N92-14085

Elimination Of Secular Terms From The Differential Equations For The Elements of Perturbed Two-Body Motion

Victor R. Bond
and
Michael F. Fraietta

McDonnell Douglas Space Systems Company
Engineering Services Division
Houston, Texas

ABSTRACT

In 1961, Sperling linearized and regularized the differential equations of motion of the two-body problem by changing the independent variable from time to fictitious time by Sundman's transformation ($r = \frac{dt}{ds}$) and by embedding the two-body energy integral and the Laplace vector. In 1968, Burdet developed a perturbation theory which was uniformly valid for all types of orbits using a variation of parameters approach on the elements which appeared in Sperling's equations for the two-body solution. In 1973, Bond and Hanssen improved Burdet's set of differential equations by embedding the total energy (which is a constant when the potential function is explicitly dependent upon time.) The Jacobian constant was used as an element to replace the total energy in a reformulation of the differential equations of motion. In the process, another element which is proportional to a component of the angular momentum was introduced.

Recently trajectories computed during numerical studies of atmospheric entry from circular orbits and low thrust beginning in near-circular orbits exhibited numerical instability when solved by the method of Bond and Gottlieb (1989) for long time intervals. It was found that this instability was due to secular terms which appear on the right-hand sides of the differential equations of some of the elements. In this paper, this instability is removed by the introduction of another vector integral called the delta integral (which replaces the Laplace Vector) and another scalar integral which remove the secular terms. The introduction of these integrals requires a new derivation of the differential equations for most of the elements. For this rederivation, the Lagrange method of variation of parameters is used making the development more concise. Numerical examples of this improvement will be presented.

This work was performed for NASA-JSC Houston, Texas under Contract No. NAS9-17885.

1.0 Summary

In 1961 Sperling linearized and regularized the differential equations of motion of the two-body problem by changing the independent variable from time to fictitious time by Sundman's transformation ($r = \frac{dt}{ds}$) and by embedding the two-body energy integral and the Laplace vector which is also an integral of the motion into the Newtonian form of the differential equations of motion. The solution of Sperling's differential equations was uniformly valid for all types of orbits. In 1968, Burdet developed a perturbation theory using a variation of parameters approach on the 14 elements which appeared in the two-body solution. In 1973, Bond and Hanssen improved Burdet's set of differential equations by using the total energy of the perturbed system as a parameter instead of the two-body energy and by reducing the number of elements to 13. In 1989 Bond and Gottlieb embedded the Jacobian integral, which is a constant when the potential function is explicitly dependent upon time as well as position in the Newtonian equations. The Jacobian constant was used as an element to replace the total energy in a reformulation of the differential equations of motion. In this process, another element which is proportional to a component of the angular momentum is introduced. This brought the total number of elements back to 14. In this paper the Laplace vector is replaced by another vector integral as well as another scalar integral which remove small secular terms which appear in the differential equations for some of the elements.

2.0 Introduction

The non-linear differential equations of motion for the cartesian coordinates of the two-body problem can be regularized and linearized by the three-step procedure of changing the independent variable form time (t) to fictitious time (s) by the application of the Sundman transformation, embedding the Laplace integral and embedding the Jacobian integral.

By regularization we mean the removal of all singularities, and by linearization we mean that the differential equations for the cartesian coordinates are transformed to harmonic oscillators. Previously, regularization and linearization were done by Burdet (1968) by embedding the two-body energy which is constant only for the two-body problem and by Bond and Hanssen (1973) by embedding the total energy which is a constant when the two-body system is perturbed by a conservative potential (function of position only). In Bond and Gottlieb (1989), the Jacobian integral, which is a constant for the case of the two-body system perturbed by a potential function that is explicitly dependent on time as well as position, was embedded in the Newtonian equations. All three of these approaches reduce to the same system of equations in the absence of perturbations.

Recent numerical studies on atmospheric entry from near circular orbits and on low thrust in near circular orbits exhibit numerical instability when solved by the method of Bond and Gottlieb (1989) for long time intervals. These two cases are similar since both have persistent, tangential, non-conservative perturbations. It was found that this instability was due to secular terms which appear on the right hand sides of the differential equations of some of the elements. In this paper this instability is removed by the introduction of another vector integral of the motion and another scalar integral which remove the secular terms. The introduction of these integrals which were included by Burdet (1968) require a new derivation of the differential equations for most of the elements. For this rederivation the Lagrange method of variation of parameters is used making the development more concise.

2.1 The Differential Equations of Motion in the Fictitious Time

The differential equation for perturbed two-body motion is

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\underline{r} = \underline{F} \qquad (2.1)$$

where $\underline{r}$ is the position vector of one of the masses with respect to the other in cartesian coordinates and r is the magnitude of $\underline{r}$ and $(\dot{\ }) = \frac{d()}{dt}$. Also the gravitational constant is

$$\mu = G(M + m) \tag{2.2}$$

where G is the universal gravitational constant and M and m are the masses of the two bodies. The quantity $\underline{F}$ is the perturbation which can be expressed by,

$$\underline{F} = \underline{P} - \frac{\partial}{\partial \underline{r}} V(\underline{r}, t) \tag{2.3}$$

where $V(\underline{r}, t)$ is the potential due to perturbing masses and $\underline{P}$ is any perturbative acceleration which is not derived from a potential.

Equation (2.1) can be linearized (except for the perturbation) in three steps:

<u>STEP</u> <u>(1)</u> Change the independent variable from time (t) to fictitious time (s) according to the transformation

$$\frac{dt}{ds} = r \tag{2.4}$$

The derivatives of $\underline{r}$ with respect to t become

$$\dot{\underline{r}} = \underline{r}' / r \tag{2.5}$$

where $(\,)' = \frac{d(\,)}{ds}$, and

$$\ddot{\underline{r}} = \underline{r}'' / r^2 - \underline{r}' r' / r^3 \tag{2.6}$$

where

$$r' = \underline{r} \cdot \underline{r}' / r \tag{2.7}$$

<u>STEP</u> <u>(2)</u> Embed the integral called the Laplace vector (a constant when $\underline{F}$=0)

$$\underline{\varepsilon} = \frac{1}{\mu}\left[\left(\dot{\underline{r}} \cdot \dot{\underline{r}}\right)\underline{r} - \left(\dot{\underline{r}} \cdot \underline{r}\right)\dot{\underline{r}}\right] - \underline{r}/r \tag{2.8}$$

which becomes

$$\underline{\varepsilon} = \frac{1}{\mu r^2}\left[\left(\underline{r}' \cdot \underline{r}'\right)\underline{r} - \left(\underline{r}' \cdot \underline{r}\right)\underline{r}'\right] - \underline{r}/r \tag{2.9}$$

when the new independent variable s is used.

