, y(0) = 0$.

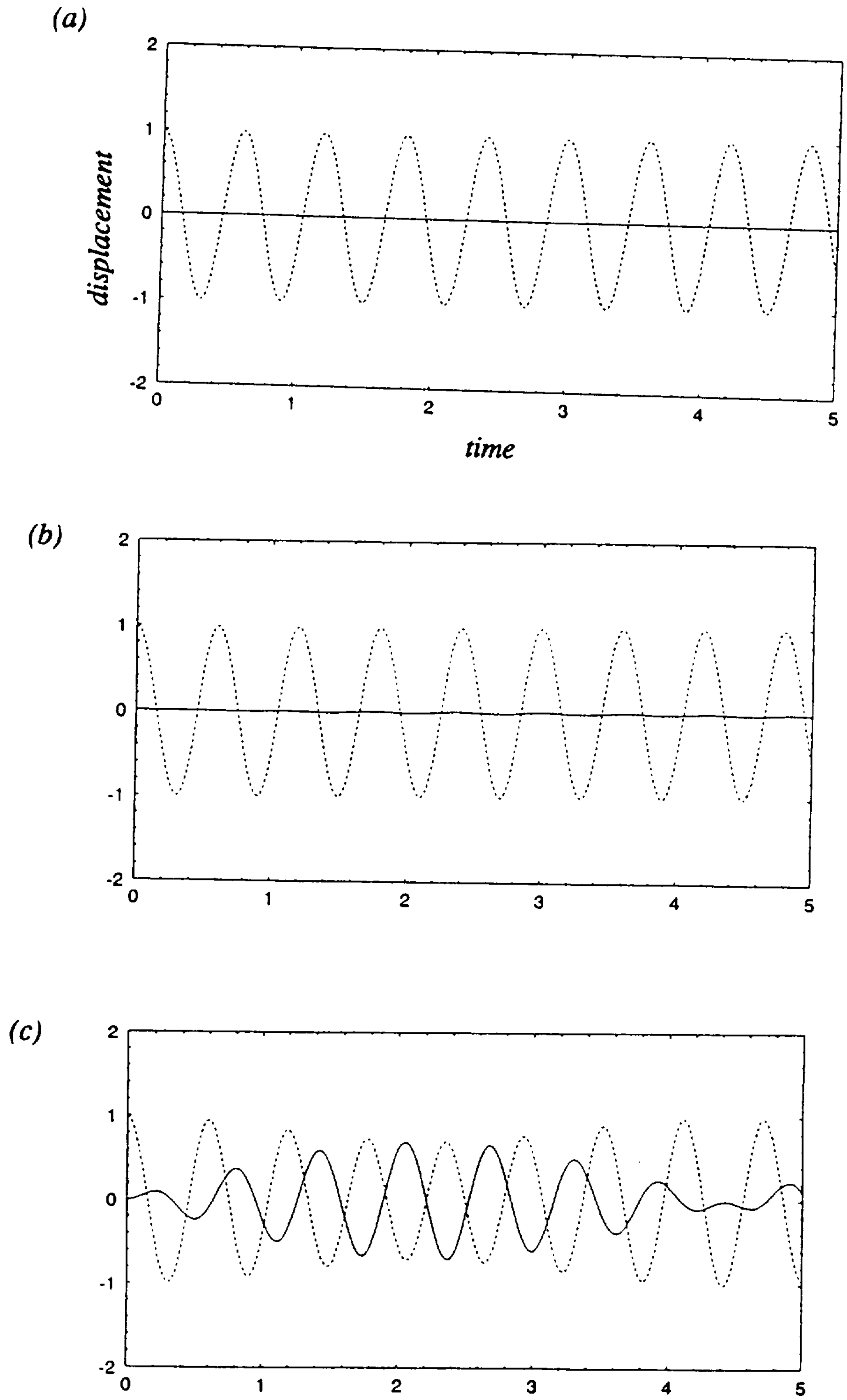
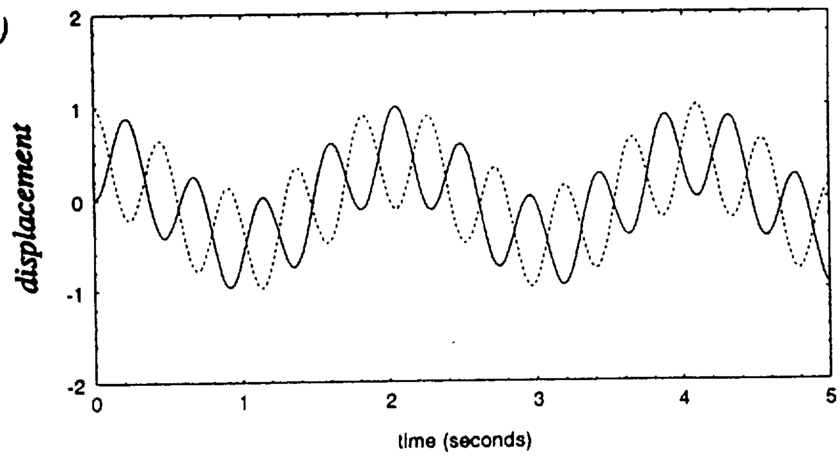


