Third International Conference on Inverse Design Concepts and Optimization in Engineering Sciences (ICIDES-III), Editor: G.S. Dulikravich, Washington D.C., October 23-25, 1991.

The Research Progress on Hodograph Method of Aerodynamic Design at Tsinghua University

Chen Zuoyi Guo Jingrong
(Tsinghua University)

1. Introduction

The Hodograph Method is a classical method in fluid dynamics. Because it can transform the nonlinear equation to the linear. So it is always used to resolve the fluid dynamic equation in the early time. For example, the well known Karmen–Jein formula is obtained from Hodograph equation. Then the Hodograph method is widely applicated in the research of transonic flow. Due to the Mixed–Type character of transonic flow equation, the Hodograph method may be the only accurate method to solve the transonic flow equation. In the 1970's, the Hodograph method start to be used in the inverse problem of fluid dynamics. (i.e. design problem, such as the work by Hobsen and karadimas (Ref.1) In the 1980's., we have done systematic reserch work on Hodograph method in Tsinghua University, and have taken much progress as the follows.

1. Research progress on the analytical solution of Hodograph Mixed–Type equation

The various analytical solutions of Hodograph Mixed–Type equation are presented in Ref.3. If we use these solutions to solve the inverse problem, especially design the cascade profile, there are limits on various boundary condition. We can't always find right solution under any boundary condition. So that the application scope may be restricted. The differences among various analytical solution of Hodograph Mixed–Type equations and their application conditions have been presented. It shows the differences are obvious (see Fig.1).

The application scope of the four nozzle–solution of Hodograph is shown in Table.1.

All the nozzle–solutions of Hodograph were straight line corresponding with the mean stream line before. If we use them in the inverse problem of cascade, the best one is prefered to be curve mean streamline. The nozzle–solution of Hodograph corresponding with the curve mean streamline is presented. We get the analytical solution of Mixed–Type Hodograph equation that is corresponding with the parabolic coordinates, as shown in Fig.2.

The relationship between the parabolic and rectangular coordinates is:

$$x = \frac{1}{2}(\xi^2 - \eta^2) \tag{1}$$

$$y = \xi\eta \tag{2}$$

Thus the vorticity of V may be represented by:

Third International Conference on Inverse Design Concepts and Optimization in Engineering Sciences (ICIDES-III). Editor: G.S. Dulikravich, Washington D.C., October 23-25, 1991.

$$\nabla \times \bar{V} = \frac{1}{\xi^2 + \eta^2}\left(\frac{\partial (V_\eta \sqrt{\xi^2 + \eta^2})}{\partial \zeta} - \frac{\partial (V_\zeta \sqrt{\xi^2 + \eta^2})}{\partial \eta} \right) \tag{3}$$

It satisfies:

$$\frac{\partial \psi}{\partial \xi} = \sqrt{\xi^2 + \eta^2}\,\rho V_\eta \tag{4}$$

$$\frac{\partial \psi}{\partial \eta} = \sqrt{\xi^2 + \eta^2}\,\rho V_\zeta \tag{5}$$

If the equivalent velocity is introduced:

$$\hat{V}_\eta = \sqrt{\xi^2 + \eta^2}\, V_\eta \tag{6}$$

$$\hat{V}_\zeta = \sqrt{\xi^2 + \eta^2}\, V_\zeta \tag{7}$$

the similar Mixed–Type Hodograph equation can be obtained:

$$\frac{\partial^2 \psi}{\partial \hat{\sigma}^2} + K(\sigma)\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{8}$$

2. Design of the transonic turbine and compressor cascade using Hodograph method

we have presented the new method to design the transonic turbine and compressor cascade using Hodograph. This method is that applicated the Chaplygin equation numerical solution to solve the subsonic area and applicated the analytical solution of Mixed–Type Hodograph equation to solve the supersonic area.