<u>STEP</u> <u>(3)</u> Embed the energy integral (a constant when $\underline{F}$=0)

$$\alpha_k = \frac{2\mu}{r} - \dot{\underline{r}} \cdot \dot{\underline{r}} \tag{2.10}$$

which becomes

$$\alpha_k = \frac{2\mu}{r} - \frac{1}{r^2}\underline{r}' \cdot \underline{r}' \tag{2.11}$$

when the new independent variable is used. Note that

$$\alpha_k = -2\, h_k \tag{2.12}$$

where h_k is the two-body or Keplerian energy. Using these three steps in order, equation (2.1) becomes

$$\underline{r}'' + \alpha_k \underline{r} = -\mu\underline{\varepsilon} + r^2 \underline{F} \tag{2.13}$$

which is the differential equation for the position vector $\underline{r}$. By taking the dot product of equation (2.13) with the position vector $\underline{r}$ we obtain

$$r'' + \alpha_k r = \mu + r\, \underline{r} \cdot \underline{F} \tag{2.14}$$

which is the differential equation for the distance r. We now change from the energy integral α_k to the Jacobian integral α_J (Bond and Gottlieb (1989)) which is given by

$$\alpha_J = \alpha_k + 2\sigma - 2V(\underline{r}, t) \tag{2.15}$$

where σ is called the axial element and is defined by

$$\sigma = \underline{\omega} \cdot (\underline{r} \times \underline{\dot{r}}) \tag{2.16}$$

The vector $\underline{\omega}$ is the constant rotational rate of the central attracting body or orbital rate of a third body giving rise to the perturbing potential $V(\underline{r}, t)$. In Section 4.0 it will be shown that $\alpha_J = constant$ when $/\underline{P}/ = 0$ and that $\sigma = constant$ when $/\underline{\omega}/ = 0$. Solving equation (2.15) for α_k and substituting into equations (2.13) and (2.14) we obtain

$$\underline{r}'' + \alpha_J \underline{r} = -\mu\underline{\varepsilon} + r^2\underline{F} + 2(\sigma - V(\underline{r}, t))\underline{r} \equiv -\mu\underline{\varepsilon} + \underline{Q} \tag{2.17}$$

and

$$r'' + \alpha_J r = \mu + r\underline{r} \cdot \underline{F} + 2(\sigma - V(\underline{r}, t))r \equiv \mu + \frac{1}{r}\underline{Q} \cdot \underline{r} \tag{2.18}$$

Note that all of the perturbation terms have been moved to the right side in equation (2.17) and (2.18).

Equation (2.17) and (2.18) are coupled only through the perturbation terms. We will refer to equation (2.17) as the spatial differential equation since its solution provides position and velocity. We will refer to equation (2.18) along with equation (2.4) as the temporal differential equations since their solutions provide time. Note that when there are no perturbations (that is $/\underline{F}/ = 0$ and $/\underline{\omega}/ = 0$) then we have the two-body differential equations

$$\underline{r}'' + \alpha_J \underline{r} = -\mu\underline{\varepsilon} \tag{2.19}$$

and

$$r'' + \alpha_J r = \mu \tag{2.20}$$

and the Jacobi constant and Keplerian energy become the same

$$\alpha_J = \alpha_k$$

3.0 Two Body Solution

The differential equation of motion for the two-body problem in the fictitious time was shown in the previous section to be

$$\underline{r}'' + \alpha_J \underline{r} = -\mu\underline{\varepsilon} \tag{3.1}$$

The solution of (3.1) in terms of the Stumpff functions of Appendix B is

$$\underline{r} = \underline{r}_o c_o + \underline{r}_o' s c_1 - \mu\underline{\varepsilon} s^2 c_2 \tag{3.2}$$

where $\underline{r}_o$ and $\underline{r}_o'$ are the initial values of $\underline{r}$ and $\underline{r}'$, and the Stumpff functions are $c_l = c_l(\alpha_J s^2)$. This can be verified by direct substitution of (3.2) into (3.1) and using the derivatives of the Stumpff functions

$$\begin{aligned} c_o' &= -\alpha_J s c_1 \\ s c_1' + c_1 &= c_o \\ s c_2' + 2c_2 &= c_1 \end{aligned} \tag{3.3}$$

The first derivative of (3.2) which is the "velocity" in the fictitious time is

$$\underline{r}' = -(\alpha_J \underline{r}_o + \mu\underline{\varepsilon}) s c_1 + \underline{r}_o' c_o \tag{3.4}$$

In place of $\mu\underline{\varepsilon}$ which is a constant of the motion we define the constant "delta vector"

$$\underline{\delta} = -\alpha_J \underline{r}_o - \mu \underline{\varepsilon} \tag{3.5}$$

Now using the Stumpff function identity

$$c_o + \alpha_J s^2 c_2 \equiv 1 \tag{3.6}$$

and equation (3.5) and (3.2) we obtain

$$\underline{r} = \underline{r}_o + \underline{r}_o' s c_1 + \underline{\delta} s^2 c_2 \tag{3.7}$$

similarly equation (3.4) becomes

$$\underline{r}' = \underline{r}_o' c_o + \underline{\delta} s c_1 \tag{3.8}$$

The differential equation of motion for the distance r was shown in the previous section to be

$$r'' + \alpha_J r = \mu \tag{3.9}$$

The solution of equation (3.9) is similar to that for (3.1). In terms of Stumpff functions the distance is

$$r = r_o c_o + r_o' s c_1 + \mu s^2 c_2 \tag{3.10}$$

and its derivative is

$$r' = (\mu - r_o \alpha_J) s c_1 + r_o' c_o \tag{3.11}$$

Now define the constant

$$\gamma = \mu - r_o \alpha_J \tag{3.12}$$

which we substitute for μ in equation (3.10) along with the identity of equation (3.6) to obtain

$$r = r_o + r_o' s c_1 + \gamma s^2 c_2 \tag{3.13}$$

Similarly equation (3.1) becomes

$$r' = r_o' c_o + \gamma s c_1 \tag{3.14}$$

Now substitute equation (3.13) for r in the independent variable transformation, equation (2.4), to obtain

$$dt = r_o ds + r_o' s c_1 ds + \gamma s^2 c_2 \, ds \tag{3.15}$$

Now use the integration formula

$$\int s^m c_m \, ds = s^{m+1} c_{m+1}$$

which is from Appendix B to obtain the equation for time (Kepler's equation),

$$t = t_o + r_o s + r_o' s^2 c_2 + \gamma s^3 c_3 \tag{3.16}$$

where t_o is the initial time.

The integration constants which were introduced in this section are $\underline{r}_o, \underline{r}_o', r_o, r_o', t_o$. The new constant $\underline{\delta}$ simply replaces the Laplace vector which is a constant of two-body motion through the definition (3.5). Similarly we note that the constant γ replaces the gravitational constant (equation (3.12)). The introduction of the constants $\underline{\delta}$ and γ was done by Burdet (1968). This fact was noted by Bond and Hanssen (1973). The Jacobian element α_J is the same as the two-body energy parameter α_k in the unperturbed case is also a constant of the motion. In addition we have the axial element σ which is also a constant of the motion (see equation 2.16). This is a total of 15 constants of the motion.

The constants $\underline{r}_o, \underline{r}_o'$ and $\underline{\delta}$ will be treated as orbital elements associated with the spatial differential equation (2.17) and r_o, r_o', γ, t_o will be treated as orbital elements associated with the temporal differential equations (2.4) and (2.18).