Figure 17: Time domain response of the canonical system with conservative coupling for various coupling strengths and fixed mistuning.
(a) $\alpha = 0.1, \beta = 0$; (b) $\alpha = 0.1, \beta = 0.01$; (c) $\alpha = 0.1, \beta = 0.1$
.... $x(t)$; ____ $y(t)$; $x(0) = 1, y(0) = 0$.

(d)



(e)

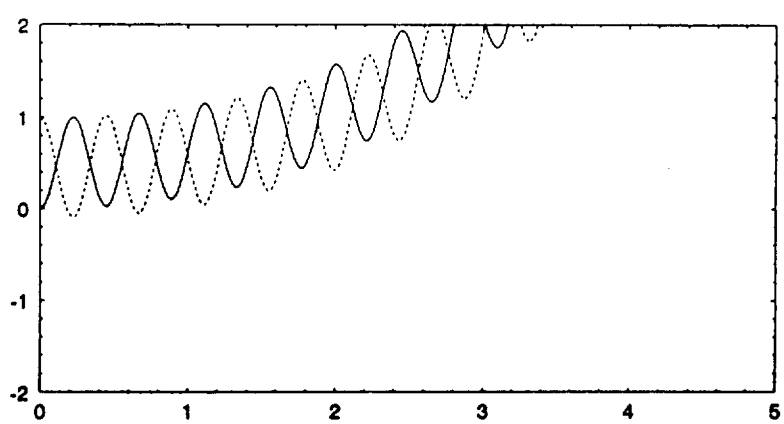


Figure 17 *(concluded)*: Time domain response of the canonical system with conservative coupling for various coupling strengths and fixed mistuning.
(d) $\alpha = 0.1$, $\beta = 0.9$; (e) $\alpha = 0.1$, $\beta = 1$ (buckling limit).