The differential equation for numerical solution of Chaplygin equation is:

$$\begin{aligned}\psi_{r_{i,k}} = \Bigg[B_{i,k} + \frac{p_{q,k}}{S_{i,k}}\psi_{r_{i+1,k}} + \left(\frac{Q_{i,k}}{l^2_{i,k}} + \frac{T_{i,k}}{2l_{i,k}}\right)\psi_{r_{i,k+1}} \\ + \frac{P_{i,k}}{S^2_{i,k}}\psi_{r_{i-1,k}} + \left(\frac{Q_{i,k}}{l^2_{i,k}} - \frac{T_{i,k}}{2l_{i,k}}\right)\psi_{r_{i,k-1}} \Bigg] / [2P_{i,k} / S^2_{i,k} + 2Q_{i,k} / l^2_{i,k}]\end{aligned} \tag{9}$$

The profile coordinates equations are:

$$x - x_0 = \int_1^{M^*_a} -h\frac{\rho_\infty}{\rho^*}\frac{Ma^*_\infty}{Ma^*}\cos\theta_\infty\left[P + Q\left(\frac{d\theta}{dMa^*}\right)^2\right]\frac{\partial p}{\partial \theta}\cos\theta dMa^* \tag{10}$$

Third International Conference on Inverse Design Concepts and Optimization in Engineering Sciences (ICIDES-III), Editor: G.S. Dulikravich, Washington D.C., October 23-25, 1991.

$$y - y_0 = \int_1^{Ma^*} -h\frac{\rho_\infty}{\rho^*}\frac{Ma^*_\infty}{Ma^*}\cos\theta_\infty\left[P + Q\left(\frac{d\theta}{dMa^*}\right)^2\right]\frac{\partial\psi}{\partial\theta}\sin\theta dMa^* \tag{11}$$

The profile coordinates equations which are obtained by the analytical solution of Mixed–Type Hodograph equation are:

$$x - x_0 = \left[\int -l^*\frac{\cos\theta}{Ma^*}\left(\frac{\partial\psi}{\partial\sigma} + K_a(\sigma)\frac{\left(\frac{\partial\psi}{\partial\theta}\right)^2}{\frac{\partial\psi}{\partial\sigma}}\right)d\theta\right]_{\psi=1} \tag{12}$$

$$y - y_0 = \left[\int_{\theta_o}^{\theta} -l^*\frac{\sin\theta}{Ma^*}\left(\frac{\partial\psi}{\partial\sigma} + K_a(\sigma)\frac{\left(\frac{\partial\psi}{\partial\theta}\right)^2}{\frac{\partial\psi}{\partial\sigma}}\right)d\theta\right]_{\psi=1} \tag{13}$$

The location of shock wave is the importantance in design. The method of profile design by the given location of shock wave is also presented. The comparision between the design shock wave locatgion and the experiment is shown in Fig.3.

3. The basic equation of three dimensional Hodograph method

On the basis of the general theory in three dimensional flow and the correspondence between the physical surface and Hodograph, the basic equations of three dimensional Hodograph method have been obtained.

Using the concept of equivalent physical surface and equivalent Hodograph, we obtain the different equation of streamfunction with Hodograph coordinates and integral equation for returning from the Hodograph to the physical surface.

The correspondence between S1 flow surface and equivalent physical surface is shown in Fig.4.

The streamfunction equation of Hodograph corresponding to the S1 flow surface is

$$A_1\frac{\partial^2\psi}{\partial\hat{\omega}^2} + A_2\frac{\partial\psi}{\partial\hat{\omega}} + A_3\frac{\partial^2\psi}{\partial\hat{\omega}\partial\hat{\theta}} + A_4\frac{\partial\psi}{\partial\hat{\theta}} + A_5\frac{\partial^2\psi}{\partial\hat{\theta}^2} = 0 \tag{14}$$

here $\hat{\omega}$ is equivalent velocity. $\hat{\theta}$ is the equivalent flow angle.