4.0 The Differential Equations For The Elements

When perturbations are present the elements are no longer constant. First we derive the differential equation for the axial element σ. Differentiate equation (2.16) with respect to time and substitute equation (2.1) and (2.3) to obtain

$$\dot{\sigma} = \underline{\omega} \cdot (\underline{r} \times \ddot{\underline{r}}) = \underline{\omega} \cdot \underline{r} \times \underline{F} = \underline{\omega} \cdot \underline{r} \times \left[\underline{P} - \frac{\partial}{\partial \underline{r}} V(\underline{r}, t) \right] \tag{4.1}$$

now use equation (2.4) to change to fictitious time

$$\sigma' = r\underline{\omega} \cdot \underline{r} \times \left[\underline{P} - \frac{\partial}{\partial \underline{r}} V(\underline{r}, t) \right] \tag{4.2}$$

Clearly $\sigma = constant$ when $/\underline{\omega}/ = 0$. Now we derive the differential equation for the Jacobian element α_J. Differentiate equation (2.15) with respect to time to obtain

$$\dot{\alpha}_J = \dot{\alpha}_k + 2\dot{\sigma} - 2\left[\frac{\partial}{\partial t} V(\underline{r}, t) + \dot{\underline{r}} \cdot \frac{\partial}{\partial \underline{r}} V(\underline{r}, t) \right]$$

From equations (2.10) and (2.1)

$$\dot{\alpha}_k = -2\dot{\underline{r}} \cdot \left[\underline{P} - \frac{\partial}{\partial \underline{r}} V(\underline{r}, t) \right]$$

and from Bond and Mulcihy (1988) also Bond and Gottlieb (1989)

$$\frac{\partial}{\partial t} V(\underline{r}, t) = -\underline{\omega} \cdot \underline{r} \times \frac{\partial}{\partial \underline{r}} V(\underline{r}, t)$$

and from equation (4.1) the expression for $\dot{\alpha}_J$ becomes

$$\dot{\alpha}_J = 2(-\dot{\underline{r}} + \underline{\omega} \times \underline{r}) \cdot \underline{P} \tag{4.3}$$

Now use equation (2.4) to change to fictitious time

$$\alpha_J' = 2(-\underline{r}' + r\underline{\omega} \times \underline{r}) \cdot \underline{P} \tag{4.4}$$

Note that $\alpha_J = constant$ when $/\underline{P}/ = 0$. The Jacobian constant α_J will be treated as an orbital element for both the spatial and temporal equations since α_J appears in the two-body equations (2.19) and (2.20). Even though we have already obtained the differential equation for α_J (equation (4.4)) we must include it in the variation of parameters procedures of the spatial and temporal equations. The axial element σ appears only as a perturbation in equations (2.17) and (2.18). We have also obtained the differential equation for σ (equation (4.2)). We will include σ in the variation of parameters procedure for convenience and completeness.

Even though the Laplace vector will be eliminated as an element we will need the derivative of the Laplace vector as a perturbation. This derivative as found by differentiating equation (2.8) will respect to time, then using equation (2.1) to eliminate $\ddot{\underline{r}}$, and finally using equation (2.4) to obtain

$$\mu\underline{\varepsilon}' = 2(\underline{r}' \cdot \underline{F})\underline{r} - (\underline{r} \cdot \underline{F})\underline{r}' - (\underline{r} \cdot \underline{r}')\underline{F} \tag{4.4a}$$

4.1 Spatial Elements

Now we apply the variation of parameters method of Lagrange to equations (2.17), (4.2) and (4.4). Define

$$\underline{x}_1 = \underline{r}$$

$$\underline{x}_2 = \underline{r}'$$

$$\underline{x}_3 = -\alpha_J\underline{r} - \mu\underline{\varepsilon} = -x_5\underline{x}_1 - \mu\underline{\varepsilon} \tag{4.5}$$

$$x_4 = \sigma$$

$$x_5 = \alpha_J$$

Now differentiate equations (4.5) and use (2.17), (4.2) and (4.4) to obtain

$$\begin{aligned}
\underline{x}_1' &= \underline{x}_2 \\
\underline{x}_2' &= \underline{x}_3 + \underline{Q} = \underline{x}_3 + \underline{G}_2 \\
\underline{x}_3' &= -x_5\underline{x}_2 - (\alpha_J'\underline{r} + \mu\underline{\varepsilon}') = -x_5\underline{x}_2 + \underline{G}_3 \\
x_4' &= \sigma' = G_4 \\
x_5' &= \alpha_J' = G_5
\end{aligned} \tag{4.6}$$

Where $\underline{G}_1 = \underline{0}$. Equations (4.6) can be separated into unperturbed (i.e., two-body or Keplerian) and perturbed parts, that is into the form of $\underline{x}' = \underline{F} + \underline{G}$, making them suitable for Lagrange's variation of parameters method as given by Appendix A. In this form equation (4.6) becomes

$$\begin{bmatrix} \underline{x}_1' \\ \underline{x}_2' \\ \underline{x}_3' \\ x_4' \\ x_5' \end{bmatrix} = \begin{bmatrix} \underline{x}_2 \\ \underline{x}_3 \\ -x_5\underline{x}_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \underline{G}_1 \\ \underline{G}_2 \\ \underline{G}_3 \\ G_4 \\ G_5 \end{bmatrix} \tag{4.7}$$

where $\underline{G}_1$, $\underline{G}_2$, $\underline{G}_3$, G_4, G_5 are defined from equations (4.6). The array of constants, which will become the new dependent variables, is defined as

$$\underline{c}^T = (\underline{\alpha}^T, \underline{\beta}^T, \underline{\delta}^T, \sigma, \alpha_J) \tag{4.8}$$

where

$$\begin{aligned}
\underline{\alpha} &= \underline{r}_o = \underline{x}_1(0) \\
\underline{\beta} &= \underline{r}_o' = \underline{x}_2(0) \\
\underline{\delta} &= -\alpha_J\underline{\alpha} - \mu\underline{\varepsilon} = \underline{x}_3(0)
\end{aligned} \tag{4.9}$$

and of course σ and α_J which have already been established as constants of the motion. The differential equation for $\underline{c}$, has the form, $\frac{\partial x}{\partial \underline{c}}\underline{c}' = \underline{G}$ where

$$\frac{\partial x}{\partial \underline{c}} = \begin{bmatrix}
\frac{\partial \underline{x}_1}{\partial \underline{\alpha}} & \frac{\partial \underline{x}_1}{\partial \underline{\beta}} & \frac{\partial \underline{x}_1}{\partial \underline{\delta}} & \frac{\partial \underline{x}_1}{\partial \sigma} & \frac{\partial \underline{x}_1}{\partial \alpha_J} \\
\frac{\partial \underline{x}_2}{\partial \underline{\alpha}} & \frac{\partial \underline{x}_2}{\partial \underline{\beta}} & \frac{\partial \underline{x}_2}{\partial \underline{\delta}} & \frac{\partial \underline{x}_2}{\partial \sigma} & \frac{\partial \underline{x}_2}{\partial \alpha_J} \\
\frac{\partial \underline{x}_3}{\partial \underline{\alpha}} & \frac{\partial \underline{x}_3}{\partial \underline{\beta}} & \frac{\partial \underline{x}_3}{\partial \underline{\delta}} & \frac{\partial \underline{x}_3}{\partial \sigma} & \frac{\partial \underline{x}_3}{\partial \alpha_J} \\
\frac{\partial x_4}{\partial \underline{\alpha}} & \frac{\partial x_4}{\partial \underline{\beta}} & \frac{\partial x_4}{\partial \underline{\delta}} & \frac{\partial x_4}{\partial \sigma} & \frac{\partial x_4}{\partial \alpha_J} \\
\frac{\partial x_5}{\partial \underline{\alpha}} & \frac{\partial x_5}{\partial \underline{\beta}} & \frac{\partial x_5}{\partial \underline{\delta}} & \frac{\partial x_5}{\partial \sigma} & \frac{\partial x_5}{\partial \alpha_J}
\end{bmatrix} \tag{4.10}$$

