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ELECTROMAGNETIC FINITE ELEMENTS BASED ON A FOUR-POTENTIAL VARIATIONAL PRINCIPLE

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SUMMARY

We derive electromagnetic finite elements based on a variational principle that uses the electromagnetic four-potential as primary variable. This choice is used to construct elements suitable for downstream coupling with mechanical and thermal finite elements for the analysis of electromagnetic/mechanical systems that involve superconductors. The key advantages of the four-potential are: the number of degrees of freedom per node remain modest as the problem dimensionality increases, jump discontinuities on interfaces are naturally accomodated, and static as well as dynamics are included without any *a priori* approximations. The new elements are tested on an axisymmetric problem under steady-state forcing conditions. The results are in excellent agreement with analytical solutions.

1. INTRODUCTION

The present work is part of a research program for the numerical simulation of electromagnetic/mechanical systems that involve superconductors. The simulation involves the interaction of the following four components:

(1) *Mechanical Fields*: displacements, stresses, strains and mechanical forces.

(2) *Thermal Fields*: temperature and heat fluxes.

(3) *Electromagnetic (EM) Fields*: electric and magnetic field strengths and fluxes, currents and charges.

(4) *Coupling Fields*: the foundamental coupling effect is the constitutive behavior of the materials involved. Particularly important are the metallurgical phase change phenomena triggered by thermal, mechanical and EM fields.

1.1 Finite Element Treatment

The first three fields (mechanical, thermal and electromagnetic) are treated by the finite element method. This treatment produces the spatial discretization of the continuum into mechanical, thermal and electromagnetic meshes of finite number of degrees of freedom. The finite element discretization may be developed in two ways:

(1) *Simultaneous Treatment.* The whole problem is treated as an indivisible whole. The three meshes noted above become tightly coupled, with common nodes and elements.

(2) *Staged Treatment.* The mechanical, thermal and electromagnetic components of the problem are treated separately. Finite element meshes for these components may be developed separately. Coupling effects are viewed as information that has to be transferred between these three meshes.

The present research follows the staged treatment. More specifically, *we develop finite element models for the fields in isolation, and then treat coupling effects as interaction forces between these models.* This "divide and conquer" strategy is ingrained in the partitioned treatment of coupled problems [4,16], which offers significant advantages in terms of computational efficiency and software modularity. Another advantage relates to the way research into complex problems can be made more productive. It centers on the observation that some aspects of the problem are either better understood or less physically relevant than others. These aspects may be then temporarily left alone while efforts are concentrated on the less developed and/or more physically important aspects. The staged treatment is better suited to this approach.

1.2 Mechanical Elements

Mechanical elements for this research have been derived using general variational principles that decouple the element boundary from the interior thus providing efficient ways to work out coupling with non-mechanical fields. The point of departure was previous research into the free-formulation variational principles reported in Ref. [5]. A more general formulation for the mechanical elements, which includes the assumed natural strain formulation, was established and reported in Refs. [5,6,14,15]. New representations of thermal fields have not been addressed as standard formulations are considered adequate for the coupled-field phases of this research.

2. ELECTROMAGNETIC ELEMENTS

The development of electromagnetic (EM) finite elements has not received to date the same degree of attention given to mechanical and thermal elements. Part of the reason is the widespread use of analytical and semianalytical methods in electrical engineering. These methods have been highly refined for specialized but important problems such as circuits and waveguides. Thus the advantages of finite elements in terms of generality have not been enough to counterweight established techniques. Much of the EM finite element work to date has been done in England and is well described in the surveys by Davies [1] and Trowbridge [21]. The general impression conveyed by these surveys is one of an unsettled subject, reminiscent of the early period (1960-1970) of finite elements in structural mechanics. A great number of formulations that combine flux, intensity, and scalar potentials are described with the recommended choice varying according to the application, medium involved (polarizable, dielectric, semiconductors, etc.) number of space dimensions, time-dependent characteristics (static, quasi-static, harmonic or transient) as well as other factors of lesser importance. The possibility of a general variational formulation has not apparently been recognized.

In the present work, the derivation of electromagnetic (EM) elements is based on a variational formulation that uses the four-potential as primary variable. The electric field is represented by a scalar potential and the magnetic field by a vector potential. The formulation of these variational principle proceeds along lines previously developed for the acoustic fluid problem [7,8].

The main advantages of using potentials as primary variables as opposed to the more conventional EM finite elements based on intensity and/or flux fields are, in order or importance:

1. Interface discontinuities are automatically taken care of without any special intervention.
2. No approximations are invoked *a priori* since the general Maxwell equations are used.
3. The number of degrees of freedom per finite element node is kept modest as the problem dimensionality increases.
4. Coupling with the mechanical and thermal fields, which involves derived fields, can be naturally evaluated at the Gauss points at which derivatives of the potentials are evaluated.

Following a recapitulation of the basic field equations, the variational principle is stated.

The discretization of these principle into finite element equations produces semidiscrete dynamical equations, which are specialized to the axisymmetric case. These equations are validated in a simulation of a cylindrical conductor wire.

3. ELECTROMAGNETIC FIELD EQUATIONS

3.1 The Maxwell Equations

The original Maxwell equations (1873) involve four spatial fields: **B**, **D**, **E** and **H**. Vectors **E** and **H** represents the electric and magnetic field strengths (also called intensities), respectively, whereas **D** and **B** represent the electric and magnetic flux densities, respectively. All of these are three-vector quantities, that is, vector fields in three-dimensional space ($x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$):

$$\mathbf{E} = \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \quad \mathbf{D} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}, \quad \mathbf{E} = \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \quad \mathbf{H} = \begin{Bmatrix} H_1 \\ H_2 \\ H_3 \end{Bmatrix}. \tag{1}$$

Other quantities are the electric current 3-vector **j** and the electric charge density ρ (a scalar). Units for these and other quantities of interest in this work are summarized in Tables 1-2.

With this notation, and using superposed dots to denote differentiation with respect to time t, we can state Maxwell equations as*

$$\boxed{\begin{aligned} \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0, \qquad \nabla \times \mathbf{H} - \dot{\mathbf{D}} = \mathrm{j}, \\ \nabla \cdot \mathbf{D} = \rho, \qquad \nabla \cdot \mathbf{B} = 0. \end{aligned}} \tag{2}$$

The first and second equation are also known as Faraday's and Ampère-Maxwell laws, respectively.

The system (2) supplies a total of eight partial differential equations, which as stated are independent of the properties of the underlying medium.

3.2 Constitutive Equations

The field intensities **E** and **H** and the corresponding flux densities **D** and **B** are not independent but are connected by the electromagnetic constitutive equations. For an electromagnetically isotropic, non-polarized material the equations are

$$\boxed{\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = \epsilon \mathbf{E}} \tag{3}$$

* Some authors, for example Eyges [2], include 4π factors and the speed of light c in the Maxwell equations. Other textbooks, *e.g.* [19,20], follow Heaviside's advice in using technical units that eliminate such confusing factors.

Table 1 Electric and Magnetic Quantities

Quantities	Symbol	MKS-Weber Units
Electric charge density	ρ	coulomb/m^2
Electric field intensity	$\mathbf{E}$	newton/coulomb
Electric flux density	$\mathbf{D}$	coulomb/m^2
Electric resistance	R	ohm
Electric conductivity	g	mho
Displacement current density	$\dot{\mathbf{D}}$	coulomb/(sec.m^2)
Susceptibility*	ϵ	coulomb/(joule.m)
Current	$\mathbf{j}$	coulomb/sec
Magnetic field intensity	$\mathbf{H}$	newton/weber or amperes/m
Magnetic flux density	$\mathbf{B}$	weber/m^2
Magnetic permeability†	μ	weber/(joule.m) or henry/m

* Also called capacitivity and permittivity
† Also called inductivity

Table 2 Equivalences Between Various MKS-Giorgi Units

1 newton $\equiv$ 1 kg.m/sec^2
1 joule $\equiv$ 1 newton.m
1 watt $\equiv$ 1 joule/sec
1 coulomb $\equiv$ 1 ampere.sec
1 ohm $\equiv$ 1 volt/ampere
1 farad $\equiv$ 1 coulomb/volt
1 henry $\equiv$ 1 (volt.sec)/ampere
1 weber $\equiv$ 1 volt.sec
1 mho $\equiv$ 1 ohm^{-1}

where μ and ϵ are the permeability and susceptibility, respectively, of the material†. These coefficients are functions of position but (for static or harmonic fields) do not depend on time. In the general case of a non-isotropic material both μ and ϵ become tensors. Even in isotropic media μ in general is a complicated function of $\mathbf{H}$; in ferromagnetic materials it depends on the previous history (hysteresis effect).

In free space $\mu = \mu_0$ and $\epsilon = \epsilon_0$, which are connected by

$$c_0^2 = \frac{1}{\mu_0 \epsilon_0} \tag{4}$$

where c_0 is the speed of light in a free vacuum. In MKS-A units, $c_0 = 3.10^9\,\mathrm{m/sec}$ and

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{henry/m}, \quad \epsilon_0 = \mu_0^{-1} c_0^{-2} = (36\pi)^{-1} \times 10^{-11}\ \mathrm{sec^2/(henry.m)} \tag{5}$$

† Other names are often used, see Table 1.

The condition $\mu \approx \mu_0$ holds well for most practical purposes in such media as air and copper; in fact $\mu_{air} = 1.0000004\mu_0$ and $\mu_{copper} = .99999\mu_0$.

The electrical field strength $\mathbf{E}$ is further related to the current density $\mathbf{j}$ by Ohm's law:

$$\mathbf{j} = g\mathbf{E} \tag{6}$$

where g is the conductivity of the material. Again for an non-isotropic material g is generally a tensor which may also contain real and imaginary components; in which case the above relation becomes the generalized Ohm's law. For good conductors $g >> \epsilon$; for bad conductors $g << \epsilon$. In free space, $g = 0$.

