1991

NASA/ASEE SUMMER FACULTY FELLOWSHIP PROGRAM

Numerical Methods for the Analysis of Sampled-Data Systems and for the Computation of System Zeros

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Contract No:	NGT-01-008-021 The University of Alabama at Huntsville

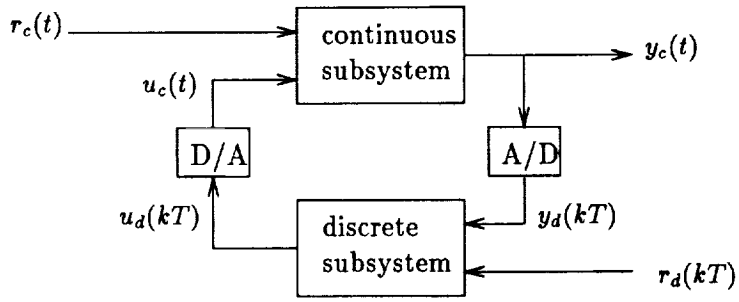

Figure 1: General sampled data system

1 MARSYAS Overview

MARSYAS is a computer-aided control system design package for the simulation and analysis of open loop and closed loop dynamic systems. Prior to the summer of 1991, MARSYAS functios provided simulation of sampled data (mixed discrete-time and continuous-time) systems; however, sampled-data system analysis of stability, frequency response and transfer function coefficients was not implemented. Analysis of purely continuous-time systems was available, but the underlying computational procedures did not employ several recent advances in numerical techniques.

This report outlines the numerical and theoretical basis behind the MARSYAS functions developed during the summer of 1991. In particular, the numerical computation of the matrix exponential $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$ as described in [11] and the numerical computation of the finite system zeros of a dynamic system (continuous-time or discrete-time) as discussed in [4], [6], and [8] are presented. These two numerical functions and their associated numerical tests comprise the bulk of the work done by the author while working in tandem with John Tiller (BCSS) and D. Pat Vallely (NASA-MSFC). The analysis of sampled-data systems is discussed in Section 2; the numerical computation of system zeros is discussed in Section 3.

2 Discretization of LTI Continuous Systems

A general sampled-data system is shown in Figure 1. The system has an external continuous-time input $r_c(t)$ and an external discrete-time input $r_d(t)$, and maps these signals to continuous-time outputs $y_c(t)$ and to discrete-time ouputs $u_c(t)$. Due to the use of A/D and D/A converters, sampled-data systems are nonlinear and time-varying in general. That is, there is no transfer function from $r_c(t)$ to $y_c(t)$, even if the constituent subsystems are linear and time-invariant (LTI)! However if the subsystems are LTI, the behavior of the A/D and D/A converters ($u_c(t) = u_d(kT)$, $t \in [kT, kT+T)$) implies that one may obtain discrete-time transfer functions $Y_d(z)/R_d(z)$ and $U_d(z)/R_d(z)$. This is accomplished by computing matrices F and G such that a continuous time system $\dot{x} = Ax + Bu$ may be equivalently represented at sampling times $0, T, \ldots, kT, \ldots$ as $x(kT+T) = Fx(kT) + Gu(kT)$. This conversion is obtained via the relations $F = e^{AT}$, $G = \left(\int_{0^T} e^{A(T-t)}dt\right) B$, where the matrix exponential e^{AT} is defined as $e^{AT} \triangleq I + AT + (AT)^2/2! + \cdots$. Van Loan [10] observes that F and G may be simultaneously computed as $\begin{bmatrix} F & G \\ 0 & I \end{bmatrix} = exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right)$; that is, only a single matrix exponential is required.