Noting from Section (3.0) that

$$\begin{aligned}
\underline{x}_1 &= \underline{r} = \underline{\alpha} + \underline{\beta}sc_1 + \underline{\delta}s^2c_2 \\
\underline{x}_2 &= \underline{r}' = \underline{\beta}c_o + \underline{\delta}sc_1
\end{aligned} \tag{4.11}$$

and from equation (3.5), (4.5) and (4.9)

$$\underline{x}_3 = \alpha_J(\underline{\alpha} - \underline{r}) + \underline{\delta}$$

also

$$x_4 = \sigma \tag{4.12}$$

$$x_5 = \alpha_J$$

The differential equations become

$$\begin{bmatrix} I & Isc_1 & Is^2c_2 & \underline{0} & \dfrac{\partial \underline{r}}{\partial \alpha_J} \\ [0] & Ic_o & Isc_1 & \underline{0} & \dfrac{\partial \underline{r}'}{\partial \alpha_J} \\ [0] & -I\alpha_J sc_1 & Ic_o & \underline{0} & \dfrac{\partial \underline{x}_3}{\partial \alpha_J} \\ \underline{0}^T & \underline{0}^T & \underline{0}^T & 1 & 0 \\ \underline{0}^T & \underline{0}^T & \underline{0}^T & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{\alpha}' \\ \underline{\beta}' \\ \underline{\delta}' \\ \sigma' \\ \alpha_J' \end{bmatrix} = \begin{bmatrix} \underline{G}_1 \\ \underline{G}_2 \\ \underline{G}_3 \\ G_4 \\ G_5 \end{bmatrix} \tag{4.13}$$

where we have used the identity from Appendix B

$$c_0 = 1 - \alpha_J s^2 c_2$$

also, I is the 3 by 3 identity matrix; [0] is the 3 by 3 null matrix; $\underline{0}$ is a column vector with 3 components; $\underline{0}^T$ is a row vector with 3 components. Equation (4.13) yields the equations

$$\underline{\alpha}' + \underline{\beta}' sc_1 + \underline{\delta}' s^2 c_2 + \alpha_J' \frac{\partial \underline{r}}{\partial \alpha_J} = 0$$

$$\underline{\beta}' c_o + \underline{\delta}' sc_1 + \alpha_J' \frac{\partial \underline{r}'}{\partial \alpha_J} = \underline{Q} \tag{4.14}$$

$$-\underline{\beta}' \alpha_J sc_1 + \underline{\delta}' c_o + \alpha_J' \frac{\partial \underline{x}_3}{\partial \alpha_J} = -\alpha_J' \underline{r} - \mu \underline{\varepsilon}'$$

$$\sigma' = r\underline{\omega} \cdot \underline{r} \times \underline{F}$$

$$\alpha_J' = 2(-\underline{r}' + r\underline{\omega} \times \underline{r}) \cdot \underline{P}$$

where we have restored the original notations for $\underline{G}_1$, $\underline{G}_2$, $\underline{G}_3$, G_4, G_5. Now using the partial derivatives,

$$\frac{\partial \underline{r}}{\partial \alpha_J} = \underline{\beta} s \frac{\partial c_1}{\partial \alpha_J} + \underline{\delta} s^2 \frac{\partial c_2}{\partial \alpha_J}$$

$$\frac{\partial \underline{r}'}{\partial \alpha_J} = \underline{\beta} \frac{\partial c_o}{\partial \alpha_J} + \underline{\delta} s \frac{\partial c_1}{\partial \alpha_J} \tag{4.15}$$

$$\frac{\partial \underline{x}_3}{\partial \alpha_J} = \underline{\alpha} - \underline{r} - \alpha_J \frac{\partial \underline{r}}{\partial \alpha_J}$$

where the Stumpff function derivatives are

$$\frac{\partial c_o}{\partial \alpha_J} = -\frac{1}{2} s^2 c_1 \tag{4.15a}$$

$$\frac{\partial c_k}{\partial \alpha_J} = \frac{1}{2\alpha_J}(c_{k-1} - kc_k)\ , \quad k \geq 1$$

and other Stumpff function identities from Appendix B equation (4.14) can be solved simultaneously, omitting several algebraic steps to give

$$\underline{\alpha}' = -\underline{Q}sc_1 - \mu\underline{\varepsilon}'s^2c_2 - \alpha_J'\left[\underline{\alpha}s^2c_2 + 2\underline{\beta}s^3\tilde{c}_3 + \frac{1}{2}\underline{\delta}s^4c_2^2\right]$$

$$\underline{\beta}' = \underline{Q}c_o + \mu\underline{\varepsilon}'sc_1 + \alpha_J'\left[\underline{\alpha}sc_1 + \underline{\beta}s^2\tilde{c}_2 - \underline{\delta}s^3(2\tilde{c}_3 - c_1c_2)\right]$$

$$\underline{\delta}' = \underline{Q}\,\alpha_J sc_1 - \mu\underline{\varepsilon}'c_o + \alpha_J'\left[-\underline{\alpha}c_o + 2\alpha_J\underline{\beta}s^3\tilde{c}_3 + \frac{1}{2}\underline{\delta}\alpha_J s^4c_2^2\right] \tag{4.16}$$

$$\sigma' = r\,\underline{\omega}\cdot\underline{r}\times\underline{F}$$

$$\alpha_J' = 2(-\underline{r}' + r\,\underline{\omega}\times\underline{r})\cdot\underline{P}$$

where $\tilde{c}_l = c_l(4\alpha_J s^2)$ as discussed in Appendix B. In the reference Bond and Gottlieb (1989) the coefficient of the factor $\alpha_J'\underline{\alpha}$ in the differential equation for $\underline{\beta}$ had a secular term. This term does not appear in equation (4.16). Note that the Laplace vector ($\mu\underline{\varepsilon}$) has been entirely removed from the formulation. The derivative of the Laplace vector ($\mu\underline{\varepsilon}'$) remains but only as an abbreviation for the perturbations given in equation (4.4a).

