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Recursive Dynamics for Flexible Multibody Systems Using Spatial Operators

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Abstract

Due to their structural flexibility, spacecraft and space manipulators are multibody systems with complex dynamics and possess a large number of degrees of freedom. This publication uses the spatial operator algebra methodology to develop a new dynamics formulation and spatially recursive algorithms for such flexible multibody systems. A key feature of the formulation is that the operator description of the flexible system dynamics is identical in form to the corresponding operator description of the dynamics of rigid multibody systems. A significant advantage of this unifying approach is that it allows ideas and techniques for rigid multibody systems to be easily applied to flexible multibody systems. The algorithms use standard finite-element and assumed modes models for the individual body deformation.

A Newton-Euler Operator Factorization of the mass matrix of the multibody system is first developed. It forms the basis for recursive algorithms such as for the inverse dynamics, the computation of the mass matrix, and the composite body forward dynamics for the system. Subsequently, an alternative Innovations Operator Factorization of the mass matrix, each of whose factors is invertible, is developed. It leads to an operator expression for the inverse of the mass matrix, and forms the basis for the recursive articulated body forward dynamics algorithm for the flexible multibody system. For simplicity, most of the development in this publication focuses on serial chain multibody systems. However, extensions of the algorithms to general topology flexible multibody systems are described.

While the computational cost of the algorithms depends on factors such as the topology and the amount of flexibility in the multibody system, in general, it appears that in contrast with the rigid multibody case, the articulated body forward dynamics algorithm is the more efficient algorithm for flexible multibody systems containing even a small number of flexible bodies. The variety of algorithms described here permits a user to choose the algorithm which is optimal for the multibody system at hand. The availability of a number of algorithms is even more important for real-time applications, where implementation on parallel processors or custom computing hardware is often necessary to maximize speed.

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1. Introduction

Due to their structural flexibility, spacecraft and space manipulators are multibody systems with complex dynamics and possessing a large number of degrees of freedom (dofs). This publication describes a spatial operator formulation for the analysis and development of efficient dynamics algorithms for such flexible multibody systems. This approach represents a reformulation and extension of the *spatial operator algebra* methodology [1] for the dynamics of flexible multibody systems [2].

The formulation developed in this publication uses spatial operators and closely parallels the corresponding formulation for rigid multibody systems. In effect it represents a unified methodology for the analysis and development of efficient dynamics algorithms for general topology, rigid/flexible multibody systems. Indeed, the analysis and recursive algorithms developed here for the inverse dynamics, the computation of the mass matrix, and the forward dynamics of the flexible multibody system closely resemble the corresponding analysis and algorithms for the rigid multibody system.

It is assumed that all deformations of the bodies are small and a linear model of elasticity is used. However, large articulation at the hinges is allowed. No special assumptions are made regarding the nature of the component bodies, and the algorithms use standard finite-element and assumed modes models for the body flexibility. For notational simplicity, and without any loss in generality, the main focus of this publication is on flexible multibody systems with serial chain topology. The extensions required for systems with tree and closed-chain topology are discussed at the end of the publication.

Section 2 summarizes the symbols and notation used in this publication. Section 3 contains the development of the equations of motion for the multibody system. The recursive relationships between the modal spatial velocities, modal spatial accelerations, and modal spatial forces are described. Using these, spatial operators are used to develop the Newton-Euler Operator Factorization of the system mass matrix.

Section 4 describes a recursive Newton-Euler inverse dynamics algorithm for the flexible multibody system. This algorithm computes the vector of generalized forces corresponding to a given state and the vector of generalized accelerations for the multibody system. In Section 5, the Newton-Euler Operator Factorization of the mass matrix is used to develop the recursive composite body forward dynamics algorithm for the system. A part of this algorithm consists of an algorithm for the recursive computation of the multibody system mass matrix.

Section 6 describes operator factorization and inversion results that form the basis for the recursive articulated body forward dynamics algorithm. First, a recursive algorithm for the computation of certain articulated body quantities is defined. These quantities are used to develop a new operator factorization, denoted the Innovations Operator Factorization of

the system mass matrix. In contrast with the Newton-Euler Operator Factorization, the factors in the Innovations Operator Factorization are square and invertible operators. This fact is used to develop an operator expression for the inverse of the mass matrix. A more efficient algorithm for computing the articulated body quantities is described at the end of the section. Using the operator expression for the mass matrix inverse, Section 7 describes the recursive articulated body forward dynamics algorithm for the multibody system. This algorithm requires neither the explicit formation of the system mass matrix nor its inversion.

Section 8 discusses the computational complexity of the various algorithms described in this publication. It is shown that the articulated body forward dynamics algorithm is more efficient than the composite body forward dynamics algorithm, even for systems with a modest number of bodies and flexible modes. Section 9 describes the extensions of the dynamics formulation and algorithms developed here to flexible multibody systems with tree and closed-chain topologies.

2. Notation

Coordinate free spatial notation [1, 3] is used throughout this publication. Briefly a *spatial velocity* of a frame is a 6-dimensional quantity, whose upper 3 elements are the angular velocity while the lower 3 elements are the linear velocity of the frame. A *spatial force* is a 6-dimensional quantity, whose upper 3 elements correspond to a moment vector while the lower 3, to a force vector.

In the stacked notation used in this publication, a variety of indices are used to identify different spatial quantities. Thus for instance, $V_s(j_k)$ denotes the spatial velocity of the j^{th} node on the k^{th} body, $V_s(k) = \text{col}\{V_s(j_k)\}$ denotes the vector of the spatial velocities of all the nodes on the k^{th} body, while $V_s = \text{col}\{V_s(k)\}$ denotes the vector of the spatial velocities of all the nodes for all the bodies in the serial chain. The index k will be used to refer to both the k^{th} body as well as the k^{th} body reference frame \mathcal{F}_k , with the usage being apparent from the context. Some key quantities used in this publication are defined below (also see Figure 1).

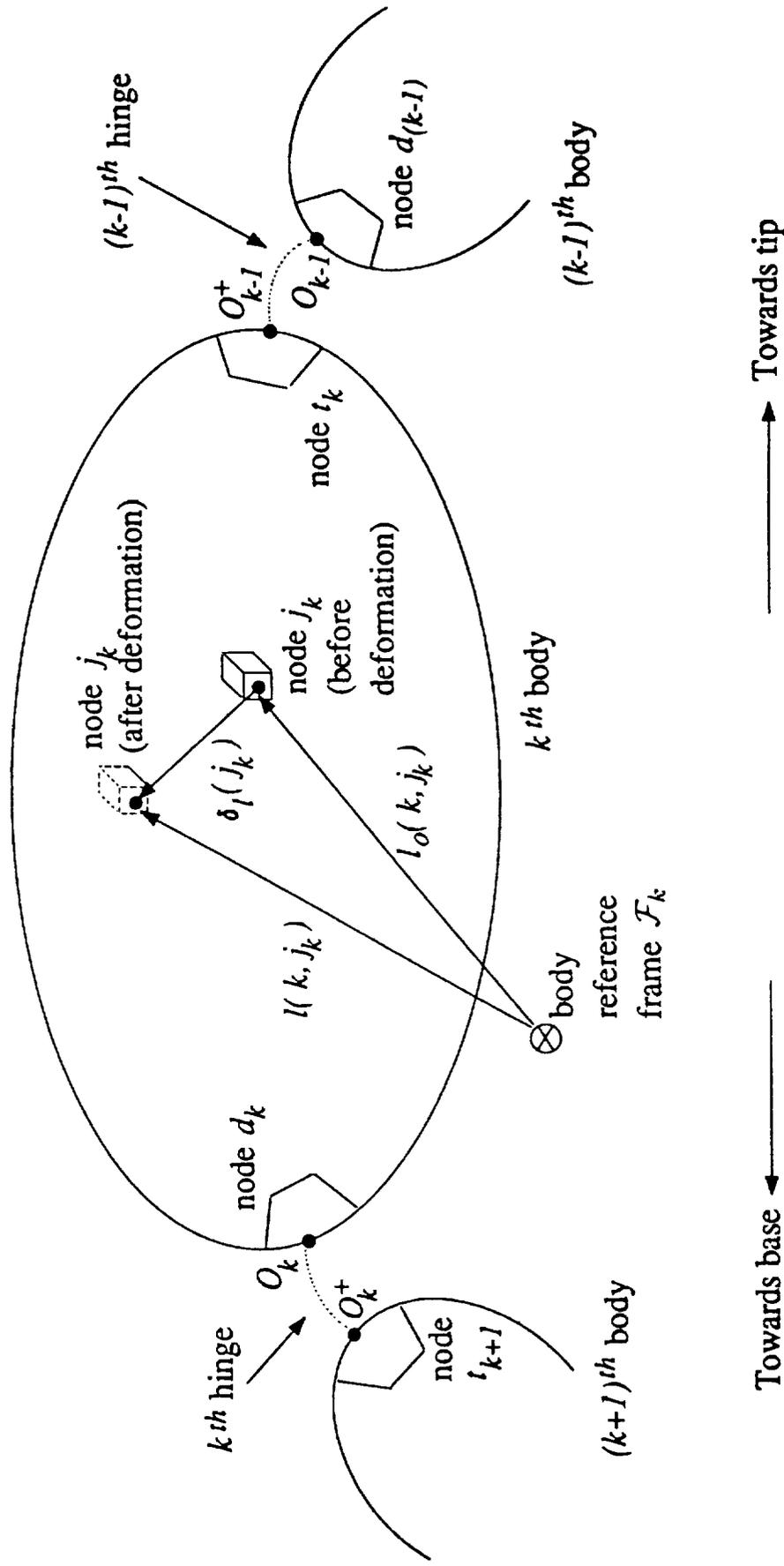


Figure 1: Illustration of links and hinges in a flexible serial multibody system

General Quantities:

- \tilde{z} = $[z]^\times \in \mathcal{R}^{3 \times 3}$ and denotes the skew-symmetric cross-product matrix associated with the 3-dimensional vector z
- \dot{x} = $\frac{dx}{dt}$ and denotes the time derivative of x with respect to an inertial frame
- $\dot{\tilde{x}}$ denotes the time derivative x with respect to the body-fixed (rotating) frame
- $\text{diag}\{x(k)\}$ denotes a block diagonal matrix whose k^{th} diagonal element is $x(k)$
- $\text{col}\{x(k)\}$ denotes a column vector whose k^{th} element is $x(k)$
- $l(x, y) \in \mathcal{R}^3$, the vector from point x to point y
- $\phi(x, y) = \begin{pmatrix} I & \tilde{l}(x, y) \\ 0 & I \end{pmatrix} \in \mathcal{R}^{6 \times 6}$, the spatial transformation operator which transforms spatial velocities and forces between points x and y

Individual Body Nodal Data:

- $n_s(k)$ number of nodes on the k^{th} body
- \mathcal{F}_k the body reference frame with respect to which the deformation field for the k^{th} body is measured. The motion of this frame characterizes the motion of the k^{th} body as a rigid body.
- j_k denotes the j^{th} node on the k^{th} body
- $l_0(k, j_k) \in \mathcal{R}^3$, the vector from \mathcal{F}_k to the location (before deformation) of the j^{th} node reference frame on the k^{th} body
- $\delta_l(j_k) \in \mathcal{R}^3$, the translational deformation of the j^{th} node on the k^{th} body
- $l(k, j_k) = l_0(k, j_k) + \delta_l(j_k) \in \mathcal{R}^3$, the vector from \mathcal{F}_k to the location (after deformation) of the j^{th} node reference frame on the k^{th} body
- $\delta_\omega(j_k) \in \mathcal{R}^3$, the deformation angular velocity of the j^{th} node on the k^{th} body with respect to the body frame \mathcal{F}_k
- $\delta_v(j_k) \in \mathcal{R}^3$, the deformation linear velocity of the j^{th} node on the k^{th} body with respect to the body frame \mathcal{F}_k
- $u(j_k) \in \mathcal{R}^6$, the spatial displacement of node j_k . The translational component of $u(j_k)$ is $\delta_l(j_k)$, while its time derivative with respect to the body frame \mathcal{F}_k is
- $$\dot{u}(j_k) = \begin{pmatrix} \delta_\omega(j_k) \\ \delta_v(j_k) \end{pmatrix}$$

- $\mathcal{J}(j_k) \in \mathcal{R}^{3 \times 3}$, the inertia tensor about the nodal reference frame for the j^{th} node on the k^{th} body
- $p(j_k) \in \mathcal{R}^3$, the vector from the nodal reference frame to the node center of mass for the j^{th} node on the k^{th} body
- $m(j_k)$ the mass of the j^{th} node on the k^{th} body
- $M_s(j_k) = \begin{pmatrix} \mathcal{J}(j_k) & m(j_k)\tilde{p}(j_k) \\ -m(j_k)\tilde{p}(j_k) & m(j_k)I \end{pmatrix} \in \mathcal{R}^{6 \times 6}$, the spatial inertia about the nodal reference frame for the j^{th} node on the k^{th} body
- $M_s(k) \text{diag}\{M_s(j_k)\} \in \mathcal{R}^{6n_s(k) \times 6n_s(k)}$, the structural mass matrix for the k^{th} body
- $K_s(k) \in \mathcal{R}^{6n_s(k) \times 6n_s(k)}$, the structural stiffness matrix for the k^{th} body

Individual Body Modal Data:

- $n_m(k)$ the number of assumed modes for the k^{th} body
- $\bar{N}(k) = n_m(k) + 6$, the number of deformation plus rigid-body dofs for the k^{th} body
- $\eta(k) \in \mathcal{R}^{n_m(k)}$, the vector of modal deformation variables for the k^{th} body
- $\lambda_r^j(k) \in \mathcal{R}^3$, the modal slope (or differential change in orientation) displacement vector for the r^{th} mode at the j_k^{th} nodal reference frame.
- $\lambda^j(k) = [\lambda_1^j(k), \dots, \lambda_{n_m(k)}^j(k)] \in \mathcal{R}^{3 \times n_m(k)}$, the modal slope displacement influence vector for the j^{th} node due to all the modes for the k^{th} body. Note that $\delta_\omega(j_k) = \lambda^j(k)\dot{\eta}(k)$.
- $\gamma_r^j(k) \in \mathcal{R}^3$, the modal translational displacement vector for the r^{th} mode at the j_k^{th} nodal reference frame
- $\gamma^j(k) = [\gamma_1^j(k), \dots, \gamma_{n_m(k)}^j(k)] \in \mathcal{R}^{3 \times n_m(k)}$, the modal translational displacement influence vector for the j^{th} node due to all the modes for the k^{th} body. Note that $\delta_l(j_k) = \gamma^j(k)\eta(k)$ and $\delta_v(j_k) = \gamma^j(k)\dot{\eta}(k)$.
- $\Pi_r^j(k) = \begin{pmatrix} \lambda_r^j(k) \\ \gamma_r^j(k) \end{pmatrix} \in \mathcal{R}^6$, the modal spatial displacement vector for the r^{th} mode at the j_k^{th} nodal reference frame
- $\Pi^j(k) = [\Pi_1^j(k), \dots, \Pi_{n_m(k)}^j(k)] \in \mathcal{R}^{6 \times n_m(k)}$, the modal spatial influence vector for the j_k^{th} node. The spatial deformation of node j_k is given by $u(j_k) = \Pi^j(k)\eta(k)$.

$\Pi(k) = \text{col}\{\Pi^j(k)\} \in \mathcal{R}^{6n_s(k) \times n_m(k)}$, the modal matrix for the k^{th} body. The r^{th} column of $\Pi(k)$ is denoted $\Pi_r(k) \in \mathcal{R}^{6n_s(k)}$ and is the mode shape function for the r^{th} assumed mode for the k^{th} body. The deformation field for the k^{th} body is given by $u(k) = \Pi(k)\eta(k)$, while $\dot{u}(k) = \Pi(k)\dot{\eta}(k)$.

$M_m(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times \overline{\mathcal{N}}(k)}$, the modal mass matrix for the k^{th} body.

$K_m(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times \overline{\mathcal{N}}(k)}$, the modal stiffness matrix for the k^{th} body.

Multibody Data:

N number of bodies in the serial flexible multibody system

$\overline{\mathcal{N}} = \sum_{k=1}^N \overline{\mathcal{N}}(k)$, the overall dofs in the serial chain obtained by disregarding the hinge constraints

$n_r(k)$ the number of dofs for the k^{th} hinge

$\mathcal{N}(k) = n_m(k) + n_r(k)$, the number of deformation plus hinge dofs for the k^{th} body

$\mathcal{N} = \sum_{k=1}^N \mathcal{N}(k)$, the overall deformation plus hinge dofs for the serial chain

d_k denotes the node on the k^{th} body to which the k^{th} hinge is attached

t_k denotes the node on the k^{th} body to which the $(k-1)^{\text{th}}$ hinge is attached

\mathcal{O}_k the reference frame for the k^{th} hinge on the k^{th} body. This frame is fixed to node d_k .

\mathcal{O}_k^+ the reference frame for the k^{th} hinge on the $(k+1)^{\text{th}}$ body. This frame is fixed to node t_{k+1} .

$\theta(k) \in \mathcal{R}^{n_r(k)}$, the vector of configuration variables for the k^{th} hinge

$\beta(k) \in \mathcal{R}^{n_r(k)}$, the vector of generalized speeds for the k^{th} hinge

$\Delta_V(k) = \begin{pmatrix} \Delta_\omega(k) \\ \Delta_v(k) \end{pmatrix} \in \mathcal{R}^6$, the relative spatial velocity for the k^{th} hinge defined as the spatial velocity of frame \mathcal{O}_k with respect to frame \mathcal{O}_k^+

$H^*(k) \in \mathcal{R}^{6 \times n_r(k)}$, the joint map matrix for the k^{th} hinge such that $\Delta_V(k) = H^*(k)\beta(k)$.

$\vartheta(k) = \begin{pmatrix} \eta(k) \\ \theta(k) \end{pmatrix} \in \mathcal{R}^{\mathcal{N}(k)}$, the vector of (deformation plus hinge) generalized configuration variables for the k^{th} body

$\chi(k) = \begin{pmatrix} \dot{\eta}(k) \\ \beta(k) \end{pmatrix} \in \mathcal{R}^{\mathcal{N}(k)}$, the vector of (deformation plus hinge) generalized motion variables (or generalized speeds) for the k^{th} body

$V(k) = V(\mathcal{F}_k) = \begin{pmatrix} \omega(k) \\ v(k) \end{pmatrix} \in \mathcal{R}^6$, the spatial velocity of the k^{th} body reference frame \mathcal{F}_k , with $\omega(k)$ and $v(k)$ denoting the angular and linear velocities respectively of frame \mathcal{F}_k

$V(\mathcal{O}_k) \in \mathcal{R}^6$, the spatial velocity of frame \mathcal{O}_k

$V(\mathcal{O}_k^+) \in \mathcal{R}^6$, the spatial velocity of frame \mathcal{O}_k^+

$V_s(j_k) \in \mathcal{R}^6$, the spatial velocity of the j^{th} node on the k^{th} body.

$\alpha_s(j_k) \in \mathcal{R}^6$, the spatial acceleration of the j^{th} node on the k^{th} body.

$V_m(k) = \begin{pmatrix} \dot{\eta}(k) \\ V(k) \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}}(k)}$, the modal spatial velocity of the k^{th} body

$\alpha_m(k) = \dot{V}_m(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k)}$, the modal spatial acceleration of the k^{th} body

$a_m(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k)}$, the modal Coriolis and centrifugal accelerations for the k^{th} body

$b_m(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k)}$, the modal gyroscopic forces for the k^{th} body

$f_m(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k)}$, the modal spatial force of interaction between the k^{th} and $(k+1)^{th}$ bodies

$f_s(j_k) \in \mathcal{R}^6$, the spatial force at node j_k

$f(k) \in \mathcal{R}^6$, the effective spatial force at frame \mathcal{F}_k

$T(k) \in \mathcal{R}^{\mathcal{N}(k)}$, the generalized force for the k^{th} body

$H_{\mathcal{F}}(k) = H(k)\phi(\mathcal{O}_k, k) \in \mathcal{R}^{n_r(k) \times 6}$, the joint map matrix referred to frame \mathcal{F}_k for the k^{th} hinge

$\mathcal{H}(k) = \begin{pmatrix} I & -[\Pi^d(k)]^* \\ 0 & H_{\mathcal{F}}(k) \end{pmatrix} \in \mathcal{R}^{\mathcal{N}(k) \times \bar{\mathcal{N}}(k)}$, the (deformation plus hinge) modal joint map matrix for the k^{th} body

$$\begin{aligned}
\mathcal{A}(k) &= \begin{pmatrix} [\Pi^t(k)]^* \\ \phi(k, t_k) \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times 6}, \text{ relates spatial forces and velocities between node } \\
&\quad t_k \text{ and frame } \mathcal{F}_k \\
\mathcal{B}(k+1, k) &= [0, \phi(t_{k+1}, k)] \in \mathcal{R}^{6 \times \bar{\mathcal{N}}(k)}, \text{ relates spatial forces and velocities between node } \\
&\quad t_{k+1} \text{ and frame } \mathcal{F}_k \\
\Phi(k+1, k) &= \mathcal{A}(k+1)\mathcal{B}(k+1, k) = \begin{pmatrix} 0 & [\Pi^t(k+1)]^* \phi(t_{k+1}, k) \\ 0 & \phi(k+1, k) \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}}(k+1) \times \bar{\mathcal{N}}(k)}, \text{ the} \\
&\quad \text{interbody transformation operator which relates modal spatial forces and ve-} \\
&\quad \text{locities between the } k^{\text{th}} \text{ and } (k+1)^{\text{th}} \text{ bodies} \\
\mathcal{C}(k, k-1) &= \begin{pmatrix} 0 \\ \vdots \\ \phi(t_k, k-1) \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{R}^{6n_s(k) \times 6} \\
\mathcal{B}(k) &= [\phi(k, 1_k), \phi(k, 2_k), \dots, \phi(k, n_s(k))] \in \mathcal{R}^{6 \times 6n_s(k)}, \text{ relates the spatial velocity} \\
&\quad \text{of frame } \mathcal{F}_k \text{ to the spatial velocities of all the nodes on the } k^{\text{th}} \text{ body when the} \\
&\quad \text{body is regarded as being rigid} \\
\mathcal{M} &\in \mathcal{R}^{\mathcal{N} \times \mathcal{N}}, \text{ the multibody system mass matrix} \\
\mathcal{C} &\in \mathcal{R}^{\mathcal{N}}, \text{ the vector of Coriolis, centrifugal and elastic forces for the multibody} \\
&\quad \text{system}
\end{aligned}$$

3. Equations of Motion for Flexible Serial Chains

In this section we develop the equations of motion for a serial flexible multibody system consisting of N flexible bodies. Recursive relationships between the modal spatial velocities, accelerations and forces are developed. Spatial operators are introduced to express these relationships in a compact form, and obtain the Newton-Euler Operator factorization of the mass matrix for the multibody system.