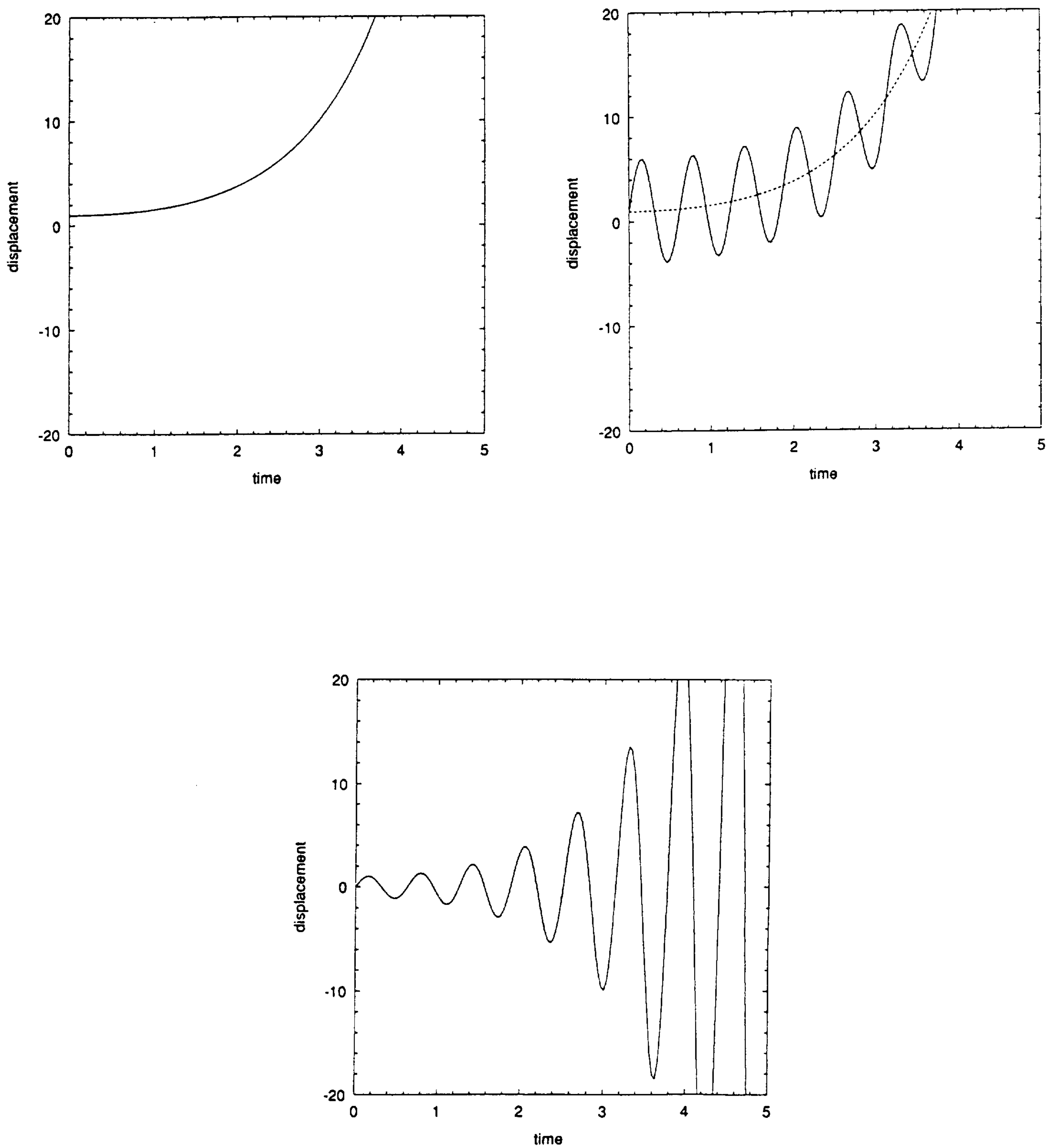


Figure 18: Different types of elastic instabilities in a mechanical system with 2 d.o.f.
(a) static divergence; (b) dynamic divergence;
(c) oscillatory divergence (flutter).