4.2 Temporal Elements

Now we apply Lagrange's variation of parameters method to equations (2.18), (2.4) and (4.4). Define

$$y_1 = r$$

$$y_2 = r'$$

$$y_3 = \mu - \alpha_J r \tag{4.17}$$

$$y_4 = t$$

$$y_5 = \alpha_J$$

Note that α_J is the only element which is common to both the spatial and temporal systems. Now differentiate equations (4.17) equation (2.18), (2.4), and (4.4) become

$$y_1' = y_2$$

$$y_2' = y_3 + \frac{1}{r}\underline{Q}\cdot\underline{r} = y_3 + g_2$$

$$y_3' = -y_5y_2 - \alpha_J'r = -y_5y_2 + g_3 \tag{4.18}$$

$$y_4' = y_1$$

$$y_5' = \alpha_J' = g_5$$

Where $g_1 = 0$ and $g_4 = 0$. Equation (4.18) can also be expressed in the form $y' = f + g$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ -y_5y_2 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix} \tag{4.19}$$

where g_1, g_2, g_3, g_4, g_5 are defined by equation (4.18). The array of constants which will become the new dependent variables are

$$\underline{\kappa}^T = (a, b, \gamma, \tau, \alpha_J) \tag{4.20}$$

where

$$a = r_o = y_1(0)$$

$$b = r_o' = y_2(0) \tag{4.21}$$

$$\gamma = \mu - \alpha_J a = y_3(0)$$

$$\tau = t_o = y_4(0)$$

and α_J has already been established as a constant of the motion.

The differential equations for $\underline{\kappa}$ (having the form $\frac{\partial y}{\partial \underline{\kappa}}\underline{\kappa}' = g$) becomes

$$\begin{bmatrix} \frac{\partial y_1}{\partial a} & \frac{\partial y_1}{\partial b} & \frac{\partial y_1}{\partial \gamma} & \frac{\partial y_1}{\partial \tau} & \frac{\partial y_1}{\partial \alpha_J} \\ \frac{\partial y_2}{\partial a} & \frac{\partial y_2}{\partial b} & \frac{\partial y_2}{\partial \gamma} & \frac{\partial y_2}{\partial \tau} & \frac{\partial y_2}{\partial \alpha_J} \\ \frac{\partial y_3}{\partial a} & \frac{\partial y_3}{\partial b} & \frac{\partial y_3}{\partial \gamma} & \frac{\partial y_3}{\partial \tau} & \frac{\partial y_3}{\partial \alpha_J} \\ \frac{\partial y_4}{\partial a} & \frac{\partial y_4}{\partial b} & \frac{\partial y_4}{\partial \gamma} & \frac{\partial y_4}{\partial \tau} & \frac{\partial y_4}{\partial \alpha_J} \\ \frac{\partial y_5}{\partial a} & \frac{\partial y_5}{\partial b} & \frac{\partial y_5}{\partial \gamma} & \frac{\partial y_5}{\partial \tau} & \frac{\partial y_5}{\partial \alpha_J} \end{bmatrix} \begin{bmatrix} a' \\ b' \\ \gamma' \\ \tau' \\ \alpha_J' \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix} \tag{4.22}$$

but from equations (3.12), (3.13), (3.14), (3.15) and (4.21)

$$\begin{aligned} y_1 &= r = a + bsc_1 + \gamma s^2 c_2 \\ y_2 &= r' = bc_o + \gamma sc_1 \\ y_3 &= \mu - \alpha_J r = \gamma + \alpha_J a - \alpha_J r = \gamma + \alpha_J(a - r) \\ y_4 &= t = \tau + as + bs^2 c_2 + \gamma s^3 c_3 \\ y_5 &= \alpha_J \end{aligned} \tag{4.23}$$

So we can evaluate the matrix elements in (4.22) to obtain

$$\begin{bmatrix} 1 & sc_1 & s^2c_2 & 0 & \frac{\partial r}{\partial \alpha_J} \\ 0 & c_o & sc_1 & 0 & \frac{\partial r'}{\partial \alpha_J} \\ 0 & -\alpha_J sc_1 & c_o & 0 & \frac{\partial y_3}{\partial \alpha_J} \\ s & s^2c_2 & s^3c_3 & 1 & \frac{\partial t}{\partial \alpha_J} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a' \\ b' \\ \gamma' \\ \tau' \\ \alpha_J' \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix} \tag{4.24}$$

Equation (4.24) when expanded yields,

$$\begin{aligned} a' + b'sc_1 + \gamma' s^2 c_2 + \alpha_J' \frac{\partial r}{\partial \alpha_J} &= 0 \\ b'c_o + \gamma' sc_1 + \alpha_J' \frac{\partial r'}{\partial \alpha_J} &= \frac{1}{r}\underline{Q} \cdot \underline{r} \\ -b'\alpha_J sc_1 + \gamma' c_o + \alpha_J' \frac{\partial y_3}{\partial \alpha_J} &= -r\alpha_J' \end{aligned} \tag{4.25}$$

$$a's + b's^2c_2 + \gamma's^3c_3 + \tau' + \alpha_J'\frac{\partial t}{\partial \alpha_J} = 0$$

$$\alpha_J' = 2(-\underline{r}' + r\ \underline{\omega} \times \underline{r}) \cdot \underline{P}$$

Where we have restored the original notations for g_2, g_3 and g_5. We evaluate the partial derivatives in equations (4.25) using equations (4.23)

$$\begin{aligned}
\frac{\partial r}{\partial \alpha_J} &= bs\frac{\partial c_1}{\partial \alpha_J} + \gamma s^2\frac{\partial c_2}{\partial \alpha_J} \\
\frac{\partial r'}{\partial \alpha_J} &= b\frac{\partial c_o}{\partial \alpha_J} + \gamma s\frac{\partial c_1}{\partial \alpha_J} \\
\frac{\partial y_3}{\partial \alpha_J} &= a - r - \alpha_J\frac{\partial r}{\partial \alpha_J} \\
\frac{\partial t}{\partial \alpha_J} &= bs^2\frac{\partial c_2}{\partial \alpha_J} + \gamma s^3\frac{\partial c_3}{\partial \alpha_J}
\end{aligned} \tag{4.26}$$

where the Stumpff function derivatives are given by equations (4.15a). Equations (4.25) can be solved simultaneously for the derivatives,

$$\begin{aligned}
a' &= -\frac{1}{r}\underline{r} \cdot \underline{Q}sc_1 - \alpha_J'\left[as^2c_2 + 2bs^3\tilde{c}_3 + \frac{1}{2}\gamma s^4c_2^2\right] \\
b' &= \frac{1}{r}\underline{r} \cdot \underline{Q}c_o + \alpha_J'\left[asc_1 + bs^2\tilde{c}_2 - \gamma s^3(2\tilde{c}_3 - c_1c_2)\right] \\
\gamma' &= \frac{1}{r}\underline{r} \cdot \underline{Q}\,\alpha_J sc_1 + \alpha_J'\left[-ac_0 + 2b\,\alpha_J s^3\tilde{c}_3 + \frac{1}{2}\gamma\alpha_J s^4c_2^2\right] \\
\tau' &= \frac{1}{r}\underline{r} \cdot \underline{Q}s^2c_2 + \alpha_J'\left[as^3c_3 + \frac{1}{2}bs^4c_2^2 - 2\gamma s^5(c_5 - 4\tilde{c}_5)\right]
\end{aligned} \tag{4.27}$$

As in the development of equations (4.16) the Stumpff function identities of Appendix B have been used. In the reference Bond and Gottlieb (1989) the coefficient of the factor $\alpha_J'a$ in the differential equation for b had a secular term. This term does not appear in equation (4.27).