It is assumed that each flexible body has a *lumped mass* model with a rigid body being located at each node. The number of nodes on the k^{th} body is denoted $n_s(k)$. The j^{th} node on the k^{th} body is referred to as the j_k^{th} node. Each body has associated with it

a *body reference frame*, denoted \mathcal{F}_k for the k^{th} body. The deformation of the nodes on the body is described with respect to this body reference frame, while the rigid body motion of the k^{th} body is characterized by the motion of frame \mathcal{F}_k .

The 6-dimensional *spatial deformation* (slope plus translational) of node j_k (with respect to frame \mathcal{F}_k) is denoted $u(j_k) \in \mathcal{R}^6$. The overall deformation field for the k^{th} body is defined as the vector $u(k) = \text{col}\{u(j_k)\} \in \mathcal{R}^{6n_s(k)}$. The vector from frame \mathcal{F}_k to the reference frame on node j_k is denoted $l(k, j_k) \in \mathcal{R}^3$.

The spatial inertia of the j^{th} node is defined as

$$M_s(j_k) = \begin{pmatrix} \mathcal{J}(j_k) & m(j_k)\tilde{p}(j_k) \\ -m(j_k)\tilde{p}(j_k) & m(j_k)I \end{pmatrix} \in \mathcal{R}^{6 \times 6} \quad (3.1)$$

where $\mathcal{J}(j_k)$, $p(j_k)$ and $m(j_k)$ are the inertia tensor about the node reference frame, the vector from the node reference frame to its center of mass, and the mass, respectively, for the j^{th} node on the k^{th} body. The *structural mass matrix* for the k^{th} body $M_s(k)$ is the block diagonal matrix

$$M_s(k) = \text{diag}\{M_s(j_k)\} \in \mathcal{R}^{6n_s(k) \times 6n_s(k)} \quad (3.2)$$

The *structural stiffness matrix* is denoted $K_s(k) \in \mathcal{R}^{6n_s(k) \times 6n_s(k)}$.

As shown in Figure 1, the bodies in the serial chain are numbered in increasing order from tip to base. We use the terminology *inboard* (*outboard*) to denote the direction along the serial chain towards (away from) the base body. The k^{th} body is attached on the inboard side to the $(k+1)^{\text{th}}$ body via the k^{th} hinge, and on the outboard side to the $(k-1)^{\text{th}}$ body via the $(k-1)^{\text{th}}$ hinge. On the k^{th} body, the node to which the outboard hinge (the $(k-1)^{\text{th}}$ hinge) is attached is referred to as node t_k , while the node to which the inboard hinge (the k^{th} hinge) is attached is denoted node d_k . Thus the k^{th} hinge couples together nodes d_k and t_{k+1} . Attached to each of these nodes are the k^{th} hinge reference frames denoted \mathcal{O}_k and \mathcal{O}_k^+ , respectively. The number of dofs for the k^{th} hinge is denoted $n_r(k)$. The vector of configuration variables for the k^{th} hinge is denoted $\theta(k) \in \mathcal{R}^{n_r(k)}$, while its vector of generalized speeds is denoted $\beta(k) \in \mathcal{R}^{n_r(k)}$. In general, when there are nonholonomic hinge constraints, the dimensionality of $\beta(k)$ may be less than that of $\theta(k)$. For notational convenience, and without any loss in generality, we assume here that the dimensions of the vectors $\theta(k)$ and $\beta(k)$ are equal. In most situations $\beta(k)$ is simply $\dot{\theta}$. However there are many cases where the use of quasicordinates simplifies the dynamical equations of motion and an alternative choice for $\beta(k)$ may be preferable. The relative spatial velocity $\Delta_V(k)$ across the hinge is given by $H^*(k)\beta(k)$, where $H^*(k)$ denotes the joint map matrix for the k^{th} hinge.

We assume that there exists a set of $n_m(k)$ assumed modes for the k^{th} body, and let $\Pi_r^j(k) \in \mathcal{R}^6$ denote the *modal spatial displacement vector* at the j_k^{th} node for the r^{th} mode. We also define the *modal spatial displacement influence vector* $\Pi^j(k) \in \mathcal{R}^{6 \times n_m(k)}$ for the j_k^{th} node and the *modal matrix* $\Pi(k) \in \mathcal{R}^{6n_s(k) \times n_m(k)}$ for the k^{th} body as follows:

$$\Pi^j(k) = [\Pi_1^j(k), \dots, \Pi_{n_m(k)}^j(k)] \quad \text{and} \quad \Pi(k) = \text{col}\{\Pi^j(k)\}$$

The r^{th} column of $\Pi(k)$ is denoted $\Pi_r(k)$ and defines the mode shape for the r^{th} assumed mode for the k^{th} body. With $\eta(k) \in \mathcal{R}^{n_m(k)}$ denoting the vector of modal deformation variables for the k^{th} body, the spatial deformation of node j_k and the spatial deformation field $u(k)$ for the k^{th} body are given by

$$u(j_k) = \Pi^j(k)\eta(k) \quad \text{and} \quad u(k) = \Pi(k)\eta(k) \quad (3.3)$$

In the multibody context, it is often convenient to choose the k^{th} body reference frame \mathcal{F}_k as fixed to node d_k at the inboard hinge. For this choice, the assumed modes are referred to as *cantilever modes*, and for which

$$\Pi_r^d(k) = 0 \quad \text{and} \quad r = 1 \dots n_m(k) \quad (3.4)$$

As a consequence, node d_k exhibits zero deformation ($u(d_k) = 0$). Alternative choices of modes, which are often preferred for control analysis and design, are the *free-free modes*. For this case, the reference frame \mathcal{F}_k is not fixed to any node, but is rather assumed to be fixed to the undeformed body, so that all nodes exhibit nonzero deformation. We assume here that when free-free assumed modes are used, only the deformation modes, and none of the rigid body modes, have been included in the mode set. The dynamics model and algorithms developed here handle both types of modes' cases, with some additional computational simplifications arising from Eq. (3.4) when cantilever modes are used. For a related discussion regarding the choice of reference frame and modal representations for a flexible body see reference [4].

The vector of *generalized configuration variables* $\vartheta(k)$ and *generalized speeds* $\chi(k)$ for the k^{th} body are defined as

$$\vartheta(k) \triangleq \begin{pmatrix} \eta(k) \\ \theta(k) \end{pmatrix} \in \mathcal{R}^{\mathcal{N}(k)} \quad \text{and} \quad \chi(k) \triangleq \begin{pmatrix} \dot{\eta}(k) \\ \beta(k) \end{pmatrix} \in \mathcal{R}^{\mathcal{N}(k)} \quad (3.5)$$

where $\mathcal{N}(k) \triangleq n_m(k) + n_r(k)$. The overall vectors of *generalized configuration variables* ϑ

and *generalized speeds* χ for the serial multibody system are defined as

$$\vartheta \triangleq \text{col}\{\vartheta(k)\} \in \mathcal{R}^{\mathcal{N}} \quad \text{and} \quad \chi \triangleq \text{col}\{\chi(k)\} \in \mathcal{R}^{\mathcal{N}} \quad (3.6)$$

where $\mathcal{N} \triangleq \sum_{k=1}^N \mathcal{N}(k)$. The number of overall dofs for the multibody system is \mathcal{N} . The *state* of the multibody system is defined by the pair of vectors $\{\vartheta, \chi\}$. For a given system state $\{\vartheta, \chi\}$, the equations of motion define the relationship between the vector of *generalized accelerations* $\dot{\chi}$ and the vector of *generalized forces* $T \in \mathcal{R}^{\mathcal{N}}$ for the system. The *inverse dynamics* problem consists of computing the vector of generalized forces T for a prescribed set of *generalized accelerations* $\dot{\chi}$. The *forward dynamics* problem is the converse one and consists of computing the set of generalized accelerations $\dot{\chi}$ resulting from a set of generalized forces T . The equations of motion for the system are developed in the remainder of this section. In Section 4, a recursive Newton-Euler inverse dynamics algorithm is developed, while Section 5 and Section 7 describe two alternative forward dynamics algorithms.

3.1 Recursive Propagation of Velocities

In this section, recursive relationships as well as operator expressions for the modal spatial velocities of the bodies in the serial chain are developed.

Let $V(k)$ denote the spatial velocity of the k^{th} body reference frame \mathcal{F}_k , i.e.,

$$V(k) = \begin{pmatrix} \omega(k) \\ v(k) \end{pmatrix} \in \mathcal{R}^6$$

where $\omega(k)$ and $v(k)$ denote the angular and linear velocities respectively of \mathcal{F}_k . The spatial velocity $V_s(t_{k+1}) \in \mathcal{R}^6$ of node t_{k+1} (on the inboard of the k^{th} hinge) is related to the spatial velocity $V(k+1)$ of the $(k+1)^{\text{th}}$ body reference frame \mathcal{F}_{k+1} , and the modal deformation variable rates $\dot{\eta}(k+1)$ as follows:

$$\begin{aligned} V_s(t_{k+1}) &= \phi^*(k+1, t_{k+1})V(k+1) + \dot{u}(t_{k+1}) \\ &= \phi^*(k+1, t_{k+1})V(k+1) + \Pi^t(k+1)\dot{\eta}(k+1) \end{aligned} \quad (3.7)$$

The spatial transformation operator $\phi(x, y) \in \mathcal{R}^{6 \times 6}$ above is defined to be

$$\phi(x, y) = \begin{pmatrix} I & \tilde{l}(x, y) \\ 0 & I \end{pmatrix} \quad (3.8)$$

where $l(x, y) \in \mathcal{R}^3$ denotes the vector between the points x and y . Note that the following important (group) property holds:

$$\phi(x, y)\phi(y, z) = \phi(x, z)$$

for arbitrary points x , y and z . As in Eq. (3.7), and all through this publication, the index k will be used to refer to both the k^{th} body as well as to the k^{th} body reference frame \mathcal{F}_k with the specific usage being evident from the context. Thus $V(k)$ may be read as $V(\mathcal{F}_k)$, and $\phi(k, t_k)$ as $\phi(\mathcal{F}_k, t_k)$.