3.3 Maxwell Equations in Terms of $\mathbf{E}$ *and* $\mathbf{B}$

To pass to the four-potential considered in Section 4 it is convenient to express Maxwell's equations in terms of the electrical field strength $\mathbf{E}$ and the magnetic flux $\mathbf{B}$. In fact this is the pair most frequently used in electromagnetic work that involve arbitrary media. On eliminating $\mathbf{D}$ and $\mathbf{H}$ through the constitutive equations (3) we obtain

$$\boxed{\begin{array}{cc} \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0, & \nabla \times \mathbf{B} - \mu\epsilon\dot{\mathbf{E}} = \mu\mathbf{j}, \\ \nabla \cdot \mathbf{E} = \rho/\epsilon, & \nabla \cdot \mathbf{B} = 0. \end{array}} \tag{7}$$

The second equation assumes that ϵ is independent of time; otherwise $\epsilon\dot{\mathbf{E}} = \epsilon\, d\mathbf{E}/dt$ should be replaced by $d(\epsilon\mathbf{E})/dt$. In charge-free vacuum the equations reduce to

$$\begin{array}{cc} \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0, & \nabla \times \mathbf{B} - \frac{1}{c_0^2}\dot{\mathbf{E}} = 0, \\ \nabla \cdot \mathbf{E} = 0, & \nabla \cdot \mathbf{B} = 0. \end{array} \tag{8}$$

3.4 The Electromagnetic Potentials

The electric scalar potential Φ and the magnetic vector potential $\mathbf{A}$ are introduced by the definitions

$$\boxed{\mathbf{E} = -\nabla\Phi - \dot{\mathbf{A}}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.} \tag{9}$$

This definition satisfies the two homogeneous Maxwell equations in (7). The definition of $\mathbf{A}$ leaves its divergence $\nabla \cdot \mathbf{A}$ arbitrary. We shall use the Lorentz gauge [13]

$$\nabla \cdot \mathbf{A} + \mu\epsilon\dot{\Phi} = 0. \tag{10}$$

With this choice the two non-homogeneous Maxwell equations in terms of Φ and $\mathbf{A}$ separate into the wave equations

$$\nabla^2\Phi - \mu\epsilon\ddot{\Phi} = -\rho/\epsilon, \qquad \nabla^2\mathbf{A} - \mu\epsilon\ddot{\mathbf{A}} = -\mu\mathbf{j}. \tag{11}$$

4. THE ELECTROMAGNETIC FOUR-POTENTIAL

Maxwell's equations can be presented in a compact manner* in the four-dimensional space-time defined by the coordinates

$$x_1 \equiv x, \quad x_2 \equiv y, \quad x_3 \equiv z, \quad x_4 = ict \tag{12}$$

where x_1, x_2, x_3 are spatial Cartesian coordinates, $i^2 = -1$ is the imaginary unit, and $c = 1/\sqrt{\mu\epsilon}$ is the speed of EM waves in the medium under consideration. In the sequel Roman subscripts will consistently go from 1 to 4 and the summation convention over repeated indices will be used unless otherwise stated.

4.1 The Field Strength Tensor

The unification can be expressed most conveniently in terms of the *field-strength tensor* $\mathbf{F}$, which is a four-dimensional *antisymmetric* tensor constructed from the components of $\mathbf{E}$ and $\mathbf{B}$ as follows:

$$\mathbf{F} = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{23} & -F_{34} & 0 \end{pmatrix} \stackrel{\text{def}}{=} \beta \begin{pmatrix} 0 & cB_3 & -cB_2 & -iE_1 \\ -cB_3 & 0 & cB_1 & -iE_2 \\ cB_2 & -cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix}. \tag{13}$$

Here β is an adjustment factor to be determined later. Similarly, introduce the *four-current* vector $\mathbf{J}$ as

$$\mathbf{J} = \begin{Bmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{Bmatrix} \stackrel{\text{def}}{=} \beta \begin{Bmatrix} c\mu j_1 \\ c\mu j_2 \\ c\mu j_3 \\ i\rho/\epsilon \end{Bmatrix} = \beta c \begin{Bmatrix} \mu j_1 \\ \mu j_2 \\ \mu j_3 \\ i\sqrt{\mu/\epsilon}\,\rho \end{Bmatrix}. \tag{14}$$

Then, for arbitrary β, the non-homogeneous Maxwell equations, namely $\nabla \times \mathbf{B} - \mu\epsilon\dot{\mathbf{E}} = \mu\mathbf{j}$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon$, may be presented in the compact "continuity" form†

$$\frac{\partial F_{ik}}{\partial x_k} = J_i. \tag{15}$$

The other two Maxwell equations, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0$, can be presented as

$$\frac{\partial F_{ik}}{\partial x_m} + \frac{\partial F_{mi}}{\partial x_k} + \frac{\partial F_{km}}{\partial x_i} = 0, \tag{16}$$

where the index triplet (i, j, k) takes on the values (1,2,3), (4,2,3), (4,3,1) and (4,1,2).

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* A form compatible with special relativity.

† The covariant form of these two equations.

4.2 *The Four-Potential*

The EM "four-potential" ϕ is a four-vector whose components are constructed with the electric and magnetic potential components of $\mathbf{A}$ and Φ:

$$\phi = \beta \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \stackrel{\text{def}}{=} \begin{Bmatrix} cA_1 \\ cA_2 \\ cA_3 \\ i\Phi \end{Bmatrix}. \tag{17}$$

It may then be verified that $\mathbf{F}$ can be expressed as the four-curl of ϕ, that is

$$F_{ik} = \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{\partial x_k}, \tag{18}$$

or in more detail and using commas to abbreviate partial derivatives:

$$\mathbf{F} = \begin{pmatrix} 0 & \phi_{2,1} - \phi_{1,2} & \phi_{3,1} - \phi_{1,3} & \phi_{4,1} - \phi_{1,4} \\ \phi_{1,2} - \phi_{2,1} & 0 & \phi_{3,2} - \phi_{2,3} & \phi_{4,2} - \phi_{2,4} \\ \phi_{1,3} - \phi_{3,1} & \phi_{2,3} - \phi_{3,2} & 0 & \phi_{4,3} - \phi_{3,4} \\ \phi_{1,4} - \phi_{4,1} & \phi_{2,4} - \phi_{4,2} & \phi_{3,4} - \phi_{4,3} & 0 \end{pmatrix}. \tag{19}$$

4.3 *The Lagrangian*

With these definitions, the basic Lagrangian of electromagnetism can be stated as‡

$$\begin{aligned} L &= \tfrac{1}{4} F_{ik} F_{ik} - J_i \phi_i = \tfrac{1}{4} \beta^2 \left(\frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{x_k} \right)^2 - J_i \phi_i \\ &= \tfrac{1}{2} \beta^2 \left(c^2 B^2 - E^2 \right) - \frac{\beta^2}{\epsilon} (j_1 A_1 + j_2 A_2 + j_3 A_3 - \rho \Phi), \end{aligned} \tag{20}$$

in which

$$B^2 = \mathbf{B}^T \mathbf{B} = B_1^2 + B_2^2 + B_3^2, \quad E^2 = \mathbf{E}^T \mathbf{E} = E_1^2 + E_2^2 + E_3^2. \tag{21}$$

Comparing the first term with the magnetic and electric energy densities [2,19,20]

$$u_m = \tfrac{1}{2} \mathbf{B}^T \mathbf{H} = \frac{1}{2\mu} B^2, \qquad u_e = \tfrac{1}{2} \mathbf{D}^T \mathbf{E} = \tfrac{1}{2} \epsilon E^2, \tag{22}$$

we must have $\beta^2 c^2 = \beta^2 / (\mu\epsilon) = 1/\mu$, from which

$$\beta = \sqrt{\epsilon}. \tag{23}$$

‡ Lanczos [12] presents this Lagrangian for free space, but the expression (24) for an arbitrary material was found in none of the textbooks on electromagnetism listed in the References.

Consequently the required Lagrangian is

$$\boxed{L = \frac{1}{2\mu}B^2 - \tfrac{1}{2}\epsilon E^2 - (j_1 A_1 + j_2 A_2 + j_3 A_3 - \rho\Phi).} \tag{24}$$

The associated variational form is

$$R = \int_{t_0}^{t_1} \int_V L\, dV\, dt \tag{25}$$

where V is the integration volume considered in the analysis. In theory V extends over the whole space, but in the numerical simulation the integration is truncated at a distant boundary or special devices are used to treat the decay behavior at infinity.

4.4 The Four-Field Equations

On setting the variation of the functional (24) to zero we recover the field equations (15–16). Taking the divergence of both sides of (15) and observing that $\mathbf{F}$ is an antisymmetric tensor so that its divergence vanishes we get

$$\frac{\partial J_i}{\partial x_i} = c\mu(\nabla \cdot \mathbf{j} + \dot{\rho}) = 0, \tag{26}$$

The vanishing term in parenthesis is the equation of continuity, which expresses the law of conservation of charge. The Lorentz gauge condition (10) may be stated as $\nabla \cdot \phi = 0$. Finally, the potential wave equations (11) may be expressed in compact form as

$$\Box \phi_i = -J_i \tag{27}$$

where $\Box$ denotes the "four-wave-operator", also called the D'Alembertian:

$$\Box \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x_k \partial x_k} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{c^2 \partial t^2}. \tag{28}$$

Hence each component of the four-potential ϕ satisfies an inhomogeneous wave equation. In free space, $J_i = 0$ and each component satisfies the homogeneous wave equation.

5. THE AXISYMMETRIC TEST EXAMPLE

The simplest example for testing the finite element formulation based on the four-potential variational principle is provided by the axisymmetric magnetic field generated by a uniform, steady current flowing through a straight, infinitely long conducting wire of circular cross section. In the present Section we derive expressions for the magnetostatic fields outside and within the conductor. These analytical solutions will be later compared with the finite element numerical solutions.

5.1 The Free-Space Magnetic Field

To take advantage of the axisymmetric geometry we choose a cylindrical coordinate system with the wire centerline as the longitudinal z-axis. The vector components in the cylindrical coordinate directions r, θ and z are denoted by

$$\begin{array}{ll} A_1,\ B_1,\ E_1 & \text{in the } r \text{ direction} \\ A_2,\ B_2,\ E_2 & \text{in the } \theta \text{ direction} \\ A_3,\ B_3,\ E_3 & \text{in the } z \text{ direction} \end{array}$$

The electromagnetic fields will then vary in the radial direction (r) but not in the angular (θ) and axial (z) directions. Similarly, the current density that flows in the wire has only one nonzero component acting in the positive or negative z direction; conventionally we select the positive direction.

In Cartesian coordinates the radial component of the electrostatic potential in free space can be calculated from the expression (see, *e.g.*, [2,10,18,19,20])

$$A_z = A_3 = \frac{\mu_0}{4\pi} \int_V \frac{j_3}{|\mathbf{r}|}\, dV, \tag{29}$$

where $|\mathbf{r}|$ is the distance between the elemental charge $j_3\, dV$ and the point in space at which we wish to find the field potential. The integral extends over the volume containing charges. This expression serves equally well in cylindrical coordinates. In fact, the transformation of z components will be one to one if the center of the systems coincide.