The relationship between the equivalent velocity and true velocity is

$$\hat{\omega}e^{i\hat{\theta}} = A + Bi$$

$$B = \frac{1}{K_3}(r\omega_o + r^2\omega - r\omega_r\frac{n_\theta}{n_r}) \tag{15}$$

Third International Conference on Inverse Design Concepts and Optimization in Engineering Sciences (ICIDES-III), Editor: G.S. Dulikravich, Washington D.C., October 23-25, 1991.

$$A = \frac{1}{K_2}(\varepsilon\omega_z + \beta\omega_\theta)$$

Similarly, we can obtain the streamfunction equation of Hodograph corresponding to the S2 flow surface, that is:

$$\tilde{A}_1\frac{\partial^2\psi}{\partial\tilde{\omega}^2} + \tilde{A}_2\frac{\partial\psi}{\partial\tilde{\omega}} + \tilde{A}_3\frac{\partial^2\psi}{\partial\tilde{\omega}\partial\tilde{\theta}} + \tilde{A}_4\frac{\partial\psi}{\partial\tilde{\theta}} + \tilde{A}_5\frac{\partial^2\psi}{\partial\tilde{\theta}^2} = 0 \tag{16}$$

The correspondence between S2 flow surface and equivalent physical surface is shown in Fig.5.

The relationship between the equivalent velocity and actual velocity is:

$$\tilde{A}K_6 = \omega_z - \frac{n_z}{n_\theta}(\omega_\theta + \omega r) \tag{17}$$

$$\tilde{B}K_5 = \omega_r - \frac{n_r}{n_\theta}(\omega_\theta + \omega r) \tag{18}$$

The integral equations for returning from the Hodograph to the S1 surface are:

$$z - z_o = \int_{\hat{\theta}_o}^{\hat{\theta}} \left\{\frac{1}{\sqrt{\varepsilon}\hat{\omega}}cos\hat{\theta}\left[\frac{\partial(\frac{\sqrt{\varepsilon}}{\rho b})}{\partial\hat{\theta}}\frac{\partial\psi}{\partial\hat{\theta}} + \frac{\sqrt{\varepsilon}}{b\rho\hat{\omega}}\frac{(\frac{\partial\psi}{\partial\hat{\theta}})^2}{(\frac{\partial\psi}{\partial\hat{\omega}})} - \frac{\partial(\frac{\sqrt{\varepsilon}}{b\rho})}{\partial\hat{\omega}}\frac{(\frac{\partial\psi}{\partial\hat{\theta}})^2}{(\frac{\partial\psi}{\partial\hat{\omega}})} + \frac{\sqrt{\varepsilon}}{b\rho}\hat{\omega}\frac{\partial\psi}{\partial\hat{\omega}}\right]\right\}d\hat{\theta} \tag{19}$$

$$\theta - \theta_o = \int_{\hat{\theta}_o}^{\hat{\theta}} \left\{\frac{1}{r\hat{\omega}}sin\hat{\theta}\left[\frac{\partial(\frac{\sqrt{\varepsilon}}{\rho b})}{\partial\hat{\theta}}\frac{\partial\psi}{\partial\hat{\theta}} + \frac{\sqrt{\varepsilon}}{b\rho\hat{\omega}}\frac{(\frac{\partial\psi}{\partial\hat{\theta}})^2}{(\frac{\partial\psi}{\partial\hat{\omega}})} - \frac{\partial(\frac{\sqrt{\varepsilon}}{b\rho})}{\partial\hat{\omega}}\frac{(\frac{\partial\psi}{\partial\hat{\theta}})^2}{(\frac{\partial\psi}{\partial\hat{\omega}})} + \frac{\sqrt{\varepsilon}}{b\rho}\hat{\omega}\frac{\partial\psi}{\partial\hat{\omega}}\right]\right\}d\hat{\theta} \tag{20}$$