The spatial velocity $V(\mathcal{O}_k^+)$ of frame \mathcal{O}_k^+ on the inboard side of the k^{th} hinge is related to $V_s(t_{k+1})$ via

$$V(\mathcal{O}_k^+) = \phi^*(t_{k+1}, \mathcal{O}_k)V_s(t_{k+1}) \quad (3.9)$$

The relative spatial velocity $\Delta_V(k)$ across the k^{th} hinge is given by $H^*(k)\beta(k)$, and so the spatial velocity $V(\mathcal{O}_k)$ of frame \mathcal{O}_k on the outboard side of the k^{th} hinge is

$$V(\mathcal{O}_k) = V(\mathcal{O}_k^+) + H^*(k)\beta(k) \quad (3.10)$$

The spatial velocity $V(k)$ of the k^{th} body reference frame is given by

$$V(k) = \phi^*(\mathcal{O}_k, k)V(\mathcal{O}_k) - \dot{u}(d_k) = \phi^*(\mathcal{O}_k, k)V(\mathcal{O}_k) - \Pi^d(k)\dot{\eta}(k) \quad (3.11)$$

Putting together Eq. (3.7), Eq. (3.9), Eq. (3.10) and Eq. (3.11), it follows that

$$\begin{aligned} V(k) = & \phi^*(k+1, k)V(k+1) + \phi^*(t_{k+1}, k)\Pi^t(k+1)\dot{\eta}(k+1) \\ & + \phi^*(\mathcal{O}_k, k)H^*(k)\beta(k) - \Pi^d(k)\dot{\eta}(k) \end{aligned} \quad (3.12)$$

Thus with $\bar{\mathcal{N}}(k) \triangleq n_m(k) + 6$, and using Eq. (3.12), the *modal spatial velocity* $V_m(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k)}$ for the k^{th} body is given by

$$V_m(k) \triangleq \begin{pmatrix} \dot{\eta}(k) \\ V(k) \end{pmatrix} = \Phi^*(k+1, k)V_m(k+1) + \mathcal{H}^*(k)\chi(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k)} \quad (3.13)$$

where the *interbody transformation operator* $\Phi(\cdot, \cdot)$ and the *modal joint map matrix* $\mathcal{H}(k)$ are

defined as

$$\Phi(k+1, k) \triangleq \begin{pmatrix} 0 & [\Pi^t(k+1)]^* \phi(t_{k+1}, k) \\ 0 & \phi(k+1, k) \end{pmatrix} \in \mathcal{R}^{\overline{\mathcal{N}}(k+1) \times \overline{\mathcal{N}}(k)} \quad (3.14)$$

$$\mathcal{H}(k) \triangleq \begin{pmatrix} I & -[\Pi^d(k)]^* \\ 0 & H_{\mathcal{F}}(k) \end{pmatrix} \in \mathcal{R}^{\mathcal{N}(k) \times \overline{\mathcal{N}}(k)} \quad (3.15)$$

where

$$H_{\mathcal{F}}(k) \triangleq H(k)\phi(\mathcal{O}_k, k) \in \mathcal{R}^{n_r(k) \times 6}$$

Note that

$$\Phi(k+1, k) = \mathcal{A}(k+1)\mathcal{B}(k+1, k) \quad (3.16)$$

where

$$\mathcal{A}(k) \triangleq \begin{pmatrix} [\Pi^t(k)]^* \\ \phi(k, t_k) \end{pmatrix} \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times 6} \quad \text{and} \quad \mathcal{B}(k+1, k) \triangleq [0, \phi(t_{k+1}, k)] \in \mathcal{R}^{6 \times \overline{\mathcal{N}}(k)} \quad (3.17)$$

The modal joint map matrix $\mathcal{H}(k)$ can be partitioned as

$$\mathcal{H}(k) = \begin{pmatrix} \mathcal{H}_f(k) \\ \mathcal{H}_r(k) \end{pmatrix} \in \mathcal{R}^{\mathcal{N}(k) \times \overline{\mathcal{N}}(k)} \quad (3.18)$$

where

$$\mathcal{H}_f(k) \triangleq [I, -[\Pi^d(k)]^*] \in \mathcal{R}^{n_m(k) \times \overline{\mathcal{N}}(k)} \quad \text{and} \quad \mathcal{H}_r(k) \triangleq [0, H(k)\phi(\mathcal{O}_k, k)] \in \mathcal{R}^{n_r(k) \times \overline{\mathcal{N}}(k)} \quad (3.19)$$

With $\bar{\mathcal{N}} = \sum_{k=1}^N \bar{\mathcal{N}}(k)$, define the spatial operator \mathcal{E}_Φ as

$$\mathcal{E}_\Phi \triangleq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \Phi(2,1) & 0 & \dots & 0 & 0 \\ 0 & \Phi(3,2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \Phi(N, N-1) & 0 \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}} \times \bar{\mathcal{N}}} \quad (3.20)$$

Noting that \mathcal{E}_Φ is nilpotent (i.e. $\mathcal{E}_\Phi^N = 0$), we define the spatial operator Φ as

$$\Phi \triangleq [I - \mathcal{E}_\Phi]^{-1} = I + \mathcal{E}_\Phi + \dots + \mathcal{E}_\Phi^{N-1} = \begin{pmatrix} I & 0 & \dots & 0 \\ \Phi(2,1) & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(N,1) & \Phi(N,2) & \dots & I \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}} \times \bar{\mathcal{N}}} \quad (3.21)$$

where

$$\Phi(i, j) \triangleq \Phi(i, i-1) \dots \Phi(j+1, j) \text{ for } i > j$$

Also define the spatial operator $\mathcal{H} \triangleq \text{diag}\{\mathcal{H}(k)\} \in \mathcal{R}^{\mathcal{N} \times \mathcal{N}}$. Using these spatial operators, and defining $V_m \triangleq \text{col}\{V_m(k)\} \in \mathcal{R}^{\bar{\mathcal{N}}}$, from Eq. (3.13) it follows that the spatial operator expression for V_m is given by

$$V_m = \Phi^* \mathcal{H}^* \chi \quad (3.22)$$

3.2 Modal Mass Matrix for a Single Body

In this section we derive an expression for the modal mass matrix of the k^{th} body. With $V_s(j_k) \in \mathcal{R}^6$ denoting the spatial velocity of node j_k , and $V_s(k) \triangleq \text{col}\{V_s(j_k)\} \in \mathcal{R}^{6n_s(k)}$ the vector of all nodal spatial velocities for the k^{th} body, it follows (as in Eq. (3.7)) that

$$V_s(k) = B^*(k)V(k) + \dot{u}(k) = [\Pi(k), B^*(k)]V_m(k) \quad (3.23)$$

where

$$B(k) \triangleq [\phi(k, 1_k), \phi(k, 2_k), \dots, \phi(k, n_s(k))] \in \mathcal{R}^{6 \times 6n_s(k)} \quad (3.24)$$

Since $M_s(k)$ is the *structural mass matrix* of the k^{th} body, the kinetic energy of the k^{th} body is given by

$$\frac{1}{2} V_s^*(k) M_s(k) V_s(k) = \frac{1}{2} V_m^*(k) M_m(k) V_m(k)$$

where

$$\begin{aligned} M_m(k) &\triangleq \begin{pmatrix} \Pi^*(k) \\ B(k) \end{pmatrix} M_s(k) [\Pi(k), B^*(k)] = \begin{pmatrix} \Pi^*(k) M_s(k) \Pi(k) & \Pi^*(k) M_s(k) B^*(k) \\ B(k) M_s(k) \Pi(k) & B(k) M_s(k) B^*(k) \end{pmatrix} \\ &= \begin{pmatrix} M_m^{ff}(k) & M_m^{fr}(k) \\ M_m^{rf}(k) & M_m^{rr}(k) \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \bar{\mathcal{N}}(k)} \end{aligned} \quad (3.25)$$

Corresponding to the generalized speeds vector $\chi(k)$, $M_m(k)$ as defined above is the *modal mass matrix* of the k^{th} body. In the block partitioning in Eq. (3.25), the superscripts f and r denote the *flexible* and *rigid* blocks respectively. Thus $M_m^{ff}(k)$ represents the flex/flex coupling block, while $M_m^{fr}(k)$ the flex/rigid coupling block of $M_m(k)$. We will use this notational convention all through this publication. Note that $M_m^{rr}(k)$ is precisely the rigid body spatial inertia of the k^{th} body. Indeed, $M_m(k)$ reduces to the rigid body spatial inertia when the body flexibility is ignored, i.e., no modes are used, since in this case $n_m(k) = 0$ (and so $\Pi(k)$ is null).

Since the vector $l(k, j_k)$ from \mathcal{F}_k to node j_k depends on the deformation of the node, the operator $B(k)$ is also deformation dependent. From Eq. (3.25) it follows that unlike the block $M_m^{ff}(k)$, which is deformation independent, both the blocks $M_m^{fr}(k)$ and $M_m^{rr}(k)$ are deformation dependent. The detailed expression for the modal mass matrix can be defined using *modal integrals* which are computed as a part of the finite–element structural analysis of the flexible bodies. Such an expression for the modal mass matrix of the k^{th} body is given in Appendix B in Eq. (B.8). Often the deformation dependent parts of the modal mass matrix are ignored, and free–free eigen–modes are used for the assumed modes $\Pi(k)$. When this is the case, $M_m^{fr}(k)$ is zero and $M_m^{ff}(k)$ is block diagonal.

3.3 Recursive Propagation of Accelerations

Differentiating the velocity recursion equation, Eq. (3.13), we obtain the following recursive expression for the *modal spatial acceleration* $\alpha_m(k) \in \mathcal{R}^{\mathcal{N}(k)}$ for the k^{th} body:

$$\alpha_m(k) \triangleq \dot{V}_m(k) = \begin{pmatrix} \ddot{\eta}(k) \\ \alpha(k) \end{pmatrix} = \Phi^*(k+1, k)\alpha_m(k+1) + \mathcal{H}^*(k)\dot{\chi}(k) + a_m(k) \quad (3.26)$$

where $\alpha(k) = \dot{V}(k)$, and the Coriolis and centrifugal acceleration term $a_m(k) \in \mathcal{R}^{\mathcal{N}(k)}$ is given by

$$a_m(k) = \frac{d\Phi^*(k+1, k)}{dt}V_m(k+1) + \frac{d\mathcal{H}^*(k)}{dt}\chi(k) \quad (3.27)$$

The detailed expressions for $a_m(k)$ can be found in Eq. (B.11) in Appendix B. Defining $a_m = \text{col}\{a_m(k)\} \in \mathcal{R}^{\mathcal{N}}$ and $\alpha_m = \text{col}\{\alpha_m(k)\} \in \mathcal{R}^{\mathcal{N}}$, Eq. (3.26) can be reexpressed using spatial operators in the form

$$\alpha_m = \Phi^*(\mathcal{H}^*\dot{\chi} + a_m) \quad (3.28)$$

The vector of spatial accelerations of all the nodes for the k^{th} body, $\alpha_s(k) \triangleq \text{col}\{\alpha_s(j_k)\} \in \mathcal{R}^{6n_s(k)}$, is obtained by differentiating Eq. (3.23):

$$\alpha_s(k) = \dot{V}_s(k) = [\Pi(k), B^*(k)]\alpha_m(k) + a(k) \quad (3.29)$$

where

$$a(k) \triangleq \text{col}\{a(j_k)\} = \frac{d[\Pi(k), B^*(k)]}{dt}V_m(k) \in \mathcal{R}^{6n_s(k)} \quad (3.30)$$

3.4 Recursive Propagation of Forces

We now develop the equations of motion for the k^{th} body. Let $f(k-1) \in \mathcal{R}^6$ denote the effective spatial force of interaction, referred to frame \mathcal{F}_{k-1} , between the k^{th} and $(k-1)^{\text{th}}$ bodies across the $(k-1)^{\text{th}}$ hinge. Recall that the $(k-1)^{\text{th}}$ hinge is between node t_k on the k^{th} body and node d_{k-1} on the $(k-1)^{\text{th}}$ body. With $f_s(j_k) \in \mathcal{R}^6$ denoting the spatial force at a node j_k , the force balance equation for node t_k is given by

$$f_s(t_k) = \phi(t_k, k-1)f(k-1) + M_s(t_k)\alpha_s(t_k) + b(t_k) + f_K(t_k) \quad (3.31)$$