Computation of the matrix exponential is not a trivial task. (See, for example, [7].) However, Ward [11] has proposed the use of Padé approximations with preconditioning for this purpose. The algorithm is shown in Figure 2. This procedure was tested at MSFC during the summer of 1991 in several numerical experiments; computed eigenvalues of $F = e^A$ were compared with the exponentials of the eigenvalues of

1	Avg eig. shift	$A := A - \phi I$	reduce dynamic range of $\lambda's$
2	balance	$A := D^{-1}AD$	reduce non-normality powers of 2, no roundoff
3	scale	$A := A * 2^{-j}$	reduce so $\|\|A\|\| < 1$; no roundoff
4	Padé:	$F := D_3(A)^{-1}N_3(A)$	experiments: D_8, N_8 don't help much
5	Square	$F := F^{(2^j)}$	$F := F^2$ j times, $\log(a^n) = n\log(a)$
6	inverse balance	$F := DFD^{-1}$	no (new) roundoff
7	inverse shift	$F := e^{\phi}F$	$e^{A-bI} = e^A e^{-b}$

Figure 2: Computation of the matrix exponential

A, with good agreement. Similarly, matrices $A = V\Lambda V^{-1}$ were constructed so that computed values of F could be compared with exact (known) solution values, again with good agreement. Experiment models had up to 109 states with wide variation in coefficients. Little difference was found between using 3^{rd} order Padé approximations and eigth order Padé approximations; hence, for the sake of computational speed, MARSYAS presently uses 3^{rd} order approximations.

3 Factored Transfer Functions

Numerical experiments involving discretized sampled-data systems with fast modes revealed that existing MARSYAS software was not adequate for the computation of associated system zeros. Prior to the summer of 1991, MARSYAS computed the zeros of $H(s) = C(sI - A)^{-1}B + D$ as

$$z_i = \lambda_i^{-1}\left(\begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right).$$

Unfortunately, discretized systems obtained from continuous time systems with fast modes (relative to the sampling interval) are nearly uncontrollable; that is, the matrix $\begin{bmatrix} -A & -B \\ C & D \end{bmatrix}$ is nearly singular.

EISPACK routine **balanc** [9] and Ward's GEP balancing procedure [12] attempt to *reduce* the dynamic range of the coefficients in the algebraic eigenvalue problem (AEP) and the generalized eigenvalue problem (GEP), respectively, *without* introducing any roundoff error. Both of these procedures may be divided into two steps: (1) permutation and (2) scaling.

For example, the permutation step **gep_perm** of Ward's balancing procedure computes permutation matrices P_1 and P_2 such that the transformed matrix pencil

$$(\hat{M} - \lambda\hat{N}) = P_1(M - \lambda N)P_2$$

may be conformably partitioned as $\hat{M} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} & \hat{M}_{13} \\ 0 & \hat{M}_{22} & \hat{M}_{23} \\ 0 & 0 & \hat{M}_{33} \end{bmatrix}$ and $\hat{N} = \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} & \hat{N}_{13} \\ 0 & \hat{N}_{22} & \hat{N}_{23} \\ 0 & 0 & \hat{N}_{33} \end{bmatrix}$ where $\hat{M}_{11}$, $\hat{N}_{11}$, $\hat{M}_{33}$ and $\hat{N}_{33}$ are upper triangular and $\hat{M}_{22}$ and $\hat{N}_{22}$ form a reduced order permutation-irreducible GEP. Following **gep_perm**, the scaling step **gep_scale** of Ward's balancing procedure computes diagonal matrices $D_i = \mathrm{diag}(r^{k_1^{(i)}}, \ldots, r^{k_n^{(i)}})$, $i = 1, 2$, where r is the machine radix. D_1 and D_2 are selected so that the elements of the matrix products $D_1\hat{M}D_2$ and $D_1\hat{N}D_2$ are of approximately the same magnitude. (The use of powers of the machine radix r allows these matrix products to be computed without roundoff.)

The AEP balancing procedure **balanc** may be similarly divided into a permutation step **aep_perm** and scaling step **aep_scale**. However, **balanc** requires that $P_1 = P_2^{-1} = P_2^T$ and $D_2 = D_1^{-1} = \mathrm{diag}(r^{-k_1^{(1)}}, \ldots, r^{-k_n^{(1)}})$. The use of similarity transforms P_1 and D_1 preserves the $N = I$ structure of the AEP.