The integral equation for returning from the Hodograph to the S2 surface is:

$$z - z_o = \int_{\tilde{\theta}_o}^{\tilde{\theta}} \frac{cos\tilde{\theta}}{\tilde{\omega}K_6}\left[\frac{\partial K_4}{\partial\tilde{\theta}} - \frac{\partial\psi}{\partial\tilde{\theta}} + \frac{K_4}{\tilde{\omega}}\frac{(\frac{\partial\psi}{\partial\tilde{\theta}})^2}{(\frac{\partial\psi}{\partial\tilde{\omega}})} - \frac{\partial K_4}{\partial\tilde{\omega}}\frac{(\frac{\partial\psi}{\partial\tilde{\theta}})^2}{(\frac{\partial\psi}{\partial\tilde{\omega}})} + \tilde{\omega}K_4\frac{\partial\psi}{\partial\tilde{\omega}}\right]d\tilde{\theta} \tag{22}$$

Third International Conference on Inverse Design Concepts and Optimization in Engineering Sciences (ICIDES-III), Editor: G.S. Dulikravich, Washington D.C., October 23-25, 1991.

$$r - r_o = \int_{\bar{\theta}_o}^{\bar{\theta}} \frac{\sin\bar{\theta}}{\bar{\omega} K_5} \left[\frac{\partial K_4}{\partial \bar{\theta}} \frac{\partial \psi}{\partial \bar{\theta}} + \frac{K_4}{\bar{\omega}} \frac{(\frac{\partial \psi}{\partial \bar{\theta}})^2}{(\frac{\partial \psi}{\partial \bar{\omega}})} - \frac{\partial K_4}{\partial \bar{\omega}} \frac{(\frac{\partial \psi}{\partial \bar{\theta}})^2}{(\frac{\partial \psi}{\partial \bar{\omega}})} + \bar{\omega} K_4 \frac{\partial \psi}{\partial \bar{\omega}} \right] d\bar{\theta} \tag{23}$$

4. The aerodynamic design of Hodograph is revolutionary surface

On the basis of the three dimensional flow Hodograph method, the aerodynamic design of Hodograph in revolutionary surface is presented.

The Hodograph equation corresponding to the revolutionary surface can be obtained from Hodograph equation corresponding to the S1 flow surface.

$$M_a^{\bullet} \frac{\partial^2 \psi}{\partial Ma^{\bullet 2}} + \left[1 - \frac{M_a^{\bullet}}{\bar{b}} \frac{\partial \bar{b}}{\partial M_a^{\bullet}} - \frac{M_a^{\bullet}}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial M_a^{\bullet}} + \frac{2}{\bar{b}^2} (\frac{\partial b}{\partial \theta})^2 - \frac{1}{\bar{b}} \frac{\partial^2 b}{\partial \theta^2} \right] \frac{\partial \psi}{\partial M_a^{\bullet}}$$
$$- \frac{1}{\bar{b}} \frac{\partial \bar{b}}{\partial \theta} \frac{\partial^2 \psi}{\partial M_a^{\bullet} \partial \theta} - \left[\frac{1}{\bar{b} M_a^{\bullet}} \frac{\partial \bar{b}}{\partial \theta} + \frac{2}{\bar{b}^2 \bar{\rho}} + \frac{\partial \bar{b}}{\partial M_a^{\bullet}} \frac{\partial \bar{b}}{\partial \theta} + \frac{\partial \bar{b}}{\partial \theta} \frac{\partial \bar{\rho}}{\partial M_a^{\bullet}} \frac{1}{\bar{b} \bar{\rho}^2} + \frac{1}{\bar{b}} \frac{\partial^2 \bar{b}}{\partial M_a^{\bullet} \partial \theta} \right] \frac{\partial \psi}{\partial \theta}$$
$$+ \left[\frac{1}{M_a^{\bullet}} + \frac{1}{\bar{b}} \frac{\partial \bar{b}}{\partial M_a^{\bullet}} + \frac{1}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial Ma^{\bullet}} \right] \frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{23}$$