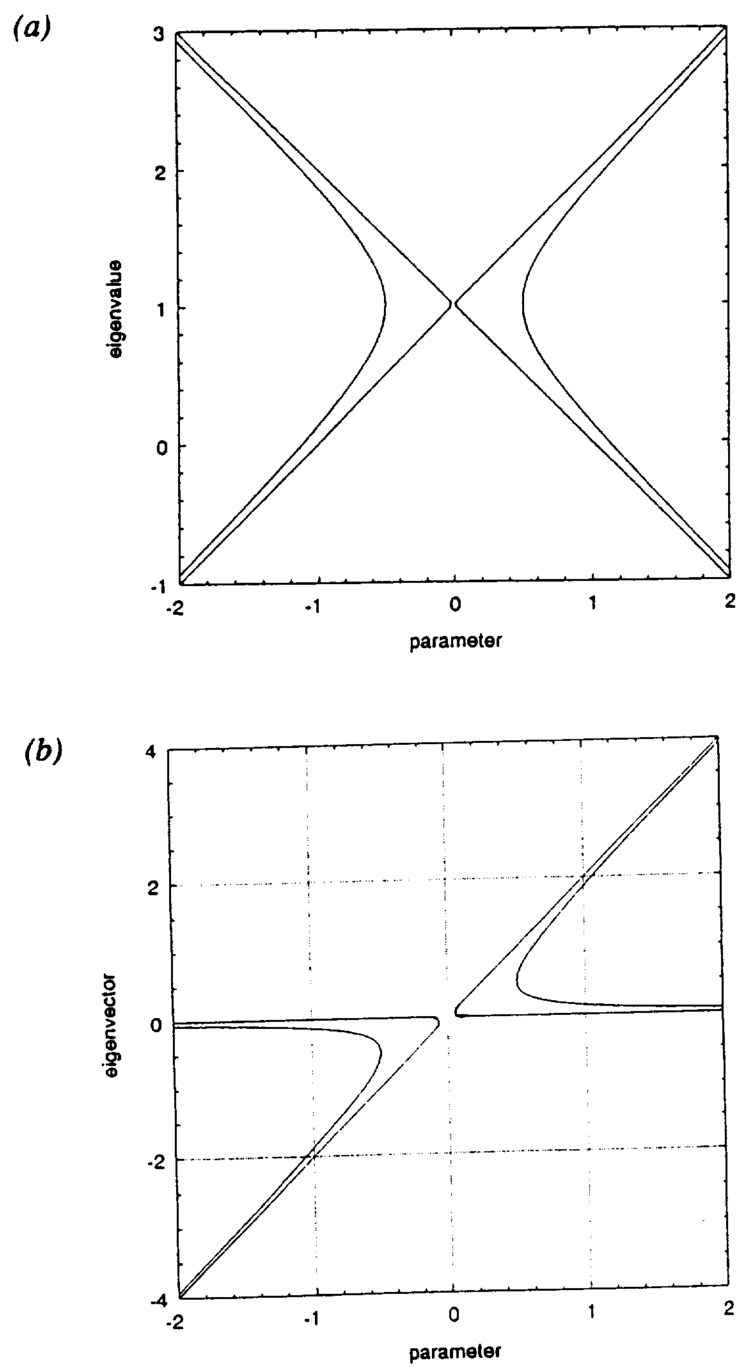


Figure 19: Loci showing attraction, and ultimate collision, in the (a) eigenvalues, and (b) eigenvectors of the canonical system with non-conservative coupling.