For all nodes other than node t_k on the k^{th} body, the force balance equation is of the form

$$f_s(j_k) = M_s(j_k)\alpha_s(j_k) + b(j_k) + f_K(j_k) \quad (3.32)$$

In the above, $f_K(k) = K_s(k)u(k) \in \mathcal{R}^{6n_s(k)}$ denotes the vector of spatial elastic strain forces for the nodes on the k^{th} body, while $b(j_k) \in \mathcal{R}^6$ denotes the spatial gyroscopic force for node j_k and is given by

$$b(j_k) = \begin{pmatrix} \tilde{\omega}(j_k)\mathcal{J}(j_k)\omega(j_k) \\ m(j_k)\tilde{\omega}(j_k)\tilde{\omega}(j_k)p(j_k) \end{pmatrix} \in \mathcal{R}^6 \quad (3.33)$$

where $\omega(j_k) \in \mathcal{R}^3$ denotes the angular velocity of node j_k . Collecting together the above equations and defining

$$C(k, k-1) \triangleq \begin{pmatrix} 0 \\ \vdots \\ \phi(t_k, k-1) \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{R}^{6n_s(k) \times 6} \quad \text{and} \quad b(k) \triangleq \text{col}\{b(j_k)\} \in \mathcal{R}^{6n_s(k)} \quad (3.34)$$

it follows from Eq. (3.31) and Eq. (3.32) that

$$f_s(k) = C(k, k-1)f_s(k-1) + M_s(k)\alpha_s(k) + b(k) + K_s(k)u(k) \quad (3.35)$$

where $f_s(k) \triangleq \text{col}\{f_s(j_k)\} \in \mathcal{R}^{6n_s(k)}$. Noting that

$$f(k) = B(k)f_s(k) \quad (3.36)$$

and using the principle of virtual work, it follows from Eq. (3.23) that the *modal spatial forces* $f_m(k) \in \mathcal{R}^{\bar{N}(k)}$ for the k^{th} body are given by

$$f_m(k) \triangleq \begin{pmatrix} \Pi^*(k) \\ B(k) \end{pmatrix} f_s(k) = \begin{pmatrix} \Pi^*(k)f_s(k) \\ f(k) \end{pmatrix} \quad (3.37)$$

Premultiplying Eq. (3.35) by $\begin{pmatrix} \Pi^*(k) \\ B(k) \end{pmatrix}$ and using Eq. (3.25), Eq. (3.29), and Eq. (3.37) leads to the following recursive relationship for the modal spatial forces:

$$\begin{aligned}
f_m(k) &= \begin{pmatrix} \Pi^*(k)C(k, k-1) \\ B(k)C(k, k-1) \end{pmatrix} f(k-1) + M_m(k)\alpha_m(k) + b_m(k) + K_m(k)\vartheta(k) \\
&= \begin{pmatrix} [\Pi^t(k)]^* \\ \phi(k, t_k) \end{pmatrix} \phi(t_k, k-1)f(k-1) + M_m(k)\alpha_m(k) + b_m(k) + K_m(k)\vartheta(k) \\
&= \Phi(k, k-1)f_m(k-1) + M_m(k)\alpha_m(k) + b_m(k) + K_m(k)\vartheta(k) \tag{3.38}
\end{aligned}$$

Here we have defined

$$b_m(k) \triangleq \begin{pmatrix} \Pi^*(k) \\ B(k) \end{pmatrix} [b(k) + M_s(k)a(k)] \in \mathcal{R}^{\bar{\mathcal{N}}(k)} \tag{3.39}$$

and the *modal stiffness matrix*

$$K_m(k) \triangleq \begin{pmatrix} \Pi^*(k)K_s(k)\Pi(k) & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \bar{\mathcal{N}}(k)} \tag{3.40}$$

The expression for $K_m(k)$ in Eq. (3.40) uses the fact that the columns of $B^*(k)$ are indeed the *deformation dependent* rigid body modes for the k^{th} body and hence they do not contribute to its elastic strain energy. Indeed, when a deformation dependent structural stiffness matrix $K_s(k)$ is used, we have that

$$K_s(k)B^*(k) = 0 \tag{3.41}$$

However the common practice (and also followed here) of using a constant, deformation-independent structural stiffness matrix leads to the anomalous situation wherein Eq. (3.41) does not hold exactly. In view of this, without any loss in accuracy we ignore these anomalous extra terms and drop them from the left-hand side of Eq. (3.41).

The velocity-dependent bias term $b_m(k)$ is formed using modal integrals generated by standard finite-element programs, and a detailed expression for it is given in Eq. (B.50) in Appendix B. From Eq. (3.38), the operator expression for the modal spatial forces

$f_m \triangleq \text{col}\{f_m(k)\} \in \mathcal{R}^{\overline{\mathcal{N}}}$ for all the bodies in the chain is given by

$$f_m = \Phi(M_m \alpha_m + b_m + K_m \vartheta) \quad (3.42)$$

where

$$\mathcal{M}_m \triangleq \text{diag}\{M_m(k)\} \in \mathcal{R}^{\overline{\mathcal{N}} \times \overline{\mathcal{N}}}, K_m \triangleq \text{diag}\{K_m(k)\} \in \mathcal{R}^{\overline{\mathcal{N}} \times \overline{\mathcal{N}}}, \text{ and } b_m \triangleq \text{col}\{b_m(k)\} \in \mathcal{R}^{\overline{\mathcal{N}}}$$

From the principle of virtual work, the *generalized forces* vector $T \in \mathcal{R}^{\mathcal{N}}$ for the multibody system is given by the expression

$$T = \mathcal{H}f_m \quad (3.43)$$

3.5 Operator Expression for the System Mass Matrix

Collecting together the operator expressions in Eq. (3.22), Eq. (3.28), Eq. (3.42) and Eq. (3.43), we obtain the following:

$$\begin{aligned} V_m &= \Phi^* \mathcal{H}^* \chi \\ \alpha_m &= \Phi^* (\mathcal{H}^* \dot{\chi} + a_m) \\ f_m &= \Phi(M_m \alpha_m + b_m + K_m \vartheta) = \Phi M_m \Phi^* \mathcal{H}^* \dot{\chi} + \Phi(M_m \Phi^* a_m + b_m + K_m \vartheta) \\ T &= \mathcal{H}f_m = \mathcal{H}\Phi M_m \Phi^* \mathcal{H}^* \dot{\chi} + \mathcal{H}\Phi(M_m \Phi^* a_m + b_m) \\ &= \mathcal{M} \dot{\chi} + \mathcal{C} \end{aligned} \quad (3.44)$$

where

$$\mathcal{M} \triangleq \mathcal{H}\Phi M_m \Phi^* \mathcal{H}^* \in \mathcal{R}^{\mathcal{N} \times \mathcal{N}} \quad \text{and} \quad \mathcal{C} \triangleq \mathcal{H}\Phi(M_m \Phi^* a_m + b_m + K_m \vartheta) \in \mathcal{R}^{\mathcal{N}} \quad (3.45)$$

Here \mathcal{M} is the system mass matrix for the serial chain and the expression $\mathcal{H}\Phi M_m \Phi^* \mathcal{H}^*$ is referred to as the *Newton-Euler Operator Factorization* of the mass matrix. The term \mathcal{C} is the vector of Coriolis, centrifugal, and elastic forces for the system.

A noteworthy fact about the operator expressions for \mathcal{M} and \mathcal{C} is that they are identical in form to the corresponding expressions for rigid multibody systems (see references [1, 5]). Indeed, the similarity is more than superficial, and the key properties of the spatial operators that are used in the analysis and algorithm development for rigid multibody systems also hold for the spatial operators defined here for flexible multibody systems. Apart from the pedagogical importance, a significant advantage of this is that a large part of the

analysis and algorithms for rigid multibody systems can be easily carried over and applied to flexible multibody systems. This is precisely the approach we adopt here.

4. Inverse Dynamics Algorithm

This section describes a recursive Newton-Euler inverse dynamics algorithm for computing the generalized forces T , for a given set of generalized accelerations $\dot{\chi}$ and system state $\{\vartheta, \chi\}$. The inverse dynamics algorithm also forms a part of forward dynamics algorithms such as those based upon composite body inertias or the conjugate gradient method [6].

Collecting together the recursive equations in Eq. (3.13), Eq. (3.26), Eq. (3.38) and Eq. (3.43) we obtain the following recursive Newton-Euler inverse dynamics algorithm:

$$\left\{ \begin{array}{l} V_m(N+1) = 0, \quad \alpha_m(N+1) = 0 \\ \text{for } k = N \dots 1 \\ \quad V_m(k) = \Phi^*(k+1, k)V_m(k+1) + \mathcal{H}^*(k)\chi(k) \\ \quad \alpha_m(k) = \Phi^*(k+1, k)\alpha_m(k+1) + \mathcal{H}^*(k)\dot{\chi}(k) + a_m(k) \\ \text{end loop} \end{array} \right. \quad (4.1)$$

$$\left\{ \begin{array}{l} f_m(0) = 0 \\ \text{for } k = 1 \dots N \\ \quad f_m(k) = \Phi(k, k-1)f_m(k-1) + M_m(k)\alpha_m(k) + b_m(k) + K_m(k)\vartheta(k) \\ \quad T(k) = \mathcal{H}(k)f_m(k) \\ \text{end loop} \end{array} \right.$$

The structure of this algorithm closely resembles the recursive Newton-Euler inverse dynamics algorithm for rigid multibody systems [7, 1]. It assumes without any loss in generality that the base body is stationary and also that the tip force is zero. Base mobility is handled by attaching an additional 6-dof hinge between the mobile base and an inertial frame. In case there is a non-zero tip force, $f_m(0)$ should be initialized to the value of the tip force in the algorithm. By taking advantage of the special structure of $\Phi(k+1, k)$ and $\mathcal{H}(k)$ in Eq. (3.14) and Eq. (3.15), the Newton-Euler recursions in Eq. (4.1) can be further simplified. Using the superscripts f and r as before to denote the flexible and rigid components, we

have the following partitioning:

$$V_m(k) = \begin{pmatrix} V_m^f(k) \\ V_m^r(k) \end{pmatrix}, \quad \alpha_m(k) = \begin{pmatrix} \alpha_m^f(k) \\ \alpha_m^r(k) \end{pmatrix}, \quad f_m(k) = \begin{pmatrix} f_m^f(k) \\ f_m^r(k) \end{pmatrix}, \quad \text{and } T(k) = \begin{pmatrix} T^f(k) \\ T^r(k) \end{pmatrix}$$

Based upon this partitioning, the simplified inverse dynamics algorithm is as follows:

$$\left\{ \begin{array}{l} V_m(N+1) = 0, \quad \alpha_m(N+1) = 0 \\ \text{for } k = N \dots 1 \\ \quad V_m^f(k) = \dot{\eta}(k) \\ \quad V_m^r(k) = \phi^*(t_{k+1}, k) \mathcal{A}^*(k+1) V_m(k+1) + H_{\mathcal{F}}^*(k) \beta(k) - \Pi^d(k) \dot{\eta}(k) \\ \\ \quad \alpha_m^f(k) = \ddot{\eta}(k) \\ \quad \alpha_m^r(k) = \phi^*(t_{k+1}, k) \mathcal{A}^*(k+1) \alpha_m(k+1) + H_{\mathcal{F}}^*(k) \dot{\beta}(k) - \Pi^d(k) \ddot{\eta}(k) + a_m^r(k) \\ \text{end loop} \end{array} \right. \quad (4.2)$$

$$\left\{ \begin{array}{l} f_m(0) = 0 \\ \text{for } k = 1 \dots N \\ \quad f_m(k) = \mathcal{A}(k) \phi(t_k, k-1) f_m^r(k-1) + M_m(k) \alpha_m(k) + b_m(k) + K_m(k) \vartheta(k) \\ \\ \quad T(k) = \begin{pmatrix} T^f(k) \\ T^r(k) \end{pmatrix} = \begin{pmatrix} f_m^f(k) - [\Pi^d(k)]^* f_m^r(k) \\ H_{\mathcal{F}}(k) f_m^r(k) \end{pmatrix} \\ \text{end loop} \end{array} \right.$$

Flexible multibody systems have actuators typically only at the hinges. Thus for the k^{th} body, only the subset of the generalized forces vector $T(k)$ corresponding to the hinge actuator forces $T^r(k)$ can be set, while the remaining generalized forces $T^f(k)$ are zero. Thus in contrast with rigid multibody systems, flexible multibody systems are *under-actuated systems* [8], since the number of available actuators is less than the number of motion dofs in the system. For such under-actuated systems, the inverse dynamics computations for the generalized force T are meaningful only when the prescribed generalized accelerations $\dot{\chi}$ form a consistent data set. For a consistent set of generalized accelerations, the inverse dynamics

computations will lead to a generalized force vector T such that $T^f(.) = 0$.