The integral equations for returning from the Hodograph to revolutionary surface are

$$z - z_0 = \int_{\theta}^{\theta} \frac{h\rho_\infty Ma_\infty^{\bullet} \cos\hat{\theta}_\infty \cos\hat{\theta}}{Ma^{\bullet} \rho^{\bullet}} \left[\frac{\partial(\frac{1}{\bar{\rho} \bar{b}})}{\partial \hat{\theta}} \frac{\partial \psi}{\partial \hat{\theta}} + \frac{1}{\bar{b} \bar{\rho} Ma^{\bullet}} \frac{(\frac{\partial \psi}{\partial \hat{\theta}})^2}{\frac{\partial \psi}{\partial Ma^{\bullet}}} - \frac{(\frac{\partial \psi}{\partial \hat{\theta}})^2}{\frac{\partial \psi}{\partial Ma^{\bullet}}} \frac{\partial(\frac{1}{\bar{\rho} \bar{b}})}{\partial Ma^{\bullet}} \right.$$
$$\left. + \frac{Ma^{\bullet}}{\bar{b} \bar{\rho}} \frac{\partial \psi}{\partial Ma^{\bullet}} \right] d\hat{\theta} \tag{24}$$

$$\theta - \theta_0 = \int_{\theta}^{\theta} \frac{h\rho_\infty Ma_\infty^{\bullet} \cos\hat{\theta}_\infty \sin\hat{\theta}}{(r_0 + z tg\alpha_1) Ma^{\bullet} \rho^{\bullet}} \left[\frac{\partial(\frac{1}{\bar{\rho} \bar{b}})}{\partial \hat{\theta}} \frac{\partial \psi}{\partial \hat{\theta}} + \frac{1}{\bar{b} \bar{\rho} Ma^{\bullet}} \frac{(\frac{\partial \psi}{\partial \hat{\theta}})^2}{\frac{\partial \psi}{\partial Ma^{\bullet}}} - \frac{(\frac{\partial \psi}{\partial \hat{\theta}})^2}{\frac{\partial \psi}{\partial Ma^{\bullet}}} \frac{\partial(\frac{1}{\bar{\rho} \bar{b}})}{\partial Ma^{\bullet}} \right.$$
$$\left. + \frac{Ma^{\bullet}}{\bar{b} \bar{\rho}} \frac{\partial \psi}{\partial Ma^{\bullet}} \right] d\hat{\theta} \tag{25}$$

In order to determine the supersonic region, the analytical solution of Hodograph Mixed–Type equation corresponding to revolutionary surface is presented.

From the streamfunction equation:

Third International Conference on Inverse Design Concepts and Optimization in Engineering Sciences
(ICIDES-III). Editor: G.S. Dulikravich, Washington D.C., October 23-25, 1991.

$$Ma^{*}\left[\frac{\partial^{2}\psi}{\partial\sigma^{2}}K^{2}-\frac{\partial\psi}{\partial\sigma}\frac{dK}{dMa^{*}}\right]+\frac{Ma^{*}}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial Ma^{*}}K\frac{\partial\psi}{\partial\sigma}+\left[\frac{2}{Ma^{*}}+\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial Ma^{*}}\right]\frac{\partial^{2}\psi}{\partial\hat{\theta}^{2}}=0 \qquad (26)$$

If put the

$$F(\sigma)=\left[\frac{2}{Ma^{*}}+\frac{1}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial Ma^{*}}\right]/(Ma^{*}K^{2}) \qquad (27)$$

thus the streamfunction equation can be transformed to the analytical solution equation which is similarly the plane flow

$$F(\sigma)\frac{\partial^{2}\psi}{\partial\hat{\theta}^{2}}+\frac{\partial^{2}\psi}{\partial\sigma^{2}}=0$$

Design example:

Design parameters:	$r=1.29$	$M_{\infty}=0.2238$	$\hat{\theta}_{\infty}=0$
	$T_{\infty}=1490$ k	$\theta_2=73$	$M_2=1.15$
	$R_0=300$mm	$\alpha_1=26.56$	$H_0=50.05$mm

The profile in the revoluitonary surface with the Hodograph Method is shown in Fig.6.