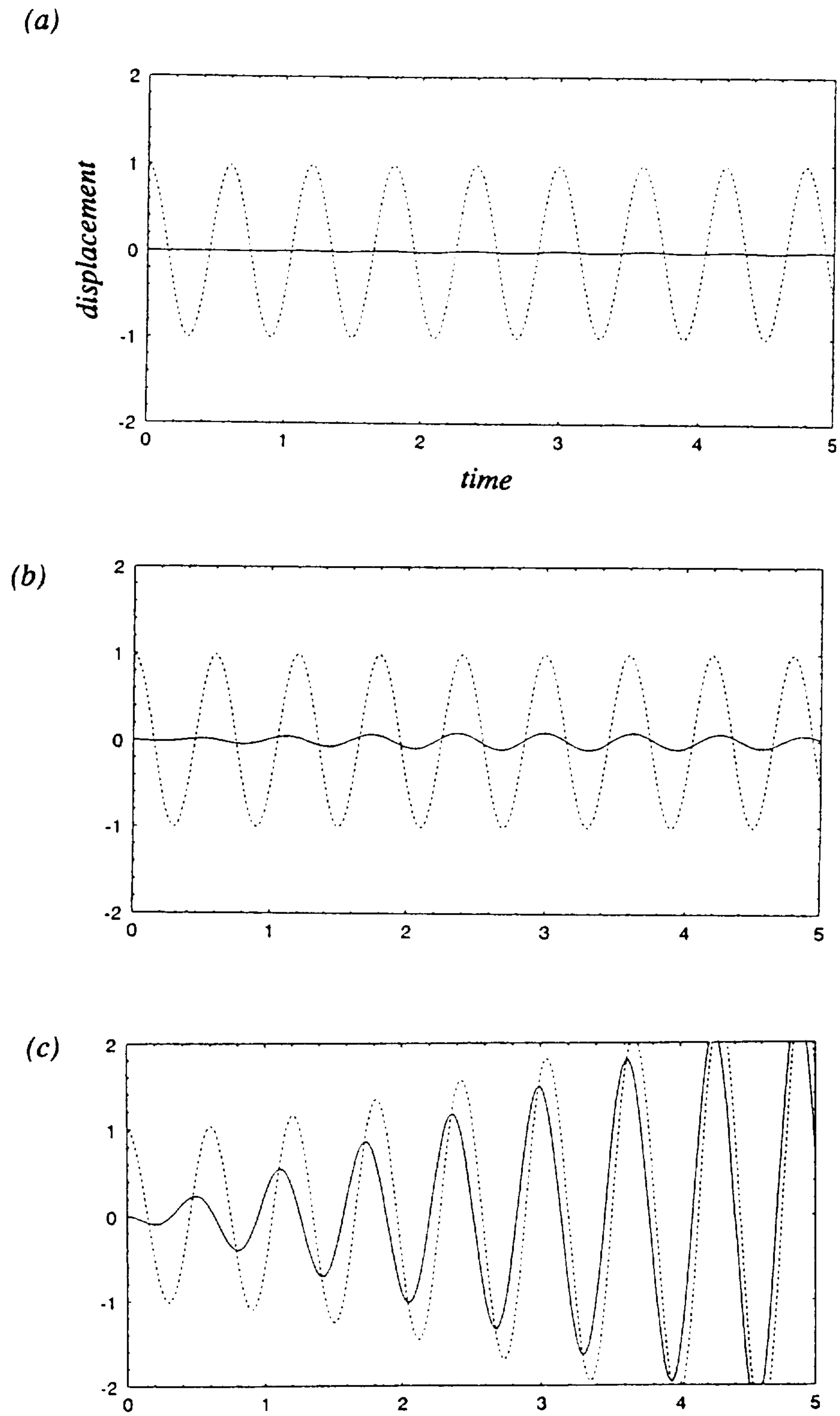


Figure 20: Time domain response of the canonical system with non-conservative coupling for various coupling strengths and fixed mistuning.
(a) $\alpha = 0.1, \beta = 0.001$; (b) $\alpha = 0.1, \beta = 0.01$; (c) $\alpha = 0.1, \beta = 0.1$
...... $x(t)$; ____ $y(t)$; $x(0) = 1, y(0) = 0$.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (*Leave blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	November 1991	Technical Memorandum

4. TITLE AND SUBTITLE
Modal Interaction in Linear Dynamic Systems Near Degenerate Modes

5. FUNDING NUMBERS
WU-505-62-21

6. AUTHOR(S)
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7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135-3191

8. PERFORMING ORGANIZATION REPORT NUMBER
E-6668

9. SPONSORING/MONITORING AGENCY NAMES(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, D.C. 20546-0001

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
NASA TM-105315
ICOMP-91-21

11. SUPPLEMENTARY NOTES
D. Afolabi, Purdue University, Indianapolis, Indiana and Institute for Computational Mechanics in Propulsion, NASA Lewis Research Center (work funded under Space Act Agreement C-99066-G). Space Act Monitor: Louis A. Povinelli, (216) 433-6714.

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified - Unlimited
Subject Category 39

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

In various problems in structural dynamics, the eigenvalues of a linear system depend on a characteristic parameter of the system. Under certain conditions, two eigenvalues of the system approach each other as the characteristic parameter is varied, leading to modal interaction. In a system with "conservative coupling", the two eigenvalues eventually repel each other, leading to the curve veering effect. In a system with "non-conservative coupling", the eigenvalues continue to attract each other, eventually colliding, leading to eigenvalue degeneracy. We study modal interaction in linear systems with conservative and non-conservative coupling using singularity theory, sometimes known as catastrophe theory. Our main result is this: eigenvalue degeneracy is a cause of instability; in systems with conservative coupling it induces only *geometric* instability, whereas in systems with non-conservative coupling eigenvalue degeneracy induces both *geometric* and *elastic* instability. Illustrative examples of mechanical systems are given.

14. SUBJECT TERMS
Flutter; Stability; Vibration localization; Catastrophe theory; Singularity theory

15. NUMBER OF PAGES
46

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102