As noted above the only non-vanishing component of the current vector is $j_3\, dS$ where dS is the elemental cross sectional area of the conductor and j_3 is the current density in the z direction. If $d\ell$ represents the differential length of the wire, then $\int_S j_3\, dV = \int_S j_3\, dS\, d\ell = I\, d\ell = I\, dz$ and $|\mathbf{r}| = \sqrt{r^2 + z^2}$. Substitution into Eq. (29) yields

$$A_3(r) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{r^2 + z^2}}. \tag{30}$$

This integral diverges, but this difficulty can be overcome by taking the wire to have a finite length $2L$ symmetric with respect to the field point, that is large with respect to its diameter. Integrating between $-L$ and $+L$ we get

$$A_3(r) = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{dz}{\sqrt{r^2 + z^2}} = \frac{\mu_0 I}{4\pi} \ln\left(z + \sqrt{r^2 + z^2}\right)\Big|_{-L}^{+L}. \tag{31}$$

Expanding this equation in powers of r/L and retaining only first-order terms gives

$$A_3 = -\left(\frac{\mu_0 I}{2\pi}\right)\ln r + C. \tag{32}$$

where C is an arbitrary constant. For subsequent developments it is convenient to select $C = (\mu_0 I/2\pi)\ln R_T$, where R_T is the "truncation radius" of the finite element mesh in the radial direction. Then

$$A_3 = -\left(\frac{\mu_0 I}{2\pi}\right)\ln\left(\frac{r}{R_T}\right). \tag{33}$$

With this normalization $A_3 = 0$ at $r = R_T$. Taking the curl of $\mathbf{A}$ gives the $\mathbf{B}$ field in cylindrical coordinates:

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \end{Bmatrix} = \begin{Bmatrix} B_r \\ B_\theta \\ B_z \end{Bmatrix} = \begin{Bmatrix} \frac{1}{r}\frac{\partial A_3}{\partial \theta} - \frac{\partial A_2}{\partial z} \\ \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial r} \\ \frac{1}{r}\frac{\partial (rA_2)}{\partial r} - \frac{1}{r}\frac{\partial A_1}{\partial \theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\frac{\partial A_3}{\partial r} \\ 0 \end{Bmatrix}. \tag{34}$$

It is seen that the only non-vanishing component of the magnetic flux density is

$$B_\theta \equiv B_2 = \mu_0 H_2 = -\frac{\partial A_3}{\partial r} = \frac{\mu_0 I}{2\pi r}. \tag{35}$$

This expression is called the law of Biot-Savart in the EM literature.

5.2 Magnetic Field Within the Conductor

Again restricting our consideration to the static case, we have from Maxwell's equations in their integral flux form

$$\oint_C \mathbf{H}\cdot d\mathbf{s} = \oint_C \mu^{-1}\mathbf{B}\cdot d\mathbf{s} = \int_S \mathbf{j}\cdot d\mathbf{S}, \tag{36}$$

where C is a contour around the field point traversed counterclockwise with an oriented differential arclength $d\mathbf{s}$ and $d\mathbf{S}$ is the oriented surface element inside the contour. The term for the electric field disappears in this analysis because $\dot{\mathbf{E}} = 0$. From before we know that the right hand side of Eq. (35) is equal to the normal component of the current that flows through the cross sectional area evaluated by the integral. In the free space case, this is the total current that flows through the conductor. But in the conductor the amount of current is a function of the distance r from the center. Again using I to represent the total current carried by the conductor, and R the radius of the conductor, and assuming an *uniform* current density $j_3 = I/(\pi R^2)$, the right hand side of (35) become_A

$$\int_S \mathbf{j}\cdot d\mathbf{S} = \int_S j_3\, dS = \frac{I}{\pi R^2}\int_S dS = I\frac{r}{R^2}. \tag{37}$$

Evaluating the left hand side of the integral and solving for B_2 gives:

$$2\pi r\mu^{-1}B_2 = I\frac{r^2}{R^2}, \qquad B_2 = \frac{\mu I\, r}{2\pi R^2}. \tag{38}$$

Comparing with (34) we see that if $\mu = \mu_0$ then B_2 is continuous at the wire surface $r = R$ and has the value $\mu_0 I/(2\pi R)$. But if $\mu \neq \mu_0$ there is a jump $(\mu - \mu_0)I/(2\pi R)$ in B_2.

The magnetic potential A_3 within the conductor is easily computed by integrating $-B_2$ with respect to r:

$$A_3 = -\frac{\mu I\, r^2}{4\pi R^2} + C. \tag{39}$$

The value of C is determined by matching (33) at $r = R$, since the potential must be continuous. The result can be written

$$A_3 = \frac{I}{2\pi}\left[\tfrac{1}{2}\mu\left(1 - \frac{r^2}{R^2}\right) - \mu_0 \ln\left(\frac{R}{R_T}\right)\right]. \tag{40}$$

The preceding expressions (33)-(40) for A_3 could also be derived in a somewhat more direct fashion by integrating the ordinary differential equation $\nabla^2 A_3 = r^{-1}(\partial(r\partial A_3/\partial r)\partial r) = \mu j_3$ to which the second of (11) reduces.

6. FINITE ELEMENT DISCRETIZATION

6.1 The Lagrangian in Cylindrical Coordinates

To construct finite element approximations we need to express the Lagrangian (24)

$$L = \frac{1}{2\mu}B^2 - \tfrac{1}{2}\epsilon E^2 - (\mathrm{j}^T\mathbf{A} - \rho\Phi), \tag{41}$$

in terms of the potentials written in cylindrical coordinates. For B^2 we can use the expression of the curl (33)

$$B^2 = \left(\frac{1}{r}\frac{\partial A_3}{\partial \theta} - \frac{\partial A_2}{\partial z}\right)^2 + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial (rA_2)}{\partial r} - \frac{1}{r}\frac{\partial A_1}{\partial \theta}\right)^2, \tag{42}$$

For E^2 we need the cylindrical-coordinate gradient formulas

$$\mathbf{E} = \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} = \begin{Bmatrix} E_r \\ E_\theta \\ E_z \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\partial \Phi}{\partial r} + \dot{A}_1 \\ \dfrac{1}{r}\dfrac{\partial \Phi}{\partial \theta} + \dot{A}_2 \\ \dfrac{\partial \Phi}{\partial z} + \dot{A}_3 \end{Bmatrix}, \tag{43}$$

so that

$$E^2 = \mathbf{E}^T\mathbf{E} = \left(\frac{\partial \Phi}{\partial r} + \frac{\partial A_1}{\partial t}\right)^2 + \left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{\partial A_2}{\partial t}\right)^2 + \left(\frac{\partial \Phi}{\partial z} + \frac{\partial A_3}{\partial t}\right)^2. \tag{44}$$

In the axisymmetric case, $A_1 = A_2 = 0$; furthermore $A_z = A_3$ is only a function of the radial distance from the wire. Therefore $\partial A_3/\partial\theta = \partial A_3/\partial z = 0$. From symmetry considerations we also know that the electric field cannot vary in the θ and z directions, which gives $\partial\Phi/\partial z = \partial\Phi/\partial\theta = 0$. Finally, the only nonvanishing current density component is j_3. Consequently the Lagrangian (41) simplifies to

$$\boxed{L = \frac{1}{2\mu}\left(\frac{\partial A_3}{\partial r}\right)^2 - \tfrac{1}{2}\epsilon\left[\left(\frac{\partial\Phi}{\partial r}\right)^2 + \left(\frac{\partial A_3}{\partial t}\right)^2\right] - (j_3 A_3 - \rho\Phi).} \tag{45}$$

6.2 Constructing EM Finite Elements

To deal with this particular axisymmetric problem a two-node "line" finite element extending in the radial r direction is sufficient. In the following we deal with an individual element identified by superscript e. The two element end nodes are denoted by i and j. The electric potential Φ and the magnetic potential $A_3 \equiv A_z$ are interpolated over each element as

$$\Phi^e = \mathbf{N}^e_\Phi \boldsymbol{\Phi}^e, \qquad A^e_3 = \mathbf{N}^e_A \mathbf{A}^e_3, \tag{46}$$

Here row vectors $\mathbf{N}^e_\Phi$ and $\mathbf{N}^e_A$ contain the finite element shape functions for Φ^e and A^e_3, respectively, which are only functions of the radial coordinate r:

$$\mathbf{N}^e_\Phi = \langle\, N^e_{\Phi i}(r) \quad N^e_{\Phi j}(r)\,\rangle, \qquad \mathbf{N}^e_A = \langle\, N^e_{Ai}(r) \quad N^e_{Aj}(r)\,\rangle, \tag{47}$$

and column vectors $\boldsymbol{\Phi}^e$ and $\mathbf{A}^e_3$ contain the nodal values of Φ and A_3, respectively, which are only functions of time t:

$$\boldsymbol{\Phi}^e = \left\{\begin{matrix}\Phi_i(t)\\ \Phi_j(t)\end{matrix}\right\}, \qquad \mathbf{A}^e_3 = \left\{\begin{matrix}A_{3i}(t)\\ A_{3j}(t)\end{matrix}\right\}. \tag{48}$$

Substitution of these finite element assumptions into the Lagrangian (45) and then into Eq. (25) yields the variational integral as sum of element contributions $R = \sum_e R^e$, where

$$\begin{aligned} R^e = \int_{t_0}^{t_1}\int_{V^e} \frac{1}{2\mu}\left(\frac{\partial \mathbf{N}^e_A}{\partial r}\mathbf{A}^e_3\right)^2 - \tfrac{1}{2}\epsilon\left[\left(\frac{\partial \mathbf{N}^e_\Phi}{\partial r}\boldsymbol{\Phi}^e\right)^2 + \left(\frac{\partial \mathbf{N}^e_A}{\partial t}\dot{\mathbf{A}}^e_3\right)^2\right] \\ - \left(j_3 \mathbf{N}^e_A \mathbf{A}^e_3 - \rho \mathbf{N}^e_\Phi \boldsymbol{\Phi}^e\right) dV^e\, dt. \end{aligned} \tag{49}$$

where V^e denotes the volume of the element. Taking the variation with respect to the element node values gives

$$\begin{aligned} \delta R^e = \int_{t_0}^{t_1}\int_{V^e} (\delta\mathbf{A}^e_3)^T\left[\frac{1}{\mu}\left(\frac{\partial \mathbf{N}^e_A}{\partial r}\right)^T\frac{\partial \mathbf{N}^e_A}{\partial r}\mathbf{A}^e_3 + \epsilon\,(\mathbf{N}^e_A)^T\mathbf{N}^e_A\,\ddot{\mathbf{A}}^e_3 - j_3\,(\mathbf{N}^e_A)^T\right] \\ + \int_{t_0}^{t_1}\int_{V^e} (\delta\boldsymbol{\Phi}^e)^T\left[-\epsilon\left(\frac{\partial \mathbf{N}^e_\Phi}{\partial r}\right)^T\frac{\partial \mathbf{N}^e_\Phi}{\partial r}\boldsymbol{\Phi}^e + \rho\,(\mathbf{N}^e_\Phi)^T\right] dV^e\, dt. \end{aligned} \tag{50}$$