Algorithm	Preconditioning	Computed Zeros
ENVD	none	-1904, $(-1500 \pm j8.744 \cdot 10^4)$, −336.2,0,0
EISPACK	none	$-5.468 \cdot 10^5$, $(2.688 \pm j4.720) \cdot 10^5$, $(9.913 \pm j9.359) \cdot 10^7$, -1500, -1500
ENVD	`zgep_scale`	-1500,-1500
EISPACK	`gep_perm`, `gep_scale`	$1500 \pm j2.7847 \cdot 10^{-5}$

Figure 3: Numerical results for Example 3.1.

Since Ward has developed an effective preconditioner for the generalized eigenvalue problem [12], it was decided to modify MARSYAS to use this balancing procedure in tandem with the QZ iteration [8] as implemented in EISPACK. This procedure resulted in great improvement in the accuracy of MARSYAS computed results. Ward's balancing procedure may be applied without roundoff error and will usually provide improvement in the computed results. However, the QZ iteration fails to isolate zeros at infinity, and so these zeros may be perturbed in the Riemann sphere to large (but finite) zeros of arbitrary phase.

Emami-Naeini and Van Dooren [4] propose a procedure for the solution of the zero-computation generalized eigenvalue problem that reduces the original problem to one of lower degree that has the same finite zeros as the original system but with no zeros at infinity. This code has been tested on several industrial system models with great success. However, implementation of this algorithm (as available from `netlib` at Oak Ridge National Labs) proved disastrous when applied to a model of the oxidzer-preburner valve dynamics of the SSME. This failure was due to widely varying magnitudes in the coefficients in the system model. As a part of the 1991 summer faculty program, a new balancing procedure for the zero-computation generalized eigenvalue problem

$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} x = \lambda \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x$$

was developed as a preconditioner to the Emami-Naeini/Van Dooren algorithm; see [6] for details. Use of this procedure in tandem with modified procedures from [12] and EISPACK provides improved numerical robustness as shown in the following example.

Example 3.1 Consider the following seventh order single-input, single-output dynamic system based upon the space shuttle main engine oxidizer-preburner valve dynamics. The system coefficients are $a_{11} = -7000$, $a_{12} = -2.5 \times 10^7$, $a_{17} = -1.82943 \times 10^9$, $a_{21} = 1$, $a_{32} = 6.4 \times 10^5$, $a_{33} = -2,240$, $a_{34} = -a_{32}$, $a_{43} = 1$, $a_{54} = 9.03934$, $a_{65} = 225,449$, $a_{66} = -3,000$,$a_{67} = -2.25 \times 10^6$, $a_{76} = 1$, $b_{11} = 1.44813 \times 10^8$, $c_{15} = 1.26582$, and all other matrix entries are zero. The EISPACK implementation of the QZ algorithm and the Emami-Naeini/Van Dooren (ENVD) algorithm were applied to this system with either no preconditioning, Ward's balancing procedure `gep_perm`, `gep_scale`, or the zero-computation generalized eigenvalue balancing procedure `zgep_scale`. A summary of the numerical results is in Figure 3. (Ward's balancing procedure [12] correctly isolates six generalized eigenvalues at infinity; the correct finite system zeros are $s = 1500 \pm j2.7847 \times 10^{-5}$. The `zgep_scale` scaled system coefficients are $a_{11} = -7000$, $a_{12} = -762.9$, $a_{17} = -1744.7$, $a_{21} = 32,768$, $a_{32} = 78.125$, $a_{33} = -2240$, $a_{34} = -156.25$, $a_{43} = 4096$, $a_{54} = 72.31472$, $a_{65} = 27.52$, $a_{66} = -3000$, $a_{67} = -34.33$, $a_{76} = 65,536$, $b_{11} = 1.0789$ and $c_{15} = 0.6329$.

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