5. Summary

The research progress of Hodograph method on aerodynamic design in Tsinghua University has summrized in this article. i.e. (1) There are some restricted conditions in application with Hodograph method to design the transonic turbine and compressor cascades. (2) The Hodograph method design is suitable not only to the transonic turbine cascade but also to the transonic compressor cascade (3) The three dimensional Hodograph method will be developed after obtaining the basic equation in three dimensional of Hodograph method, as the example the transonic turbine cascade design of Hodograph in revolutionary surface is presented.

Reference

1. D.E.Hobson. CUED / A–Turbo / TR 40. NOV.1972.
2. Karadimas Von–Karman Institute for Fluid Dynamics Lecture Series 59.
3. C. FERRARI and F.G.TRICOMI. Transonic Aerodynamics Academic press York and Landon.
4. Chen Zuoyi, papers of ISABE. 1983, Paris.
5. Chen Zuoyi. Journal of Engineering Thermophysics, Vol.4. No.4. 1983.
6. Chen Zuoyi, Journal of Tsinghua Univ., Vol.22, No.1, 1982.
7. Chen Zuoyi, Journal of Tsinghua Univ., Vol.21, No.4, 1981.

Third International Conference on Inverse Design Concepts and Optimization in Engineering Sciences
(ICIDES-III). Editor: G.S. Dulikravich, Washington D.C., October 23-25, 1991.

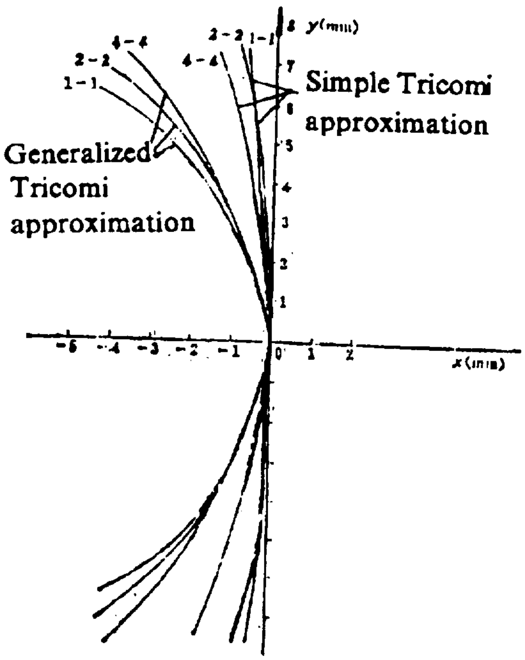

Fig.1 The Comparison between the simple Tricomi approximation and generalized Tricomi approximation under various condition

Table 1 The Comparison of application scope under various nozzle solution

$\frac{\partial M_a^*}{\partial x^*}$ \ θ_{q1}	−0.043	−0.06	−0.08	−0.10	−0.12	−0.14	−0.16
0.2	◇ ○ □ ☆	○ ☆					
0.26	◇ ○ □ ☆	◇ ○ □ ☆	◇ ○ ☆				
0.32		◇ □ ☆	◇ ○ □ ☆	◇ ○ □ ☆	◇ ○ □ ☆		
0.39					◇ □ ☆	□ ☆	□ ☆
0.45							□

Note: approximation
normal solution
- Generalized
- Homographic
- Tomotika–Tamada
- Generalized Tomotika–Tamade