On applying fixed-end initial conditions at $t = t_0$ and $t = t_1$ and the lemma of the calculus of variations, we proceed to equate each of the expressions in brackets to zero. From the first bracket we obtain for each element the following second-order dynamic equations for the magnetic potential at the nodes, which are purposedly written in a notation resembling the mass-stiffness-force equations of mechanics:

$$\mathbf{M}_A^e \ddot{\mathbf{A}}_3^e + \mathbf{K}_A^e \mathbf{A}_3^e = \mathbf{f}_A^e, \tag{51}$$

where

$$\mathbf{M}_A^e = \int_{V^e} \epsilon \, (\mathbf{N}_A^e)^T \mathbf{N}_A^e \, dV^e, \qquad \mathbf{K}_A^e = \int_{V^e} \frac{1}{\mu} \left(\frac{\partial \mathbf{N}_A^e}{\partial r} \right)^T \frac{\partial \mathbf{N}_A^e}{\partial r} \, dV^e, \tag{52}$$

$$\mathbf{f}_A^e = \int_{V^e} j_3 \, (\mathbf{N}_A^e)^T \, dV^e. \tag{53}$$

From the second bracket we obtain for the electric potential a simpler relation which does not involve time derivatives, *i.e*, is static in nature:

$$\mathbf{K}_\Phi^e \mathbf{\Phi}^e = \mathbf{f}_\Phi^e, \tag{54}$$

where

$$\mathbf{K}_\Phi^e = \int_{V^e} \epsilon \left(\frac{\partial \mathbf{N}_\Phi^e}{\partial r} \right)^T \frac{\partial \mathbf{N}_\Phi^e}{\partial r} \, dV^e, \qquad \mathbf{f}_\Phi^e = \int_{V^e} \rho \, (\mathbf{N}_\Phi^e)^T \, dV^e. \tag{55}$$

Assembling these equations in the usual way we obtain the semidiscrete *master finite element equations*:

$$\begin{aligned} \mathbf{M}_A \ddot{\mathbf{A}}_3 + \mathbf{K}_A \mathbf{A}_3 &= \mathbf{f}_A, \\ \mathbf{K}_\Phi \mathbf{\Phi} &= \mathbf{f}_\Phi. \end{aligned} \tag{56}$$

6.3 The Static Case

In time-independent problems, the term $\ddot{\mathbf{A}}_3$ disappears from (56) and the master finite element equations of electromagnetostatics become

$$\mathbf{K}_A \mathbf{A}_3 = \mathbf{f}_A, \qquad \mathbf{K}_\Phi \mathbf{\Phi} = \mathbf{f}_\Phi. \tag{57}$$

If the current density and charge distributions are known *a priori* then these two equations may be solved separately. If only the charge distribution ρ is known then the second equation should be solved first to obtain the electric field $\mathbf{E}$ as gradient of the computed electric potential $\mathbf{\Phi}$; then the current density $\mathbf{j}$ can be obtained from Ohm's law (6) and used to computed the force vector $\mathbf{f}_A$ of the first equation, which is then solved for the magnetic potential. Conversely, if only the current density distribution is known *a priori* the preceding steps are reversed.

For the present test problem the current distribution is assumed to be known, and we shall be content with solving the first equation for the magnetic flux.

6.4 An Alternative Semidiscretization

If upon setting the brackets of the variation (50) to zero we multiply them through by μ and $1/\epsilon$, respectively, the expressions for the mass, stiffness and force matrices become

$$\mathbf{M}_A^e = \int_{V^e} \frac{1}{c^2} (\mathbf{N}_A^e)^T \mathbf{N}_A^e \, dV^e, \ \mathbf{K}_A^e = \int_{V^e} \left(\frac{\partial \mathbf{N}_A^e}{\partial r}\right)^T \frac{\partial \mathbf{N}_A^e}{\partial r} \, dV^e, \ \mathbf{f}_A^e = \int_{V^e} \mu j_3 \mathbf{N}_A^{e\,T} \, dV^e,$$
$$\mathbf{K}_\Phi^e = \int_{V^e} \left(\frac{\partial \mathbf{N}_\Phi^e}{\partial r}\right)^T \frac{\partial \mathbf{N}_\Phi^e}{\partial r} \, dV, \qquad \mathbf{f}_\Phi^e = \int_{V^e} \frac{1}{\epsilon} \rho \, (\mathbf{N}_\Phi^e)^T \, dV. \tag{58}$$

The matrices $\mathbf{M}$ and $\mathbf{K}$ above are quite similar to the capacitance and reactance matrices, respectively, obtained in the potential analysis of acoustic fluids [7,8]. Another attractive feature of (58) is that $\mathbf{K}_A = \mathbf{K}_\Phi$ if the shape functions of both potentials coalesce, as is natural to assume. These advantages are, however, more than counterbalanced by the fact that "jump forces" contributions to $\mathbf{f}_A$ and $\mathbf{f}_\Phi$ arise on material interfaces where μ and ϵ change abruptly, and the proper handling of such forces substantially complicates the programming logic. Note that this issue does not arise in the treatment of homogeneous acoustic fluids.

6.5 Applying Boundary Conditions

The finite element mesh is necessarily terminated at a finite size, which for the test problem is defined as the truncation radius R_T alluded to in Section 5.1. In static calculations the material outside the FE mesh may be viewed as having zero permeability μ, or, equivalently, infinite stiffness or zero potential. It follows that the potential value at the node located on the truncation radius may be prescribed to be zero. This is the only essential boundary condition necessary for this particular problem.

7. NUMERICAL VALIDATION

7.1 Finite Element Model

The test problem consists of a wire conductor of radius R transporting a unit current density. For this problem the finite element mesh is completely defined if we specify the radial node coordinates $r_i^e = r_n^e$ and $r_j^e = r_{n+1}^e$ for each element e. If the mesh contains N_{ec} elements inside the conductor, those elements are numbered $e = 1, 2, \ldots N_{ec}$ and nodes $n = 1, 2, \ldots N_{ec} + 1$ starting from the conductor center outwards. The first node ($n = 1$) is at the conductor center $r = 0$ and node $n = N_{ec} + 1$ is placed at the conductor boundary $r = R$. The mesh is then continued with N_{ef} elements into free space, with a double node at the counductor boundary. The last node is placed at $r = R_T$ at which point the free space mesh is truncated; usually $R_T = 4R$ to $5R$. Although the mesh appears to be one-dimensional, a typical element actually forms a "tube" of longitudinal axis z, internal radius r_i^e and external radius r_j^e, extending a unit distance along z.

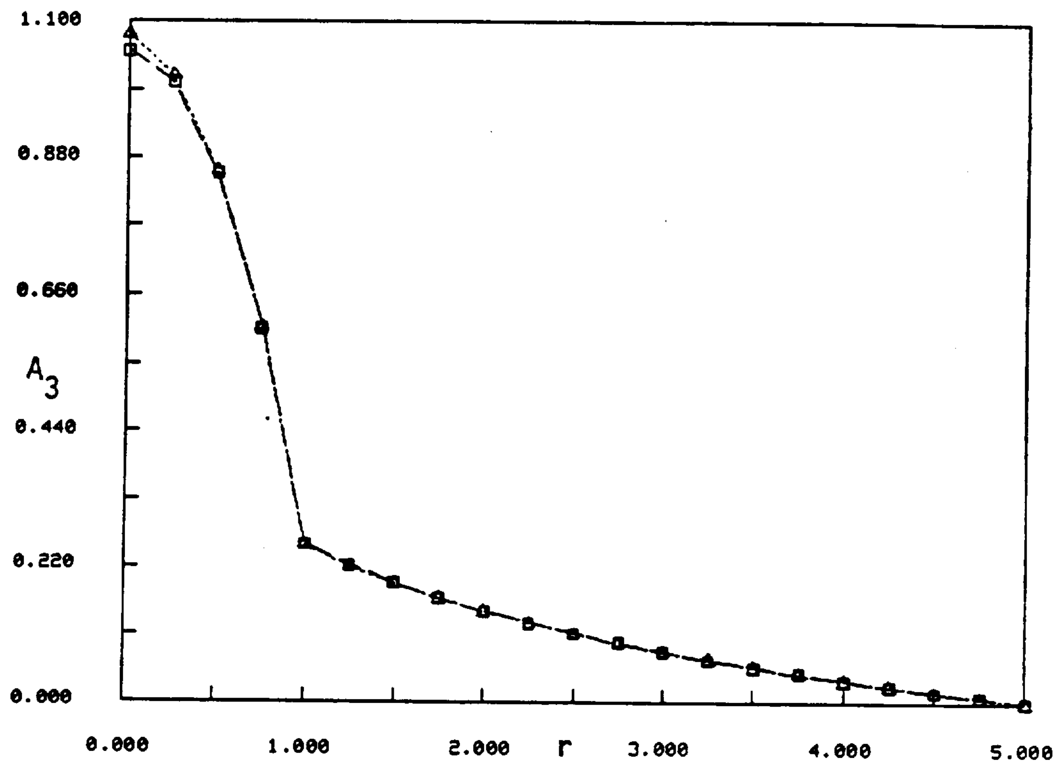

Figure 1. Magnetic potential A_3 vs. distance from center r, $\mu_{wire} = 10.0$: finite element values (triangles) and analytical values (squares).