It is useful to note that

$$\mu = \gamma + \alpha_J a \tag{4.28}$$

is an integral of the system of equations (4.27). From equations (4.27) it is easy to show that

$$\gamma' + a'\alpha_J + a\alpha_J' = 0 \tag{4.29}$$

which can be integrated to give

$$\gamma + a\,\alpha_J = constant \tag{4.30}$$

By comparison of equation (4.30) to equation (4.21) the constant of integration is the gravitational constant μ. Therefore it is not necessary to compute γ from its differential equation. We can compute γ from equation (4.28),

$$\gamma = \mu - \alpha_J a \tag{4.31}$$

5.0 Minimization Of Perturbations

The variation of parameters approach is not dependent on the magnitude of the perturbation. No assumption on the size of the perturbation is required in order that the method be rigorous. However, small perturbations enhance the efficiency, speed, and accuracy of any perturbation method. In the method described in this paper, the embedding of the Jacobi integral has the effect of introducing a perturbation parameter $/\underline{\omega}/$ that is the rotational speed of the planet, or the mean motion of the perturbing third body. To prevent this perturbation from becoming too large the following modification is offered:

Let,

$$\sigma = \sigma_o + \Delta\sigma \tag{5.1}$$

where σ_o is the initial value of σ and $\Delta\sigma$ is the change in σ. In effect we can let $\Delta\sigma$ replace σ so that the differential equations reflect only changes in σ. Substitute equation (5.1) into equation (2.17) to obtain

$$\underline{r}'' + \alpha_J \underline{r} = -\mu\underline{\varepsilon} + r^2\underline{F} + 2(\sigma_o + \Delta\sigma - V(\underline{r}, t))\underline{r}$$

Now since σ_o is constant we can move it to the left side of this differential equation to get

$$\underline{r}'' + (\alpha_J - 2\sigma_o)\underline{r} = -\mu\underline{\varepsilon} + r^2\underline{F} + 2(\Delta\sigma - V(\underline{r}, t))\underline{r} \tag{5.2}$$

Similarly equation (2.18) becomes

$$r'' + (\alpha_J - 2\sigma_o)r = \mu + r\underline{r} \cdot \underline{F} + 2(\Delta\sigma - V(\underline{r}, t))r \tag{5.3}$$

This change does not affect the outcome of the variation of parameters approach taken here. This change is only a computational convenience and is in effect in the computational procedure of Section 6.1 where the element α_J is actually $\alpha_J - 2\sigma_o$ and σ is actually $\Delta\sigma$. Note that the initial value of $\Delta\sigma$ is

$$\Delta\sigma = 0 \tag{5.4}$$

6.0 Application

In this section the most important equations are collected and listed in a logical order suitable for computation. Also two numerical examples are presented.

6.1 Computational Procedure

Given $\underline{r}_o$, $\underline{v}_o$, t_o find $\underline{r}(t)$ and $\underline{v}(t)$.

STEP 1 Initialization

$$s = 0$$

$$r_o = (\underline{r}_o \cdot \underline{r}_o)^{1/2}$$

$$a = r_o$$

$$b = \underline{r}_o \cdot \underline{v}_o$$

$$\tau = t_o$$

$$\underline{\alpha} = \underline{r}_o$$

$$\underline{\beta} = a\ \underline{v}_o$$

Evaluate Perturbations V_o, $\left[\dfrac{\partial V}{\partial \underline{r}}\right]_o$, $\underline{P}_o$.

$$\alpha_J = \frac{2\mu}{r_o} - \underline{v}_o \cdot \underline{v}_o - 2\,V_o$$

$$\gamma = \mu - \alpha_J\, a$$

$$\underline{\delta} = -(\underline{v}_o \cdot \underline{v}_o)\underline{r}_o + (\underline{r}_o \cdot \underline{v}_o)\underline{v}_o + \frac{\mu}{r_o}\underline{r}_o - \alpha_J \underline{r}_o$$

$$\sigma = 0$$

<u>STEP</u> <u>2</u> Transform Elements to Coordinates

$$\underline{r} = \underline{\alpha} + \underline{\beta} s c_1 + \underline{\delta} s^2 c_2$$

$$\underline{r}' = \underline{\beta} c_o + \underline{\delta} s c_1$$

$$\underline{x}_3 = \alpha_J(\underline{\alpha} - \underline{r}) + \underline{\delta}$$

$$\gamma = \mu - \alpha_J\, a$$

$$r = a + b s c_1 + \gamma s^2 c_2$$

$$\underline{v} = \underline{r}'/r$$

$$r' = b c_o + \gamma s c_1$$

$$t = \tau + a s + b s^2 c_2 + \gamma s^3 c_3$$

<u>STEP</u> <u>3</u> Evaluate Differential Equations For The Elements

$$\underline{F} = \underline{P} - \frac{\partial V}{\partial \underline{r}}$$

$$\underline{Q} = r^2 \underline{F} + 2\underline{r}(-V + \sigma)$$

$$\alpha_J' = 2(-\underline{r}' + r\,\underline{\omega} \times \underline{r}) \cdot \underline{P}$$

$$\mu\underline{\varepsilon}' = 2\,(\underline{r}' \cdot \underline{F})\underline{r} - (\underline{r} \cdot \underline{F})\underline{r}' - (\underline{r} \cdot \underline{r}')\underline{F}$$

$$\underline{\alpha}' = -\underline{Q} s c_1 - \mu\underline{\varepsilon}' s^2 c_2 - \alpha_J'\left[\underline{\alpha} s^2 c_2 + 2\underline{\beta} s^3 \tilde{c}_3 + \frac{1}{2}\underline{\delta} s^4 c_2^2\right]$$

$$\underline{\beta}' = \underline{Q} c_o + \mu\underline{\varepsilon}' s c_1 + \alpha_J'\left[\underline{\alpha} s c_1 + \underline{\beta} s^2 \tilde{c}_2 - \underline{\delta} s^3 (2\tilde{c}_3 - c_1 c_2)\right]$$

$$\underline{\delta}' = \underline{Q}\,\alpha_J s c_1 - \mu\underline{\varepsilon}' c_o + \alpha_J'\left[-\underline{\alpha} c_o + 2\alpha_J \underline{\beta} s^3 \tilde{c}_3 + \frac{1}{2}\underline{\delta}\alpha_J s^4 c_2^2\right]$$

$$\sigma' = r\,\underline{\omega} \cdot \underline{r} \times \underline{F}$$

$$a' = -\frac{1}{r}\underline{r} \cdot \underline{Q} s c_1 - \alpha_J'\left[a s^2 c_2 + 2b s^3 \tilde{c}_3 + \frac{1}{2}\gamma s^4 c_2^2\right]$$

$$b' = \frac{1}{r}\underline{r} \cdot \underline{Q} c_o + \alpha_J'\left[a s c_1 + b s^2 \tilde{c}_2 - \gamma s^3 (2\tilde{c}_3 - c_1 c_2)\right]$$

$$\gamma' = \frac{1}{r}\underline{r} \cdot \underline{Q}\,\alpha_J s c_1 + \alpha_J'\left[-a c_0 + 2b\,\alpha_J s^3 \tilde{c}_3 + \frac{1}{2}\gamma\alpha_J s^4 c_2^2\right] \qquad \textit{(optional)}$$

$$\tau' = \frac{1}{r}\underline{r} \cdot \underline{Q} s^2 c_2 + \alpha_J'\left[a s^3 c_3 + \frac{1}{2} b s^4 c_2^2 - 2\gamma s^5 (c_5 - 4\tilde{c}_5)\right]$$