5. Composite Body Forward Dynamics Algorithm

The forward dynamics problem for a multibody system consists of computing the generalized accelerations $\dot{\chi}$ for a given set of generalized forces T and state of the system $\{\vartheta, \chi\}$. The *composite body forward dynamics algorithm* consists of the following steps: (a) computing the system mass matrix \mathcal{M} , (b) computing the bias vector \mathcal{C} , and (c) numerically solving the following linear matrix equation for $\dot{\chi}$:

$$\mathcal{M}\dot{\chi} = T - \mathcal{C} \quad (5.1)$$

Later in Section 6 we describe the recursive articulated body forward dynamics algorithm that does not require the explicit computation of either \mathcal{M} or \mathcal{C} .

It is evident from Eq. (5.1) that the components of the vector \mathcal{C} are the generalized forces for the system when the generalized accelerations $\dot{\chi}$ are all zero. Thus \mathcal{C} can be computed using the inverse dynamics algorithm in Eq. (4.2). An efficient composite-body-based recursive algorithm for the computation of the mass matrix \mathcal{M} is described in this section. This algorithm is based upon the following lemma which describes a decomposition of the mass matrix into block diagonal, block upper triangular and block lower triangular components.

Lemma 5.1: Define the *composite body inertias* $R(k) \in \mathcal{R}^{\mathcal{N}(k) \times \mathcal{N}(k)}$ recursively for all the bodies in the serial chain as follows:

$$\left\{ \begin{array}{l} R(0) = 0 \\ \text{for } k = 1 \cdots N \\ \quad R(k) = \Phi(k, k-1)R(k-1)\Phi^*(k, k-1) + M_m(k) \\ \text{end loop} \end{array} \right. \quad (5.2)$$

Also define $R \triangleq \text{diag}\{R(k)\} \in \mathcal{R}^{\mathcal{N} \times \mathcal{N}}$. Then we have the following spatial operator decomposition where $\tilde{\Phi} \triangleq \Phi - I$:

$$\Phi M_m \Phi^* = R + \tilde{\Phi} R + R \tilde{\Phi}^* \quad (5.3)$$

Proof: See Appendix A. ■

Physically, $R(k)$ is the modal mass matrix of the composite body formed from all the bodies outboard of the k^{th} hinge by freezing all their (deformation plus hinge) dofs. It follows from Eq. (3.45) and Lemma 5.1 that

$$\mathcal{M} = \mathcal{H}\Phi M_m \Phi^* \mathcal{H}^* = \mathcal{H}R\mathcal{H}^* + \mathcal{H}\tilde{\Phi}R\mathcal{H}^* + \mathcal{H}R\tilde{\Phi}^*\mathcal{H}^* \quad (5.4)$$

Note that the three terms on the right of Eq. (5.4) are block diagonal, block lower triangular and block upper triangular respectively. The algorithm for computing the mass matrix \mathcal{M} computes these terms recursively. The main recursion proceeds from tip to base, and computes the blocks along the diagonal of \mathcal{M} . As each such diagonal element is computed, a new recursion to compute the off-diagonal elements is spawned. Its structure is similar to that of the composite body algorithm for computing the mass matrix of rigid multibody systems [6, 9], and is as follows:

$$\left\{ \begin{array}{l} R(0) = 0 \\ \text{for } k = 1 \cdots N \\ \quad R(k) = \Phi(k, k-1)R(k-1)\Phi^*(k, k-1) + M_m(k) \\ \quad \quad = \mathcal{A}(k)\phi(t_k, k-1)R^r(k-1)\phi^*(t_k, k-1)\mathcal{A}^*(k) + M_m(k) \\ \quad X(k) = R(k)\mathcal{H}^*(k) \\ \quad \mathcal{M}_s(k, k) = \mathcal{H}(k)X(k) \\ \quad \left\{ \begin{array}{l} \text{for } j = (k+1) \cdots N \\ \quad X(j) = \Phi(j, j-1)X(j-1) = \mathcal{A}(j)\phi(t_j, j-1)X^r(j-1) \\ \quad \mathcal{M}(j, k) = \mathcal{M}^*(k, j) = \mathcal{H}(j)X(j) \\ \quad \text{end loop} \end{array} \right. \\ \text{end loop} \end{array} \right. \quad (5.5)$$

The structure of the above algorithm for computing the mass matrix closely resembles the composite rigid body algorithm for computing the mass matrix of rigid multibody systems [6, 9]. Using the sparsity of both $\mathcal{H}_f(k)$ and $\mathcal{H}_r(k)$, additional computational simplification of the steps in the above algorithm is easy to implement.

6. Factorization and Inversion of the Mass Matrix

An operator factorization of the system mass matrix \mathcal{M} , denoted the *Innovations Operator Factorization*, is derived in this section. This factorization is an alternative to the Newton-Euler factorization in Eq. (3.45) and, in contrast with the latter, the factors in the Innovations factorization are square and invertible. Operator expressions for the inverse of these factors are developed and these immediately lead to an operator expression for the inverse of the mass matrix. Using further operator identities, we obtain an operator expression for the generalized accelerations $\dot{\chi}$ in terms of the applied generalized forces T . The recursive implementation of this expression leads to the recursive articulated body forward dynamics algorithm described in Section 7. The operator factorization and inversion results here closely resemble the corresponding results for rigid multibody systems (see [1]).

Given below is a recursive algorithm which defines some required articulated body quantities:

$$\left\{ \begin{array}{l}
 P^+(0) = 0 \\
 \text{for } k = 1 \cdots N \\
 \quad P(k) = \Phi(k, k-1)P^+(k-1)\Phi^*(k, k-1) + M_m(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \bar{\mathcal{N}}(k)} \\
 \quad D(k) = \mathcal{H}(k)P(k)\mathcal{H}^*(k) \in \mathcal{R}^{\mathcal{N}(k) \times \mathcal{N}(k)} \\
 \quad G(k) = P(k)\mathcal{H}^*(k)D^{-1}(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \mathcal{N}(k)} \\
 \quad K(k+1, k) = \Phi(k+1, k)G(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \mathcal{N}(k)} \\
 \quad \bar{\tau}(k) = I - G(k)\mathcal{H}(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \bar{\mathcal{N}}(k)} \\
 \quad P^+(k) = \bar{\tau}(k)P(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \bar{\mathcal{N}}(k)} \\
 \quad \Psi(k+1, k) = \Phi(k+1, k)\bar{\tau}(k) \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \bar{\mathcal{N}}(k)} \\
 \text{end loop}
 \end{array} \right. \quad (6.1)$$

The operator $P \in \mathcal{R}^{\bar{\mathcal{N}} \times \bar{\mathcal{N}}}$ is defined as the block diagonal matrix with the k^{th} diagonal element being $P(k)$. The quantities defined in Eq. (6.1) form the component elements of the following spatial operators:

$$\begin{aligned}
 D &\triangleq \mathcal{H}P\mathcal{H}^* = \text{diag}\{D(k)\} \in \mathcal{R}^{\mathcal{N} \times \mathcal{N}} \\
 G &\triangleq P\mathcal{H}^*D^{-1} = \text{diag}\{G(k)\} \in \mathcal{R}^{\bar{\mathcal{N}} \times \mathcal{N}} \\
 K &\triangleq \mathcal{E}_\Phi G \in \mathcal{R}^{\bar{\mathcal{N}} \times \mathcal{N}}
 \end{aligned}$$

$$\begin{aligned}
\bar{\tau} &\triangleq I - G\mathcal{H} = \text{diag}\{\bar{\tau}(k)\} \in \mathcal{R}^{\bar{\mathcal{N}} \times \bar{\mathcal{N}}} \\
\mathcal{E}_\Psi &\triangleq \mathcal{E}_\Phi \bar{\tau} \in \mathcal{R}^{\bar{\mathcal{N}} \times \bar{\mathcal{N}}}
\end{aligned} \tag{6.2}$$

The only nonzero block elements of K and \mathcal{E}_Ψ are the elements' $K(k+1, k)$'s and $\Psi(k+1, k)$'s respectively along the first sub-diagonal.

As in the case for \mathcal{E}_Φ , \mathcal{E}_Ψ is nilpotent, so we can define

$$\Psi \triangleq (I - \mathcal{E}_\Psi)^{-1} = \begin{pmatrix} I & 0 & \dots & 0 \\ \Psi(2,1) & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Psi(N,1) & \Psi(N,2) & \dots & I \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}} \times \bar{\mathcal{N}}} \tag{6.3}$$

where

$$\Psi(i, j) \triangleq \Psi(i, i-1) \cdots \Psi(j+1, j) \text{ for } i > j$$

The structure of the operators \mathcal{E}_Ψ and Ψ is identical to that of the operators \mathcal{E}_Φ and Φ respectively except that the component elements are now $\Psi(i, j)$ rather than $\Phi(i, j)$. Also, the elements of Ψ have the same semigroup properties as the elements of the operator Φ , and as a consequence, high-level operator expressions involving them can be directly mapped into recursive algorithms, and the explicit computation of the elements of the operator Ψ is not required.

The Innovations Operator Factorization of the mass matrix is defined in the following lemma.

Lemma 6.1:

$$\mathcal{M} = [I + \mathcal{H}\Phi K]D[I + \mathcal{H}\Phi K]^* \tag{6.4}$$

Proof: See Appendix A. ■

Note that the factor $[I + \mathcal{H}\Phi K] \in \mathcal{R}^{\mathcal{N} \times \mathcal{N}}$ is square, block lower triangular and non-singular, while D is a block diagonal matrix. This factorization may be regarded as a block LDL^* decomposition of \mathcal{M} . The following lemma gives the closed form operator expression for the inverse of the factor $[I + \mathcal{H}\Phi K]$.

Lemma 6.2:

$$[I + \mathcal{H}\Phi K]^{-1} = [I - \mathcal{H}\Psi K] \quad (6.5)$$

Proof: See Appendix A. ■

It follows from Lemmas 6.1 and 6.2 that the operator expression for the inverse of the mass matrix is given by:

Lemma 6.3:

$$\mathcal{M}^{-1} = [I - \mathcal{H}\Psi K]^* D^{-1} [I - \mathcal{H}\Psi K] \quad (6.6)$$

Once again, note that the factor $[I - \mathcal{H}\Psi K]$ is square, block lower triangular and nonsingular and so Lemma 6.3 may be regarded as providing a block LDL^* decomposition of \mathcal{M}^{-1} . ■

7. Articulated Body Forward Dynamics Algorithm

We first use the operator expression for the mass matrix inverse developed in Section 6 to obtain an operator expression for the generalized accelerations $\dot{\chi}$. This expression directly leads to a recursive algorithm for the forward dynamics of the system. The structure of this algorithm is completely identical in form to the articulated body algorithm for serial *rigid* multibody systems. The computational cost of this algorithm is further reduced by separately processing the flexible and hinge dofs at each step in the recursion, and this forms the *articulated body forward dynamics algorithm* for serial flexible multibody systems.