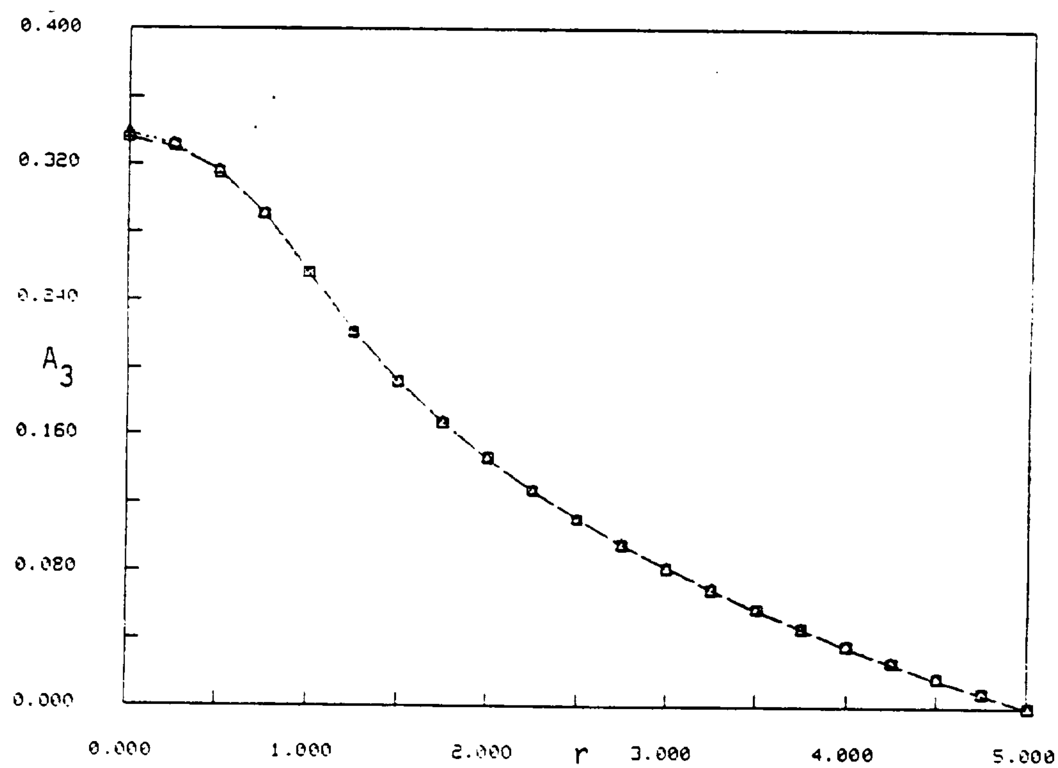

Figure 2. Magnetic potential A_3 vs. distance from center r, $\mu_{wire} = 1.0$: finite element values (triangles) and analytical values (squares).

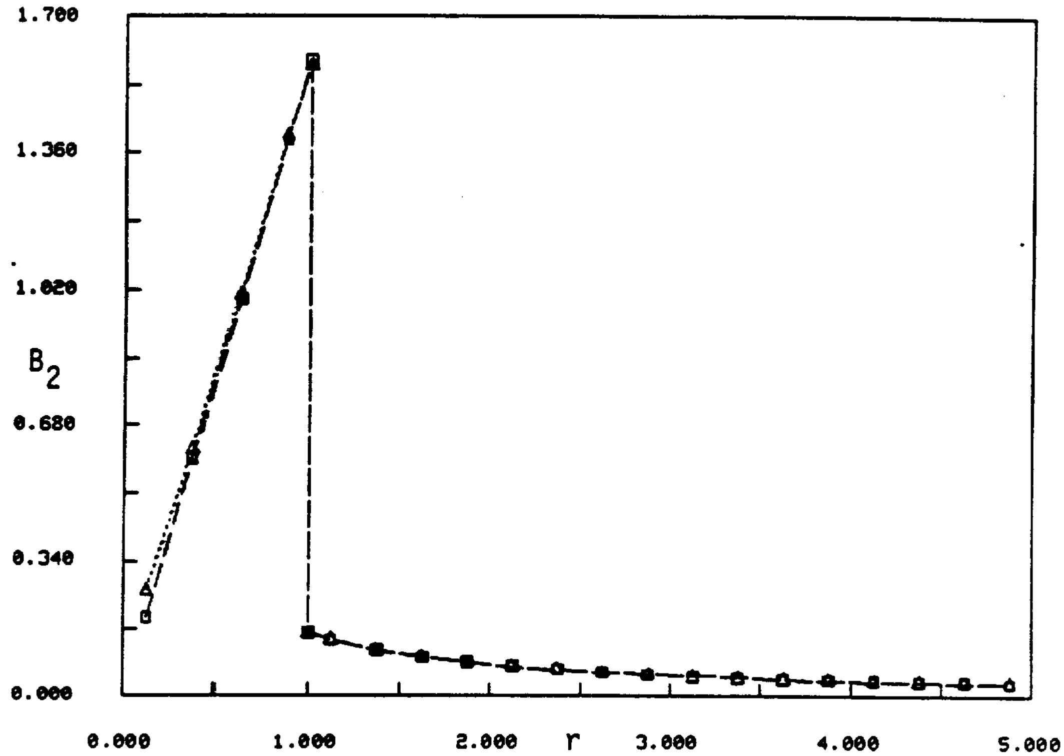

Figure 3. Magnetic flux density B_2 vs. distance from center r, $\mu_{wire} = 10.0$: finite element values (triangles) and analytical values (squares). Values shown on the interface $r = 1$ with dark symbols have been extrapolated from element center values to display the jump more accurately; this extrapolation scheme has not been used elsewhere.

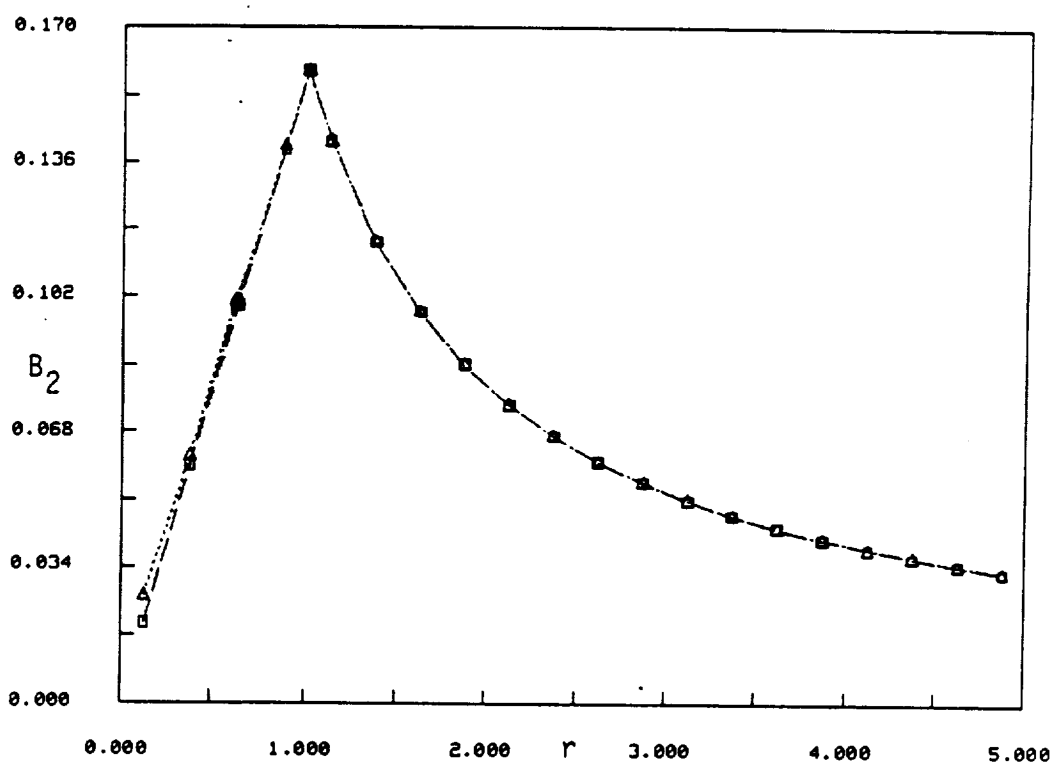

Figure 4. Magnetic flux density B_2 vs. distance from center r, $\mu_{wire} = 1.0$: finite element values (triangles) and analytical values (squares).

For the present study the magnetic potential was linearly interpolated in r, using the linear shape functions

$$\mathbf{N}_A^e = \langle \tfrac{1}{2}(1-\xi) \quad \tfrac{1}{2}(1+\xi) \rangle, \tag{59}$$

where ξ is the dimensionless isoparametric coordinate that varies from -1 at node i to $+1$ at node j. This interpolation provides for C^0 continuity of the potential inside the conductor and in free space.

For the calculation of the element stiffnesses and force vectors, it was assumed that the permeability μ and the current density j_3 were uniform over the element. Then analytical integration over the element geometry gives

$$\mathbf{K}_A^e = \frac{\mu r_m}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{f}_A^e = j_3 \ell \left\{ \begin{matrix} \frac{1}{6}(2r_i^e + r_j^e) \\ \frac{1}{6}(r_i^e + 2r_j^e) \end{matrix} \right\}, \tag{60}$$

where $r_m = \frac{1}{2}(r_i^e + r_j^e)$ is the mean radius and $\ell = r_j^e - r_i^e$ the radial length. For the test problem, μ is constant inside the conductor whereas outside it $\mu = \mu_0$ was assumed to be unity. The longitudinal current density is $j_3 = I/(\pi R^2)$ inside the conductor whereas outside it j_3 vanishes.

The master stiffness matrix and force vector were assembled following standard finite element techniques. The only essential boundary condition was the setting of the nodal potential on the truncation boundary to zero, as explained in Section 6.5. The modified master equations were processed by a conventional symmetric skyline solver, which provided the value of the magnetic potential at the mesh nodes. The magnetic flux density B_2, which is constant over each element, was recovered in element by element fashion through the simple finite difference scheme

$$B_2^e = -\frac{\partial A_3}{\partial r} \approx \frac{A_{3i}^e - A_{3j}^e}{\ell}. \tag{61}$$

This value is assigned to the center of element e.

7.2 *Numerical Results*

The numerical results shown in Figures 1 through 6 pertain to a unit radius conductor ($R = 1$), with the external (free space) mesh truncated at $R_T = 5$. The element radial lengths $r_j^e - r_i^e$ were kept constant and equal to 0.25, which corresponds to 4 internal and 16 external elements.