<u>STEP</u> <u>4</u> Numerically Integrate Over Δs To Obtain Elements At $s + \Delta s$

STEP 5

$$s = s + \Delta s$$

Go back to step 2.

6.2 Numerical Applications

The equations of the BG14 $\underline{\delta}$ element method given above in Section 6.1 were programmed as nearly as possible in the same format as the older BG14 $\underline{\varepsilon}$ method (Bond and Gottlieb, 1989). The two methods were then compared to reference cases. The RK45 numerical method (Fehlberg, 1969) was used as the numerical integration method in both examples.

6.2.1 Example 1

The first example is that of a highly eccentric ($e \approx 0.95$) orbit about the Earth. The orbit is subject to the J_2 (Earth oblateness) perturbing potential, which is conservative, plus lunar perturbations. This orbit was computed by both BG14 $\underline{\delta}$ and BG14 $\underline{\varepsilon}$ methods. This example was also computed by Stiefel and Scheifele (1971) with extremely high precision and will be used as the reference. Table I shows the components of the position vector in Cartesian coordinates as computed by each method after 50 revolutions of the satellite. It is seen that both methods compare very closely with the reference but the new BG14 $\underline{\delta}$ method being slightly closer to the reference.

The problem description for the first example is:

Coordinate system: X and Y fixed in Earth equatorial plane; Z perpendicular to Earth equatorial plane.

Initial conditions:

Initial State Vector				
Position	0.0	-5888.9727	-3400.0	km
Velocity	10.691338	0.0	0.0	km/sec

The time of comparison is at 288.12768941 days, after approximately 50 revolutions.

TABLE I - Comparison of BG14 $\underline{\delta}$ and BG14 $\underline{\varepsilon}$ Methods
Final Value Of Position Vector

Method	X (km)	Y (km)	Z (km)	Steps/Rev (Avg)
REFERENCE Stiefel and Sheifele (1971)	-24219.0503	227962.1064	129753.4424	500
BG14 (RK45 Fixed Step) $\underline{\delta}$ Method	-24218.8175	227961.9146	129753.3431	62
BG14 (RK45 Fixed Step) $\underline{\varepsilon}$ Method	-24218.8069	227961.9186	129753.3344	62

The Earth oblateness and lunar models used are somewhat idealized and are taken from Stiefel and Scheifele (1971). These models are specified as follows:

The Earth oblateness perturbations were compared from the potential model

$$V = \frac{3}{2}\,(GE)\,J_2\,\frac{a_e^2}{r^3}\left[\frac{Z^2}{r^2} - \frac{1}{3}\right]$$

where

$$GE = 398601\ km^3/\sec^2\ (gravitational\ constant\ of\ Earth)$$

$$a_e = 6371.22\ km\ \ (equatorial\ radius\ of\ Earth)$$

$$J_2 = 1.08265\ x\ 10^{-3}\ \ (second\ harmonic\ of\ geopotential)$$

The lunar perturbation was computed from

$$\underline{P} = -\ GM \left[\frac{\underline{r} - \underline{a}}{/\underline{r} - \underline{a}/^3} + \frac{\underline{a}}{\rho^3} \right]$$

and the lunar ephemeris is approximated by

$$a_x = \rho \sin \Omega t$$

$$a_y = -\frac{\sqrt{3}}{2} \rho \cos \Omega t$$

$$a_z = -\frac{1}{2} \rho \cos \Omega t$$

$$\rho = 384400\ km\ \ (the\ Earth-Moon\ distance)$$

$$\Omega = 2.665315780887\ x\ 10^{-6}\ rad/\sec\ \ (Moon\ orbital\ rate)$$

$$GM = 4902.66\ km^3/\sec^2\ \ (gravitational\ constant\ of\ Moon)$$

6.2.2 Example 2

The second example (Adamo, 1989) is that of a near circular geocentric satellite orbit numerically integrated by the BG14 $\underline{\delta}$ method from an initial altitude of 300 *km* down to entry interface altitude of 123.278 *km* (66.565 nautical miles). The perturbations considered were the Jacchia 1970 atmospheric model and the GEM-10 (Lerch, 1979) geopotential restricted to second order and degree. The time of flight was about 29.736111 days and the ballistic number was 78.606675 kg/m^2. This case failed at an altitude of approximately 135 *km* (72.894 nautical miles) with the older BG14 $\underline{\varepsilon}$ method.

Coordinate System: True Equator and Greenwich Meridian Of Epoch

Initial conditions:

Initial State Vector at UT1 = 0 on 3 September 1991.				
Position	6677832.962	-62810.44513	-27301.63472	m
Velocity	78.98607579	6821.102837	3626.863958	m/sec

TABLE II - Comparison of BG14 $\underline{\delta}$ and BG14 $\underline{\varepsilon}$ Methods Final Value Of Position Vector				
Method	X (m)	Y (m)	Z (m)	Steps/Rev (Avg)
BG14 (RK45 Variable Step) $\underline{\delta}$ Method	2664837.2	-5838760.8	1033865.4	29
BG14 (RK45 Variable Step) $\underline{\varepsilon}$ Method	FAILED	FAILED	FAILED	-

Additional stress cases (not shown) have been computed in which the solution was propagated down to the surface of the Earth (assuming no change in atmospheric density below 90 km).

7.0 Final Comments

Recent numerical studies on atmospheric entry from near circular orbits and on low thrust in near circular orbits exhibit numerical instability when solved by the method of Bond and Gottlieb (1989) for long time intervals. These two cases are similar since both have persistent, tangential, non-conservative perturbations. It was found that this instability was due to secular terms which appear on the right hand sides of the differential equations of some of the elements. In this paper this instability is removed by the introduction of another vector integral of the motion and another scalar integral which remove the secular terms. The introduction of these new integrals require a new derivation of the differential equations for most of the elements. For this rederivation the Lagrange method of variation of parameters is used making the development more concise.

8.0 References

Adamo, D.: Personal Communication, April 1989.

Bond, Victor R. and Hanssen, Veit,: *The Burdet Formulation of the Perturbed Two-Body Problem with Total Energy as an Element,* NASA Johnson Space Center Publication JSC Internal Note No. 73-FM-86 (JSC-08004), June 1973.