The following lemma describes the operator expression for the generalized accelerations $\dot{\chi}$ in terms of the generalized forces T .

Lemma 7.1:

$$\dot{\chi} = [I - \mathcal{H}\Psi K]^* D^{-1} [T - \mathcal{H}\Psi \{KT + Pa_m + b_m + K_m \vartheta\}] - K^* \Psi^* a_m \quad (7.1)$$

Proof: See Appendix A. ■

As in the case of rigid multibody systems [1, 3], the direct recursive implementation of Eq. (7.1) leads to the following recursive forward dynamics algorithm:

$$\left\{ \begin{array}{l} z^+(0) = 0 \\ \text{for } k = 1 \cdots n \\ \quad z(k) = \Phi(k, k-1)z^+(k-1) + P(k)a_m(k) + b_m(k) + K_m(k)\vartheta(k) \\ \quad \epsilon(k) = T(k) - \mathcal{H}(k)z(k) \\ \quad \nu(k) = D^{-1}(k)\epsilon(k) \\ \quad z^+(k) = z(k) + G(k)\epsilon(k) \\ \text{end loop} \end{array} \right. \quad (7.2)$$

$$\left\{ \begin{array}{l} \alpha_m(n+1) = 0 \\ \text{for } k = n \cdots 1 \\ \quad \alpha_m^+(k) = \Phi^*(k+1, k)\alpha_m(k+1) \\ \quad \dot{\chi}(k) = \nu(k) - G^*(k)\alpha_m^+(k) \\ \quad \alpha_m(k) = \alpha_m^+(k) + \mathcal{H}^*(k)\dot{\chi}(k) + a_m(k) \\ \text{end loop} \end{array} \right. \quad (7.3)$$

All the dofs for each body as characterized by its joint map matrix $\mathcal{H}^*(\cdot)$ are processed together at each recursion step in this algorithm. However, by taking advantage of the sparsity and special structure of the joint map matrix, additional reduction in computational cost is obtained by processing the flexible dofs and the hinge dofs separately. These simplifications are described in the following sections.

7.1 Simplified Algorithm for the Articulated Body Quantities

We describe intuitively the basis for the separation of the modal and hinge dofs for the body. First we recall the velocity recursion equation in Eq. (3.13)

$$V_m(k) = \Phi^*(k+1, k)V_m(k+1) + \mathcal{H}^*(k)\chi(k) \quad (7.4)$$

and the partitioned form of $\mathcal{H}(k)$ in Eq. (3.15)

$$\mathcal{H}(k) = \begin{pmatrix} \mathcal{H}_f(k) \\ \mathcal{H}_r(k) \end{pmatrix} \quad (7.5)$$

Introducing a dummy variable k' , we can rewrite Eq. (7.4) as

$$\begin{aligned} V_m(k') &= \Phi^*(k+1, k')V_m(k+1) + \mathcal{H}_f^*(k)\dot{\eta}(k) \\ V_m(k) &= \Phi^*(k', k)V_m(k') + \mathcal{H}_r^*(k)\beta(k) \end{aligned} \quad (7.6)$$

where

$$\Phi(k+1, k') \triangleq \Phi(k+1, k) \quad \text{and} \quad \Phi(k', k) \triangleq I$$

Conceptually, each flexible body is now associated with two bodies. The first one has the same kinematical and mass/inertia properties as the real body and is associated with the flexible dofs. The second body is a fictitious body and it is massless and has zero extent. It is associated with the hinge dofs. The serial chain now contains twice the number of bodies as the original one, with half the new bodies being fictitious ones. The new \mathcal{H}^* operator will have the same number of columns but twice the number of rows as the original \mathcal{H}^* operator. The new Φ operator will have twice the number of rows as well as columns as the original one. Going through the same analysis as described in the previous sections, we once again obtain the same operator expression as Eq. (7.1). This expression also leads to a recursive forward dynamics algorithm as in Eq. (7.3). However each sweep in the algorithm will contain twice as many steps as the original algorithm. However each step will be processing only a smaller number of dofs leading to a reduction in the overall cost. Note that a new set of articulated body quantities also needs to be defined using an algorithm whose structure is similar to that of the algorithm in Eq. (6.1).

For low-spin multibody systems, an additional source of computational simplification is possible. Given the inherent linearization that results from using modes for modeling body deformation, there is typically little loss in model fidelity for such systems when the deformation and deformation rate dependent terms in \mathcal{M} and \mathcal{C} are dropped from the dynamical equations of motion [10]. Such models have been dubbed *ruthlessly linearized models*. The ruthlessly linearized models are considerably less complex, and their use results in a substantial reduction in the computational complexity of the dynamical algorithms. For these models, the modal mass matrices for the component bodies are constant in the body frame. In this case, given matrices A , B and C , the following matrix identity

$$[A + BCB^*]^{-1} = A^{-1} - A^{-1}B[C^{-1} + B^*A^{-1}B]^{-1}B^*A^{-1} \quad (7.7)$$

can be used to simplify the formation of D^{-1} in Eq. (6.1). As a result of this simplification, the computational complexity of each recursion step for the articulated body inertias becomes a quadratic rather than a cubic function of the number of modes.

Separating the flexible and hinge dofs for each body, and using the simplifications from Eq. (7.7) for a ruthlessly linearized model, a simplified recursive algorithm for the computation of the articulated body quantities is described below. When a ruthlessly linearized model is not desired the only change required in the algorithm below is that instead of computing $D_f^{-1}(k)$ indirectly, it must be computed by directly inverting the matrix $D_f(k)$. The following matrices can be used only for the ruthless model and need to be precomputed just once prior to the dynamical simulation:

$$\left\{ \begin{array}{l}
 \text{for } k = 1 \dots N \\
 \Lambda(k) = [\mathcal{H}_f(k)M_m(k)\mathcal{H}_f^*(k)]^{-1} \in \mathcal{R}^{N \times N} \\
 \zeta(k) = \mathcal{H}_f(k)\mathcal{A}(k) \in \mathcal{R}^{N \times 6} \\
 \Upsilon(k) = \Lambda(k)\zeta(k) \in \mathcal{R}^{N \times 6} \\
 \Omega(k) = \zeta^*(k)\Upsilon(k) \in \mathcal{R}^{6 \times 6} \\
 \text{end loop}
 \end{array} \right. \quad (7.8)$$

The remainder of the algorithm for computing the articulated body quantities is as follows:

$$\left. \begin{aligned}
 & P^+(0) = 0 \\
 \text{for } k &= 1 \cdots N \\
 & \Gamma(k) = \mathcal{B}(k, k-1)P^+(k-1)\mathcal{B}^*(k, k-1) \in \mathcal{R}^{6 \times 6} \\
 & P(k) = \mathcal{A}(k)\Gamma(k)\mathcal{A}^*(k) + M_m(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times \overline{\mathcal{N}}(k)} \\
 & D_f(k) = \mathcal{H}_f(k)P(k)\mathcal{H}_f^*(k) \in \mathcal{R}^{n_m(k) \times n_m(k)} \\
 D_f^{-1}(k) &= \Lambda(k) - \Upsilon(k)[\Gamma^{-1}(k) + \Omega(k)]^{-1}(k)\Upsilon^*(k) \in \mathcal{R}^{n_m(k) \times n_m(k)} \\
 G_f(k) &= P(k)\mathcal{H}_f^*(k)D_f^{-1}(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times n_m(k)} \\
 \bar{\tau}_f(k) &= I - G_f(k)\mathcal{H}_f(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times \overline{\mathcal{N}}(k)} \\
 P_r(k) &= \bar{\tau}_f(k)P(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times \overline{\mathcal{N}}(k)} \\
 \\
 D_r(k) &= \mathcal{H}_r(k)P_r(k)\mathcal{H}_r^*(k) \in \mathcal{R}^{n_r(k) \times n_r(k)} \\
 G_r(k) &= P_r(k)\mathcal{H}_r^*(k)D_r^{-1}(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times n_r(k)} \\
 \bar{\tau}_r(k) &= I - G_r(k)\mathcal{H}_r(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times \overline{\mathcal{N}}(k)} \\
 P^+(k) &= \bar{\tau}_r(k)P_r(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times \overline{\mathcal{N}}(k)} \\
 \Psi(k+1, k) &= \Phi(k+1, k)\bar{\tau}(k) \in \mathcal{R}^{\overline{\mathcal{N}}(k) \times \overline{\mathcal{N}}(k)} \\
 \text{end loop}
 \end{aligned} \right\} \quad (7.9)$$

The sparsity of $\mathcal{B}(k+1, k)$, $\mathcal{H}_f(k)$ and $\mathcal{H}_r(k)$ leads to still further simplification of the above algorithm. Using the symbol “ \times ” to indicate “don’t care” blocks, the structure in block partitioned form of some of the quantities in Eq. (7.9) is given below:

$$\begin{aligned}
 \Gamma(k) &= \phi(t_k, k-1)P_R^+(k-1)\phi^*(t_k, k-1), \quad (P_R^+(k) \text{ is defined below}) \\
 G_f(k) &= \begin{pmatrix} \times \\ g(k) \end{pmatrix}, \quad \text{where } g(k) = \mu(k)D_f^{-1}(k) \in \mathcal{R}^{6 \times n_m(k)} \\
 \text{and } \mu(k) &\triangleq [P^{rJ}(k), P^{rr}(k)]\mathcal{H}_f^*(k) \in \mathcal{R}^{6 \times n_m(k)}
 \end{aligned}$$

$$P_r(k) = \begin{pmatrix} \times & \times \\ \times & P_R(k) \end{pmatrix}, \quad \text{where } P_R(k) = P^{rr}(k) - g(k)\mu^*(k) \in \mathcal{R}^{6 \times 6}$$

$$D_r(k) = H_{\mathcal{F}}(k)P_R(k)H_{\mathcal{F}}^*(k) \in \mathcal{R}^{n_r(k) \times n_r(k)}$$

$$G_r(k) = \begin{pmatrix} \times \\ G_R(k) \end{pmatrix}, \quad \text{where } G_R(k) \triangleq P_R(k)H_{\mathcal{F}}^*(k)D_r^{-1}(k) \in \mathcal{R}^{6 \times n_r(k)}$$

$$\bar{\tau}_r(k) = \begin{pmatrix} I & \times \\ 0 & \bar{\tau}_R(k) \end{pmatrix}, \quad \text{where } \bar{\tau}_R(k) = I - G_R(k)H_{\mathcal{F}}(k) \in \mathcal{R}^{6 \times 6}$$

$$P^+(k) = \begin{pmatrix} \times & \times \\ \times & P_R^+(k) \end{pmatrix}, \quad \text{where } P_R^+(k) = \bar{\tau}_R(k)P_R(k) \in \mathcal{R}^{6 \times 6}$$