The computed values of the potential A_3 are compared with the analytical solution given by Eqs. (33) and (40). As can be seen the agreement is excellent. The comparison between computed and analytical values of the magnetic flux density B_2 shows excellent agreement except for the last element near the wire center, at which point the difference scheme (61) loses accuracy. The permeability of free space is conventionally selected to be unity. Figures 1, 3, and 5 illustrate the case where the wire permeability μ_{wire} is set to 10.0, whereas Figures 2, 4, and 6 are for the case in which μ_{wire} is 1.0, that is, same as in free space. (The value of the susceptibility ϵ does not appear in these magnetostatic

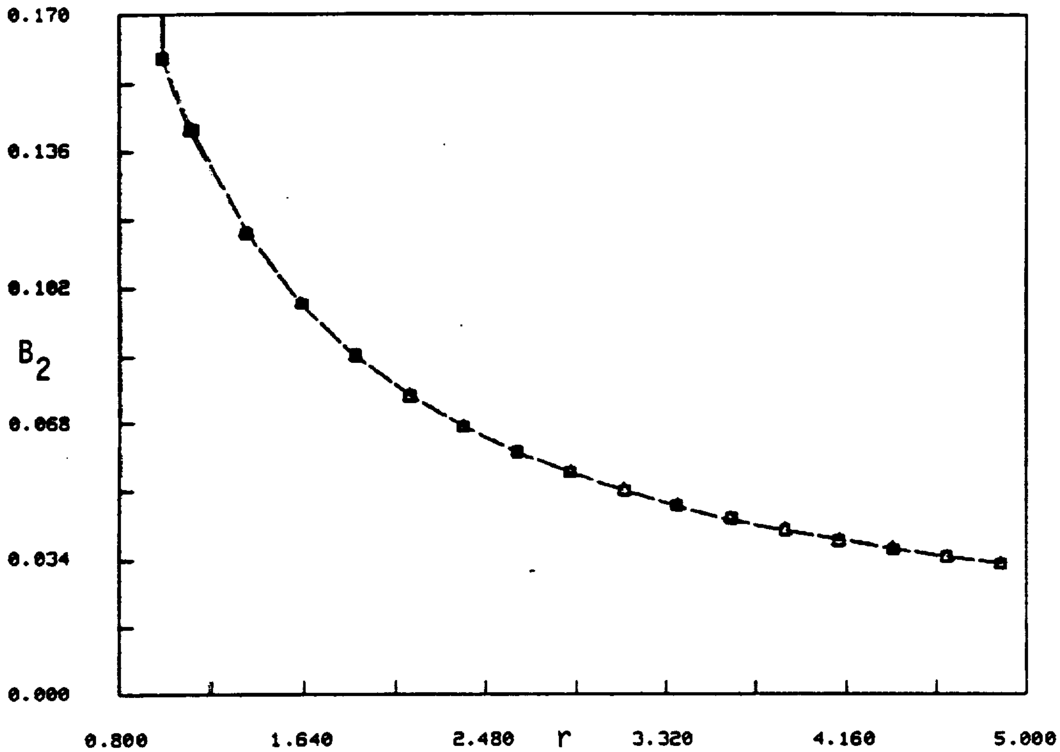

Figure 5. Restriction of Figure 3 to $r > R = 1$, $\mu_{wire} = 10.0$, showing free space magnetic flux density in more detail.

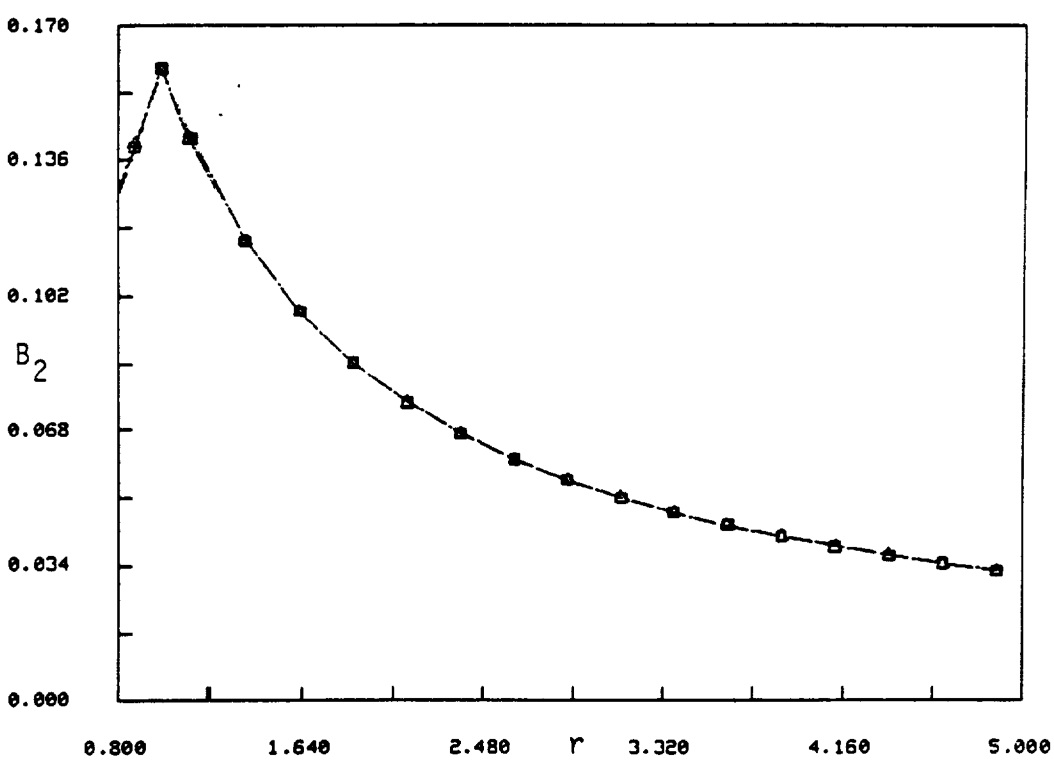

Figure 6. Restriction of Figure 4 to $r > R = 1$, $\mu_{wire} = 1.0$, showing free space magnetic flux density in more detail.

computations.) Figures 1 and 2 show computed and analytical magnetic potentials. The slope discontinuity at $r = 1$ in Figure 1 is a consequence of the change in permeability μ from the wire material to free space. Figures 3 and 4 show the computed and analytical magnetic flux densities. As discussed in Section 5.2, the jump at $r = 1$ in Figure 3 is due to the change in permeability μ from the material to free space. Figures 5 and 6 show the computed and analytical magnetic flux densities in free space with more detail. Note that Figures 5 and 6 for $r > 1$ are identical; this is the expected result because, as shown in Section 5.1, the free-space magnetic flux field depends only upon the current enclosed by a surface integral around the wire and not on the details of the interior field distribution.

In summary, the finite element model performed very accurately in the test problem and converged, as expected, to the analytical solution as the size of the elements decreased.

8. CONCLUSIONS

The results obtained in the one-dimensional steady-state case are encouraging, and appear to be extensible to two- and three-dimensional problems without major difficulties. The electric field remains effectively decoupled from the magnetic field except through Ohm's law. Care must be taken, however, in modeling the forcing function terms so as to avoid the appearance of discontinuity-induced forces at physical interfaces.

The next step in achieving the goal of a finite element model for a superconductor field is to study the time-dependent case, starting with harmonic currents and proceeding eventually to general transients. The code for this is currently written, but a suitable analytical solution for comparison with computed responses is still being developed.

If encouraging results are obtained in the dynamic case, thermocoupling effects will be added to the code. References [3,17,22] discuss several different approaches applicable to various contexts (*e.g.* eddy currents) and these will have to be investigated for suitability for capturing the couplings effects that are relevant to the superconducting problem.

After modeling the coupling effects, the next step will be to model the superconducting fields. The feasibility of using the current model for superconductor applications is great, as the current density of a superconductor can be approximated by the standard current density multiplied by a constant squared. This constant is called the *London penetration depth*. Other analytical models that possess similar characteristics have been developed and are presented in Ref. [11].

Acknowledgements

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APPENDIX: COMPUTER PROGRAM

This Appendix lists the computer program used to test the new electromagnetic elements on the axisymmetric test example. Sections of the program that pertain to the in-core skyline solver SKYFAC/SKYSOL and the command language reader TinyClip are not listed here. Their source code is presented in the following publications:

Felippa, C. A., Solution of Equations with Skyline–Stored Symmetric Coefficient Matrix, *Computers & Structures*, **5**, 1975, pp. 13–25