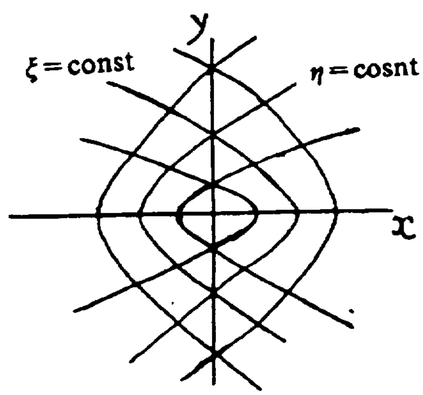

Fig.2 The parabolic coordinates in physic plane

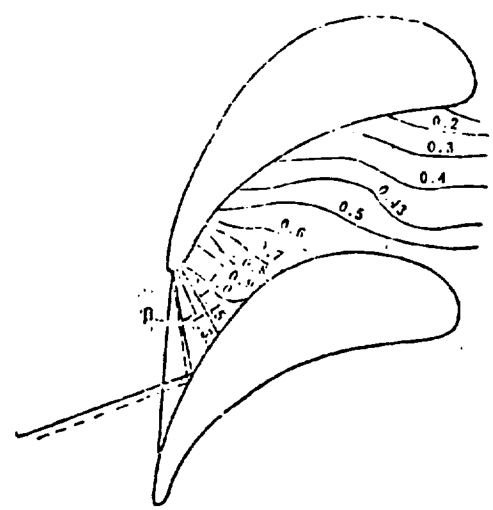

Fig.3 Comparison of shock wave Location Between Design and Experiment Q9 cascade profile
——experimental shock wave
——design shock wave

Third International Conference on Inverse Design Concepts and Optimization in Engineering Sciences (ICIDES-III), Editor: G.S. Dulikravich, Washington D.C., October 23-25, 1991.

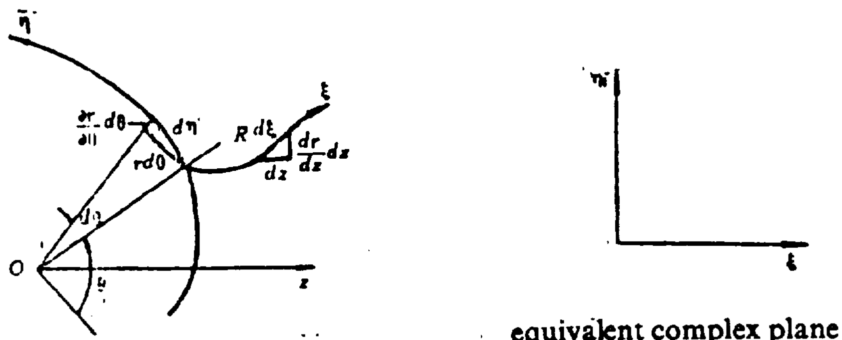

Fig.4 S$_1$ flow surface and equivalent complex plane

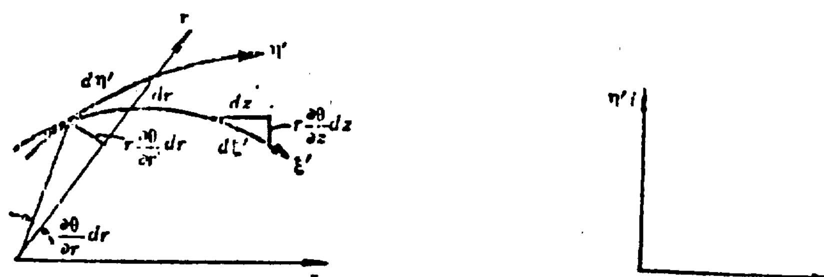

Fig.5 S$_2$ flow surface and equivalent complex plane

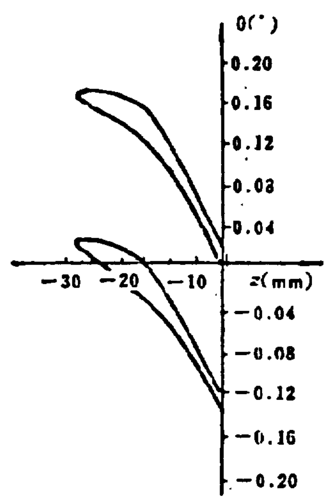

Fig.6 The transonic cascade profile in revolutionary surface using Hodograph method.