Bond, Victor R. and Mulcihy, David D.,: *Computation of Orbits Using Total Energy,* Presented at the Flight Mechanics and Estimation Theory Symposium-1988, Goddard Space Flight Center, Greenbelt, Maryland, May 1988 (NASA CP 3011).

Bond, Victor R. and Gottlieb, Robert G.,: *A Perturbed Two-Body Element Method Utilizing The Jacobian Integral,* NASA Johnson Space Center Publication JSC Internal Note No. 89-FM-3 (JSC-23495), May 1989.

Burdet, C.A.,: *Theory of Kepler Motion,* Zeitschrift fur Angewandte Mathematic und Physik, Vol. **19** , 1968.

Fehlberg, E.,: *Low Order Classical Runge-Kutta Formulas With Step Size Control.* NASA TR-315, 1969.

Jacchia, L.G.: New Static Models of Thermosphere and Exosphere with Empirical Temperature Profiles. Smithsonian Astrophysical Observatory. Special Report No. 313 (1970).

Lerch, F.J.; et al: Gravity Model Improvement Using GEOS3 (GEM9 and 10). J. Geophysical Research, Vol. **84** No. B8 (30 July 1979).

Sperling, Hans,: *Computation of Keplerian Conic Sections,* ARS Journal, May 1961

Stiefel, E. and Scheifele, G., Linear And Regular Celestial Mechanics. Springer-Verlag, 1971.

Szebehely, V., Theory Of Orbits. Academic Press, 1967.

Appendix A - The Variation Of Parameters Method Of Lagrange

Assume that we have a mechanical system given by

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) \tag{A1}$$

where

$$\underline{x}^T = (x_1, \cdots, x_n)$$
$$\underline{f}^T = (f_1, \cdots, f_n)$$

and t is the independent variable.

We also assume that the solution of the system of equations (A1) is possible and can be expressed

$$\underline{x} = \underline{x}(\underline{c}, t) \tag{A2}$$

where the integration constants, or parameters, are given by

$$\underline{c}^T = (c_1, \cdots, c_n) \tag{A3}$$

Now consider another system similar to the system (A1),

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) + \underline{g}(\underline{x}, t) \tag{A4}$$

where the new term is called a perturbation and is given by

$$\underline{g}^T(\underline{x}, t) = (g_1, \cdots, g_n) \tag{A5}$$

The objective is to make the solution, equation (A2) of the system (A1), valid for the perturbed system (A4) by allowing the parameter $\underline{c}$ to be a function of the independent variable. In other words the solution (A2) still applies but with the constant ($\underline{c}$) replaced by the function ($\underline{c}(t)$). So we have

$$\underline{x} = \underline{x}(\underline{c}(t), t) \tag{A6}$$

Now take the total derivative of equation (A6)

$$\dot{\underline{x}} = \frac{\partial x}{\partial \underline{c}}\dot{\underline{c}} + \frac{\partial x}{\partial t} \tag{A7}$$

Also take the total derivative of (A2) and use (A1) to obtain

$$\frac{\partial x}{\partial t} = \dot{\underline{x}} = f(\underline{x}, t) \tag{A8}$$

Note we have used the fact that for unperturbed case the total and partial derivatives of $\underline{x}$ are the same. Using equation (A8) we can eliminate the partial derivative $\frac{\partial x}{\partial t}$ from equation (A7) obtaining,

$$\dot{\underline{x}} = \frac{\partial x}{\partial \underline{c}}\dot{\underline{c}} + \underline{f}(\underline{x}, t) \tag{A9}$$

Now compare equation (A9) with equation (A4) to obtain

$$\dot{\underline{x}} = \frac{\partial x}{\partial \underline{c}}\dot{\underline{c}} + \underline{f}(\underline{x}, t) = \underline{f}(\underline{x}, t) + \underline{g}(\underline{x}, t)$$

After the obvious cancellation

$$\frac{\partial x}{\partial \underline{c}}\dot{\underline{c}} = \underline{g} \tag{A10}$$

where the matrix $\frac{\partial x}{\partial \underline{c}}$ is obtained from the solution, equation (A2). The matrix must be invertible. That is

$$DET\left[\frac{\partial x}{\partial \underline{c}}\right] \neq 0$$

The system of differential equations for the parameter $\underline{c}$ is therefore

$$\dot{\underline{c}} = \left[\frac{\partial x}{\partial \underline{c}}\right]^{-1} \underline{g} \qquad \text{(A11)}$$

Appendix B - The Stumpff Functions

These functions are related to the trigonometric and hyperbolic functions. The general equation for the n th Stumpff function is,

$$c_n(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(2k+n)!} \ , \ n=0,1,2,... \tag{B1}$$

When these series are compared to the series of the trigonometric and hyperbolic functions, the following relations exist:

$$\begin{aligned}
c_o(x^2) &= \cos x \quad , or \quad c_o(-x^2) = \cosh x \\
c_1(x^2) &= \frac{\sin x}{x} \quad , or \quad c_1(-x^2) = \frac{\sinh x}{x} \\
c_2(x^2) &= \frac{1-\cos x}{x^2} \quad , or \quad c_2(-x^2) = \frac{\cosh x - 1}{x^2} \\
c_3(x^2) &= \frac{x - \sin x}{x^3} \quad , or \quad c_3(-x^2) = \frac{\sinh x - x}{x^3} \\
c_4(x^2) &= \frac{\cos x - \left[1 - \frac{x}{2}\right]}{x^4} \quad , or \quad c_4(-x^2) = \frac{\cosh x - \left[1 + \frac{x^2}{2}\right]}{x^4}
\end{aligned} \tag{B2}$$

etc.

The following identities may also be easily verified:

$$\begin{aligned}
c_o(z)^2 + zc_1(z)^2 &= 1 \\
c_o(z)^2 - zc_1(z)^2 &= c_o(4z) \\
c_o(z)^2 &= 1 - 2zc_2(4z) \\
c_1(z) &= 2c_2(4z) \\
c_0(z)c_1(z) &= c_1(4z) \\
c_2(z) &= c_1(z)^2 - c_2(z)c_o(z)
\end{aligned} \tag{B3}$$

The more general identities

$$c_{n+2}(z) = \frac{1}{n}c_{n+1}(z) + \frac{1}{z}\left[-c_n(z) + \frac{1}{n}c_{n-1}(z)\right] \ , \ n>0 \tag{B4}$$

and

$$c_n(z) + zc_{n+2}(z) = \frac{1}{n!} \tag{B5}$$

are also valid.

The derivatives of these functions may be expressed as

$$2z\frac{dc_n(z)}{dz} = c_{n-1}(z) - nc_n(z) \ , \ n>0 \tag{B6}$$

and

$$\frac{dc_n(z)}{dz} = \frac{1}{2}\left[nc_{n+2}(z) - c_{n+1}(z)\right] \tag{B7}$$

A convenient integration formula is

$$\int s^k c_k(\rho s^2)ds = s^{k+1}c_{k+1}(\rho s^2) \tag{B8}$$