Felippa, C. A., A Command Reader for Interactive Programming, *Engineering Computations*, **2**, No. 3, 1985, pp. 203–238

```
C=DECK AAWIRE
C=BLOCK FORTRAN
      program    WIRE
C
      integer        MUMEL, MUMNP, MDOF
      parameter     (MUMEL=100, MUMNP=MUMEL+1)
      parameter     (MDOF=MUMNP)
      integer        numel, numnp, ndof
C
      character          CCLVAL
      character          status*60
      integer            nodelm(2,MUMEL), bctag(MUMNP)
      double precision   kmuelm(MUMEL), kepselm(MUMEL)
      integer            dlp(0:MDOF)
      double precision   a(MUMNP), b(MUMEL)
      double precision   r(MUMNP), f(MUMNP), fbc(MUMNP)
      double precision   sm(MDOF*3)
      double precision   aex(MUMNP), bex(MUMEL), fex(MUMNP)
      double precision   v1(MUMNP), v2(MUMEL)
      double precision   kmu, keps, wrad, trad, inten
      integer            nelwir, nelext
C
 1000 continue
C
      call    MATERIAL   (kmu, keps)
      call    PRINTMAT   (kmu, keps)
      call    DIMENSIONS (wrad, trad)
      call    PRINTDIM   (wrad, trad)
      call    CURRENT    (inten)
      call    PRINTCUR   (inten)
C
 1500   continue
C
        call    SUBDIVIDE (nelwir, nelext, numel, numnp, ndof)
        call    PRINTSUB  (nelwir, nelext, numel, numnp, ndof)
        call    GENELEMS
```

```
     $          (nelvir, nelext, kmu, keps, nodelm, kmuelm, kepselm)
        call    PRINTELM  (numel, nodelm, kmuelm, kepselm)
        call    GENNODES  (nelvir, nelext, wrad, trad, r)
        call    GENBCTAG  (nummp, bctag)
        call    PRINTNOD  (nummp, r, bctag)
        call    GENEXACT  (numel, nummp, r, wrad, trad,
     $                     kmu, inten, aex, bex)
        call    ASSEMSTF
     $     (numel, nodelm, kmuelm, kepselm,
     $      nummp, r, ndof, bctag, sm, dlp, status)
        if (status .ne. ' ')      go to 4000
        call    SKYMUL (sm, ndof, dlp, aex, fex, 0, v1, v2)
        call    ASSEMFOR
     $     (numel, nodelm, kmuelm, kepselm, inten,
     $      nummp, r, wrad, trad, ndof, bctag, f, fbc, status)
        if (status .ne. ' ')      go to 4000
        call    CLREAD    (' Go ahead and solve (y/n)? ', ' ')
        if (CCLVAL(1) .ne. 'Y')   go to 4000
        call   SKYCOV (sm, ndof, dlp, fbc, a, status)
        if (status .ne. ' ')      then
          call  ERROR ('SKYCOV', status)
        end if
        call   PRINTSOL (nummp, r, bctag, f, a, aex, fex)
        call   MAGFIELD (numel, nodelm, r, a, b)
        call   PRINTMAG (numel, nodelm, r, b, bex)
C
 4000   continue
C
        call    CLREAD    (' New FE subdivision (y/n)? ', ' ')
        if (CCLVAL(1) .eq. 'Y')   go to 1500
        call    CLREAD    (' New problem data (y/n)? ', ' ')
        if (CCLVAL(1) .eq. 'Y')   go to 1000
      stop
      end
C=END FORTRAN
C=DECK ASSEMFOR
C=PURPOSE Assemble force vector
C=BLOCK FORTRAN
      subroutine  ASSEMFOR
     $      (numel, nodelm, kmuelm, kepselm, inten,
     $       nummp, r, wrad, trad, ndof, bctag, f, fbc, status)
      integer         numel, nodelm(2,*), nummp
      integer         ndof, bctag(ndof)
      integer         eldof(2)
      double precision  r(*), kmuelm(*), kepselm(*)
      double precision  inten, wrad, trad
      double precision  f(*), fbc(*)
      character*(*)  status
      double precision  re(2), fe(2), mu
      integer     i, j, n, ne
C
      status =   ' '
      do 1500  j = 1,ndof
```

```
        f(j) =  0.0
 1500   continue
C
      do 3000  ne = 1,numel
        do 2200  i = 1,2
          n =    nodelm(i,ne)
          re(i) =   r(n)
          eldof(i) =  n
 2200     continue
        mu =   kmuelm(ne)
        call  FORCE (ne, re, inten, wrad, fe, status)
        if (status .ne. ' ')        then
          call  ERROR ('ASSEMFOR', status)
        end if
C
        do 2500  i = 1,2
          j    =   eldof(i)
          f(j) =  f(j) + fe(i)
 2500     continue
C
 3000   continue
      do 4000  j = 1,ndof
        fbc(j) = f(j)
        if (bctag(j) .ne. 0)     fbc(j) = 0.0
 4000   continue
      return
      end
C=END FORTRAN
C=DECK ASSEMSTF
C=PURPOSE Assemble master stiffness matrix
C=BLOCK FORTRAN
      subroutine  ASSEMSTF
     $    (numel, nodelm, kmuelm, kepselm,
     $     numnp, r, ndof, bctag, sm, dlp, status)
      character*(*)  status
      integer         numel, nodelm(2,numel), numnp
      integer         ndof, bctag(ndof), dlp(0:ndof)
      double precision  kmuelm(numel), kepselm(numel)
      double precision  r(ndof), sm(*)
      double precision  re(2), sme(2,2)
      integer         eldof(2)
      integer         i, j, k, ii, jj, n, ne
C
      status =   ' '
C
      call     FORMDLP (numel, nodelm, ndof, bctag, dlp)
      do 2500  i = 1,abs(dlp(ndof))
        sm(i) =   0.0
 2500   continue
C
      do 4000  ne = 1,numel
        do 2200  i = 1,2
          n =    nodelm(i,ne)
```

```
          re(i) =  r(n)
          eldof(i) =  n
 2200     continue
C
        call    STIFF  (ne, re, kmueln(ne), sme, status)
        if (status .ne. ' ')          then
          call   ERROR ('ASSEMSTF', status)
        end if
C
        do 3600  i = 1,2
          ii   = eldof(i)
          do 3500  j = 1,2
            jj   = eldof(j)
            if (jj .le. ii)    then
              k =   abs(dlp(ii)) - ii + jj
              sm(k) =   sm(k) + sme(i,j)
            end if
 3500       continue
 3600     continue
C
 4000   continue
C
      return
      end
C=END FORTRAN
C=DECK CURRENT
C=PURPOSE Read current intensity
C=BLOCK FORTRAN
      subroutine  CURRENT (inten)
      double precision  DCLVAL
      double precision  inten
      call     CLREAD (' Enter current intensity: ', ' ')
      inten =   DCLVAL(1)
      return
      end
C=END FORTRAN
C=DECK DIMENSIONS
C=PURPOSE Read problem dimensions (wire and truncation radius)
C=BLOCK FORTRAN
      subroutine  DIMENSIONS (wrad, trad)
      double precision  DCLVAL
      double precision  wrad, trad
      call     CLREAD (' Enter wire radius, trunc radius: ', ' ')
      wrad =   DCLVAL(1)
      trad =   DCLVAL(2)
      return
      end
C=END FORTRAN
C=DECK ERROR
C=PURPOSE Fatal error termination subroutine
C=BLOCK FORTRAN
      subroutine ERROR (name, message)
      character*(*) name, message
```

```
      integer   i, l
C
      l =    len(message)
      do 1200  i = len(message),1,-1
        if (message(i:i) .ne. ' ') go to 1300
        l =   i
 1200   continue
 1300 continue
      print *, ' '
      print *, '*** Fatal error condition detected ***'
      print *, message(1:l)
      print *, 'Error detected by ', name
      stop  '***  Error stop ***'
      end
C=END FORTRAN
C=DECK FORCE
C=PURPOSE Compute node forces for axisymm EM element due to j
C=BLOCK FORTRAN
      subroutine  FORCE
     $          (ne, re, inten, wrad, fe, status)
      integer     ne
      double precision  re(2), inten, wrad, fe(2)
      character*(*)  status
      double precision  ri, rj, rm, fn
C
      status =  ' '
      ri =      re(1)
      rj =      re(2)
      if (rj .le. ri)      then
        write (status,'(A,I5)')
     $     'FORCE: Negative or zero length, element',ne
        return
      end if
      rm =      0.5*(ri+rj)
      if (rm .lt. wrad)        then
        fn =        (inten/(3.14159*wrad**2))*(rj-ri)
        fe(1) =   fn*(ri+ri+rj)/6.
        fe(2) =   fn*(ri+rj+rj)/6.
      else
        fe(1) =   0.0
        fe(2) =   0.0
      end if
      return
      end
C=END FORTRAN
C=DECK FORMDLP
C=PURPOSE Form diagonal location pointer (DLP) array
C=BLOCK FORTRAN
      subroutine  FORMDLP
     $          (numel, nodelm, ndof, bctag, dlp)
      integer     numel, nodelm(2,numel), ndof
      integer     bctag(ndof), dlp(0:*)
      integer     i, j, k, n, ne, eldof(2), nsky
```

```
C
      do 1200  i = 0,ndof
        dlp(i) =  0
 1200   continue
      do 2000  ne = 1,numel
        do 1600  i = 1,2
          n =   nodelm(i,ne)
          eldof(i) =  n
 1600     continue
        do 1800  i = 1,2
          k =     eldof(i)
          do 1800  j = 1,2
            if (eldof(j) .le. k)  then
            dlp(k) =  max(dlp(k),k-eldof(j)+1)
            end if
 1800       continue
 2000     continue
      do 2200  i = 1,ndof
        dlp(i) =  dlp(i-1) + dlp(i)
 2200   continue
      nsky =     abs(dlp(ndof))
C
      print '(/'' No of equations:          '',I10)',ndof
      print '('' Average bandwidth:     '',F12.1)',float(nsky)/ndof
      print '('' Entries to store skyline:'',I10)',nsky
      print '('' '')'
C
      do 3000  i = 1,ndof
        if (bctag(i) .ne. 0)    dlp(i) = -abs(dlp(i))
 3000   continue
      return
      end
C=END FORTRAN
C=DECK GENBCTAG
C=PURPOSE Generate potential BC data by fixing externmost node
C=BLOCK FORTRAN
      subroutine  GENBCTAG  (numnp, bctag)
      integer     numnp, bctag(*)
      integer     n
      do 2000  n = 1,numnp
        bctag(n) = 0
 2000   continue
      bctag(numnp) =  1
      return
      end
C=END FORTRAN
C=DECK GENELEMS
C=PURPOSE Generate element data
C=BLOCK FORTRAN
      subroutine  GENELEMS
     $            (nelwir, nelext, kmu, keps, nodelm, kmuelm, kepselm)
      integer     nelwir, nelext
      integer     nx, n, ne, nodelm(2,*)
```

```
      double precision     kmu, keps, kmuelm(*), kepselm(*)
      n =     0
      ne =    0
      do 2000  nx = 1,nelwir
        n =    n + 1
        ne =   ne + 1
        nodelm(1,ne) =  n
        nodelm(2,ne) =  n+1
        kmuelm(ne) =    kmu
        kepselm(ne) =   keps
 2000   continue
      do 3000  nx = 1,nelext
        n =    n + 1
        ne =   ne + 1
        nodelm(1,ne) =  n
        nodelm(2,ne) =  n+1
        kmuelm(ne) =    1.0
        kepselm(ne) =   1.0
 3000   continue
      return
      end
C=END FORTRAN
C=DECK GENEXACT
C=PURPOSE Generate exact magnetic potential/field solutions
C=BLOCK FORTRAN
      subroutine  GENEXACT (numel, nummp, r, wrad, trad,
     $                     kmu, inten, aex, bex)
      integer     numel, nummp
      double precision  r(*), wrad, trad, kmu, inten
      double precision  aex(nummp), bex(numel)
      integer   n, ne
      double precision  c, rm
C
      c    = -(inten/(2*3.14159))*log(wrad/trad)
      do 2000  n = 1,nummp
        if (r(n) .lt. wrad)    then
          aex(n) = (kmu*inten/(4*3.141596))*(1.-(r(n)/wrad)**2) + c
        else
          aex(n) = -(inten/(2*3.14159))*log(r(n)/trad)
        end if
 2000   continue
      do 3000  ne = 1,numel
        rm =   0.5*(r(ne)+r(ne+1))
        if (rm .le. wrad)      then
          bex(ne) =  (kmu*inten/(2*3.14159))*(rm/wrad**2)
        else
          bex(ne) =  (inten/(2*3.14159))/rm
        end if
 3000   continue
      return
      end
C=END FORTRAN
C=DECK GENNODES
```

```
C=PURPOSE Generate node data
C=BLOCK FORTRAN
      subroutine  GENNODES (nelwir, nelext, wrad, trad, r)
      integer     nelwir, nelext
      integer     n, ne
      double precision  wrad, trad, r(*)
C
      n =     0
      do 2000  ne = 1,nelwir
        n =   n + 1
        r(n) =   (ne-1)*wrad/nelwir
 2000   continue
      r(n+1) =   wrad
      do 3000  ne = 1,nelext
        n =   n + 1
        r(n) =   wrad + (ne-1)*(trad-wrad)/nelext
 3000   continue
      r(n+1) =   trad
      return
      end
C=END FORTRAN
C=DECK MAGFIELD
C=PURPOSE Computed magnetic field (B) at element center
C=BLOCK FORTRAN
      subroutine  MAGFIELD (numel, nodelm, r, a, b)
      integer     numel, nodelm(2,numel)
      double precision    r(*), a(*), b(numel)
      integer     ne, ni, nj
C
      do 2000  ne = 1,numel
        ni =   nodelm(1,ne)
        nj =   nodelm(2,ne)
        b(ne) =   -(a(nj)-a(ni))/(r(nj)-r(ni))
 2000   continue
      return
      end
C=END FORTRAN
C=DECK MATERIAL
C=PURPOSE Read material properties
C=BLOCK FORTRAN
      subroutine   MATERIAL (kmu, keps)
      double precision   kmu, keps
      double precision   DCLVAL
      call    CLREAD (' Enter kmu, keps for wire: ', ' ')
      kmu =   DCLVAL(1)
      keps =  DCLVAL(2)
      return
      end
C=END FORTRAN
C=DECK PRINTCUR
C=PURPOSE Print current intensity
C=BLOCK FORTRAN
      subroutine  PRINTCUR (inten)
```

```
      double precision  inten
      print '('' Current intensity:'',F10.3)',inten
      return
      end
C=END FORTRAN
C=DECK PRINTDIM
C=PURPOSE Print problem dimensions (wire and truncation radius)
C=BLOCK FORTRAN
      subroutine  PRINTDIM (wrad, trad)
      double precision  wrad, trad
      print '('' Wire       radius:'',F10.3)',wrad
      print '('' Truncation radius:'',F10.3)',trad
      return
      end
C=END FORTRAN
C=DECK PRINTELM
C=PURPOSE Print element data
C=BLOCK FORTRAN
      subroutine  PRINTELM (numel, nodelm, kmuelm, kepselm)
      integer     i, n, numel, nodelm(2,*)
      double precision     kmuelm(*), kepselm(*)
      print *, ' ------------------------'
      print *, ' E l e m e n t    D a t a'
      print *, ' ------------------------'
      print *,
     $  ' Elem    I    J     kmu     keps'
      do 2000  n = 1,numel
        print '(3I5,2F9.3)', n,(nodelm(i,n),i=1,2),kmuelm(n),kepselm(n)
 2000   continue
      return
      end
C=END FORTRAN
C=DECK PRINTMAG
C=PURPOSE Print computed and exact magnetic field (B)
C=BLOCK FORTRAN
      subroutine  PRINTMAG (numel, nodelm, r, b, bex)
      integer     numel, nodelm(2,numel)
      double precision   r(*),b(numel),bex(numel)
      integer     ne, ni, nj
C
      print *, ' --------------------------'
      print *, ' M a g n e t i c   F i e l d'
      print *, ' --------------------------'
      print *,
     $  ' Elem   r-center   Comp-B2   Exact-B2'
      do 2000  ne = 1,numel
        ni =   nodelm(1,ne)
        nj =   nodelm(2,ne)
        print '(I5,F10.3,2F11.4)',ne,0.5*(r(ni)+r(nj)),
     $          b(ne),bex(ne)
 2000   continue
      return
      end
```

```
C=END FORTRAN
C=DECK PRINTMAT
C=PURPOSE Print material properties used in problem
C=BLOCK FORTRAN
      subroutine  PRINTMAT (kmu, keps)
      double precision  kmu, keps
      print '('' Rel. permeability of wire (vacuum=1):'',F10.3)',kmu
      print '('' Rel. permittivity of wire (vacuum=1):'',F10.3)',keps
      return
      end
C=END FORTRAN
C=DECK PRINTNOD
C=PURPOSE Print element data
C=BLOCK FORTRAN
      subroutine  PRINTNOD (nummp, r, bctag)
      integer     n, nummp, bctag(*)
      double precision      r(*)
      print *, ' ------------------------'
      print *, '  N o d e         D a t a'
      print *, ' ------------------------'
      print *, '  Node    r-coord   bctag'
      do 2000  n = 1,nummp
        print '(I6,F10.3,I6)', n,r(n),bctag(n)
 2000   continue
      return
      end
C=END FORTRAN
C=DECK PRINTSOL
C=PURPOSE Print computed and exact solution
C=BLOCK FORTRAN
      subroutine  PRINTSOL (nummp, r, bctag, f, a, aex, fex)
      integer     n, nummp, bctag(nummp)
      double precision    r(nummp), f(nummp)
      double precision    a(nummp), aex(nummp), fex(nummp)
      print *, ' ----------------------------------'
      print *, ' C o m p u t e d    S o l u t i o n'
      print *, ' ----------------------------------'
      print *,
     $  '  Node       r   bctag   Comp-for',
     $  '     Comp-A3   Exact-A3  Exact-for'
      do 2000  n = 1,nummp
        print '(I6,F10.3,I6,4F11.4)',n, r(n),bctag(n),
     $          f(n),a(n),aex(n),fex(n)
 2000   continue
      return
      end
C=END FORTRAN
C=DECK PRINTSUB
C=PURPOSE Print subdivision data
C=BLOCK FORTRAN
      subroutine  PRINTSUB (nelwir, nelext, numel, nummp, ndof)
      integer   nelwir, nelext, numel, nummp, ndof
      print '('' Subdivisions in wire      :'',I6)', nelwir
```

```
      print '('' Subdivisions in free space:'',I6)', nelext
      print '('' Number of elements        :'',I6)', numel
      print '('' Number of node points     :'',I6)', nummp
      print '('' Number of dofs            :'',I6)', ndof
      return
      end
C=END FORTRAN
C=DECK SKYCOV
C=PURPOSE Cover routine for stiffness solver
C=AUTHOR C. A. Felippa, March 1972
C=VERSION November 1982 (Fortran 77)
C=THISVERSION Condensed on November 86 for ME593 HW
C=EQUIPMENT Machine independent
C=KEYWORDS  solve skyline stiffness equation
C=BLOCK ABSTRACT
C
C     SKYCOV is a cover routine that solves the master
C     stiffness equations
C                           K u = f
C     SKYCOV calls SKYFAC to factor the skyline-stored
C     master stiffness matrix K.  If the factorization is
C     successful SKYCOV then calls SKYSOL to solve for u.
C
C=END ABSTRACT
C=BLOCK USAGE
C
C     The calling sequence is
C
C       CALL SKYCOV (S, N, DLP, F, U, STATUS)
C
C     Input arguments:
C
C       S          Skyline stored stiffness matrix
C                  Overwritten by factorization.
C       N          Number of equations
C       P          Nodal force vector
C       DLP        Skyline diagonal location pointer
C
C     Output arguments:
C
C       U          Computed displacements if no error detected.
C       STATUS     Status character variable.
C                   blank     no error detected
C                   nonblank  explanatory error message
C
C=END USAGE
C=BLOCK FORTRAN
      subroutine   SKYCOV
     $             (s, n, dlp, f, u, status)
C
C                   A R G U M E N T S
C
      integer           n, dlp(0:*)
```

```
      double precision   s(*), f(*), u(*)
      character*(*) status
C
C                   T Y P E   &   D I M E N S I O N
C
      integer        idetex, negeig, ifail, NMAX
      parameter     (NMAX=3000)
      double precision   aux(NMAX), detcf, delta, DOTPRD
      external        DOTPRD
C
C                   L O G I C
C
      status =  ' '
      if (n .gt. NMAX)        then
        write (status,'(A,I6)')'No. of equations exceeds ',NMAX
        return
      end if
      call    SKYFAC
     $ (s, 0, n, n, dlp, aux, DOTPRD, .true., .false.,
     $  0, 0, 0.0, detcf, idetex, negeig,  ifail)
      if (ifail .gt. 0)       then
        write (status,'(A,I6,A)')
     $ 'Factorization aborted at equation ',ifail,
     $ ' (matrix appears singular)'
        return
      end if
C
      call    SKYSOL
     $  (s, n, dlp, DOTPRD, 0, 1, f, u, 0, 0, aux, delta)
C
      return
      end
C=END FORTRAN
C=DECK STIFF
C=PURPOSE Construct stiffness matrix of axisymmetric EM element
C=BLOCK FORTRAN
      subroutine STIFF (ne, re, mu, s, status)
      integer        ne
      double precision  re(2), mu, s(2,2)
      character*(*)  status
      double precision  ri, rj, rl, rm
      status =  ' '
      ri =      re(1)
      rj =      re(2)
      rl =      rj - ri
      if (rl .le. 0.0)      then
        write (status,'(A,I5)')
     $        'STIFF: Negative or zero length, element',ne
        return
      end if
      rm =     0.5*(ri+rj)
      s(1,1)=   rm/(rl*mu)
      s(2,2)=   s(1,1)
```

```
      s(1,2)=  -s(1,1)
      s(2,1)=   s(1,2)
      return
      end
C=END FORTRAN
C=DECK SUBDIVIDE
C=PURPOSE Read subdivision data
C=BLOCK FORTRAN
      subroutine  SUBDIVIDE (nelwir, nelext, numel, nummp, ndof)
      integer   nelwir, nelext, numel, nummp, ndof
      integer   ICLVAL
      call     CLREAD (' Subdivisions in wire: ', ' ')
      nelwir = ICLVAL(1)
      call     CLREAD (' Subdivisions in free space: ', ' ')
      nelext = ICLVAL(1)
      numel =  nelwir + nelext
      nummp =  numel + 1
      ndof =   nummp
      return
      end
C=END FORTRAN
```

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We derive electromagnetic finite elements based on a variational principle that uses the electromagnetic four-potential as primary variable. This choice is used to construct elements suitable for downstream coupling with mechanical and thermal finite elements for the analysis of electromagnetic/mechanical systems that involve superconductors. The key advantages of the four-potential are: the number of degrees of freedom per node remain modest as the problem dimensionality increases, jump discontinuities on interfaces are naturally accomodated, and static as well as dynamics are included without any *a priori* approximations. The new elements are tested on an axisymmetric problem under steady-state forcing conditions. The results are in excellent agreement with analytical solutions.

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