Using the structure described above, the final simplified algorithm for computing the artic-

ulated body quantities is as follows:

$$\left. \begin{aligned}
 & P_R^+(0) = 0 \\
 & \text{for } k = 1 \cdots N \\
 & \quad \Gamma(k) = \phi(t_k, k-1)P_R^+(k-1)\phi^*(t_k, k-1) \\
 & \quad P(k) = \mathcal{A}(k)\Gamma(k)\mathcal{A}^*(k) + M_m(k) \\
 & \quad D_f(k) = \mathcal{H}_f(k)P(k)\mathcal{H}_f^*(k) \\
 & \quad D_f^{-1}(k) = \Lambda(k) - \Upsilon(k)[\Gamma^{-1}(k) + \Omega(k)]^{-1}(k)\Upsilon^*(k) \\
 & \quad \mu(k) = [P^{rf}(k), P^{rr}(k)]\mathcal{H}_f^*(k) \\
 & \quad g(k) = \mu(k)D_f^{-1}(k) \\
 & \quad P_R(k) = P^{rr}(k) - g(k)\mu^*(k) \\
 & \quad D_R(k) = H_{\mathcal{F}}(k)P_R(k)H_{\mathcal{F}}^*(k) \\
 & \quad G_R(k) = P_R(k)H_{\mathcal{F}}^*(k)D_R^{-1}(k) \\
 & \quad \bar{\tau}_R(k) = I - G_R(k)H_{\mathcal{F}}(k) \\
 & \quad P_R^+(k) = \bar{\tau}_R(k)P_R(k) \\
 & \text{end loop}
 \end{aligned} \right\} \quad (7.10)$$

7.2 Simplified Articulated Body Forward Dynamics Algorithm

The simplified recursive *articulated body forward dynamics algorithm* for a serial flexible multibody system follows directly from the recursive implementation of the expression in Eq. (7.1) for the multibody system with fictitious bodies described in Section 7.1. It consists of the following steps: (a) computation of the articulated body quantities using Eq. (7.8) and Eq. (7.10), (b) a base-to-tip recursion as in Eq. (4.2) for computing the modal spatial velocities $V_m(k)$, and the bias terms $a_m(k)$ and $b_m(k)$ for all the bodies, and (c) a tip-to-base

dynamics algorithm described in Section 5, this algorithm does not require the explicit computation of either \mathcal{M} or \mathcal{C} . The structure of this articulated body algorithm closely resembles the recursive articulated body forward dynamics algorithm for rigid multibody systems described in references [11, 1].

8. Computational Cost

This section discusses the computational complexity of the composite body and the articulated body forward dynamics algorithms.

Flexible multibody systems typically involve both rigid and flexible bodies and, in addition, different sets of modes are used to model the flexibility of each body. As a consequence, where possible, we describe the contribution of a typical (non-extremal) flexible body, denoted the k^{th} body, to the overall computational cost. Note that the computational cost for extremal bodies as well as for rigid bodies will typically be lower than that for a non-extremal flexible body. Summing up this cost for all the bodies in the system will give a figure close to the true computational cost for the algorithm. Without any loss in generality, we have assumed here that all the hinges are single dof rotary joints and that free-free assumed modes are being used.

All costs given below are based on the use of the *ruthlessly linearized* dynamics model of the flexible body [10, 12] wherein all deformation and deformation rate dependent terms are dropped from \mathcal{M} and \mathcal{C} . It has been pointed out in recent literature [13, 10] that the use of modes for modeling body flexibility leads to “premature linearization” of the dynamics, in the sense that while the dynamics model will contain deformation dependent terms, the geometric stiffening terms will be missing. Indeed, these missing geometric stiffening terms are typically the dominant first-order (deformation) dependent terms. When the body spin rates are high, it is necessary to take additional steps to include the stiffness terms to obtain a “consistently” linearized model with the proper degree of fidelity. However for systems with low spin rate, the contribution of the first order terms to the dynamics model is negligible. Dropping the deformation $\eta(k)$ and deformation rate $\dot{\eta}$ dependent terms leads to the “ruthlessly” linearized dynamics model and significant reductions in the computational complexity of the algorithms. In the ruthlessly linearized model, Eq. (B.8), Eq. (B.11) and Eq. (B.50) are approximated as follows:

$$M_m(k) \approx M_m^0(k), \quad a_m(k) \approx \begin{pmatrix} 0 \\ a_{mR}^0(k) \end{pmatrix}, \quad \text{and} \quad b_m(k) \approx b_m^0(k) \quad (8.1)$$

Note that with this approximation, $M_m(k)$ is constant in the body frame, while $a_m(k)$ and $b_m(k)$ are independent of $\eta(k)$ and $\dot{\eta}(k)$. A large number of flexible multibody systems

are indeed low spin systems for which the ruthlessly linearized model is adequate. The computational costs described below are for such a system and are based on the use of a ruthlessly linearized model.

8.1 Computational Cost of the Composite Body Forward Dynamics Algorithm

The composite body forward dynamics algorithm described in Section 5 is based on solving the linear matrix equation

$$\mathcal{M}\dot{\chi} = T - \mathcal{C}$$

The computational cost of this forward dynamics algorithm is given below:

1. Cost of computing $R(k)$ for the k^{th} body using the algorithm in Eq. (5.5):

$$\text{products} = 48n_m(k) + 90$$

$$\text{additions} = n_m^2(k) + \frac{97}{2}n_m(k) + 116$$

2. Contribution of the k^{th} body to the cost of computing \mathcal{M} (excluding cost of $R(k)$'s) using the algorithm in Eq. (5.5):

$$\text{products} = k[12n_m^2(k) + 34n_m(k) + 13]$$

$$\text{additions} = k[11n_m^2(k) + 24n_m(k) + 13]$$

3. Setting the generalized accelerations $\dot{\chi} = 0$, the vector \mathcal{C} can be obtained by using the inverse dynamics algorithm described in Eq. (4.2) for computing the generalized forces T . The contribution of the k^{th} body to the computational cost for $\mathcal{C}(k)$ is

$$\text{products} = 2n_m^2(k) + 54n_m(k) + 206$$

$$\text{additions} = 2n_m^2(k) + 50n_m(k) + 143$$

4. The cost of computing $T - \mathcal{C}$ is

$$\text{products} = 0$$

$$\text{additions} = \mathcal{N}$$

5. The cost of solving the linear equation in Eq. (5.1) for the accelerations $\dot{\chi}$ is

$$\text{products} = \frac{1}{6}\mathcal{N}^3 + \frac{3}{2}\mathcal{N}^2 - \frac{2}{3}\mathcal{N}$$

$$\text{additions} = \frac{1}{6}\mathcal{N}^3 + \mathcal{N}^2 - \frac{7}{6}\mathcal{N}$$

The overall complexity of the composite body forward dynamics algorithm is $O(\mathcal{N}^3)$.

8.2 Computational Cost of the Articulated Body Forward Dynamics Algorithm

The articulated body forward dynamics algorithm is based on the recursions described in Eq. (7.8), Eq. (7.10), Eq. (7.11) and Eq. (7.12). Since the computations in Eq. (7.8) can be carried out prior to the dynamics simulation, the cost of this recursion is not included in the cost of the overall forward dynamics algorithm described below:

1. The algorithm for the computation of the articulated body quantities is given in Eq. (7.10). The step involving the computation of $D^{-1}(k)$ can be carried out either by an explicit inversion of $D(k)$ with $O(n_m^3(k))$ cost, or by the indirect procedure described in Eq. (7.10) with $O(n_m^2(k))$ cost. The first method is more efficient than the second one for $n_m(k) \leq 7$.

- Cost of Eq. (7.10) for the k^{th} body based on the explicit inversion of $D(k)$ (used when $n_m(k) \leq 7$):

$$\text{products} = \frac{5}{6}n_m^3(k) + \frac{25}{2}n_m^2(k) + \frac{764}{3}n_m(k) + 180$$

$$\text{additions} = \frac{5}{6}n_m^3(k) + \frac{21}{2}n_m^2(k) + \frac{548}{3}n_m(k) + 164$$

- Cost of Eq. (7.10) for the k^{th} body based on the indirect computation of $D^{-1}(k)$ (used when $n_m(k) \geq 8$):

$$\text{products} = 12n_m^2(k) + 255n_m(k) + 572$$

$$\text{additions} = 13n_m^2(k) + 182n_m(k) + 445$$

Number of Bodies N	Number of Modes n_m		
	5	10	15
3	$5223m + 4195a$	$11432m + 9677a$	$18587m + 16287a$
6	$12591m + 10075a$	$27611m + 23315a$	$44846m + 39225a$
9	$19591m + 15955a$	$43790m + 36953a$	$71105m + 62163a$

Table 1: Computational cost for the articulated body forward dynamics algorithm

2. The cost for the tip-to-base recursion sweep in Eq. (7.11) for the k^{th} body is:

$$\text{products} = n_m^2(k) + 25n_m(k) + 49$$

$$\text{additions} = n_m^2(k) + 24n_m(k) + 50$$

3. The cost for the base-to-tip recursion sweep in Eq. (7.12) for the k^{th} body is:

$$\text{products} = 18n_m(k) + 52$$

$$\text{additions} = 19n_m(k) + 42$$

The overall complexity of this algorithm is $O(Nn_m^2)$, where n_m is an upper bound on the number of modes per body in the system. Thus it is to be expected that the articulated body algorithm will be more efficient than the composite body algorithm as the number of bodies increases. Indeed a comparison of the computational costs of the two forward dynamics algorithms reveals that the articulated body algorithm is the more efficient even when a small number of assumed modes is used.

Table 1 describes the computational cost of the articulated body forward dynamics algorithm for a different number of bodies N and number of modes n_m for a serial chain system. In the table, m and a denote floating point multiplications and additions respectively.

9. Extensions to General Topology Flexible Multibody Systems

For a rigid multibody system, reference [5] describes the extensions to the dynamics formulation and algorithms that are required as the topology of the system goes from a serial chain topology, to a tree topology and finally to a closed-chain topology system. The key to this progression is that the operator description of the system dynamics does not change despite the increase in the topological complexity of the system. Indeed, as seen here, the operator description of the dynamics remains the same even when the multibody system contains flexible rather than rigid component bodies. Thus, using the approach in [5] for rigid multibody systems, the dynamics formulation and algorithms for flexible multibody systems with serial topology can be extended in a straightforward manner to systems with tree or closed-chain topology. Based on these observations, extending the serial chain dynamics algorithms described in this publication to tree topology flexible multibody systems requires the following steps:

1. For any outward sweep involving a base to tip(s) recursion, at each body, the outward recursion must be continued along each outgoing branch emanating from the current body.
2. For an inward sweep involving a tip(s) to base recursion, at each body, the recursion must be continued inwards only after summing up contributions from each of the incoming branches at the body.

A closed-chain topology flexible multibody system can be regarded as a tree topology system with additional closure constraints. As described in reference [5], the dynamics algorithm for closed-chain systems consists of recursions involving the dynamics of the tree topology system, and in addition the computation of the closure constraint forces. The computation of the constraint forces requires the effective inertia of the tree topology system reflected to the points of closure. The algorithm for flexible multibody systems for computing these inertias is identical in form to the recursive algorithm described in [5].

10. Conclusions

This publication uses the spatial operator algebra methodology to develop a new dynamics formulation and spatially recursive algorithms for flexible multibody systems. A key feature of the formulation is that the operator description of the flexible system dynamics is identical in form to the corresponding operator description of the dynamics of rigid multibody systems. A significant advantage of this unifying approach is that it allows ideas and techniques for

rigid multibody systems to be easily applied to flexible multibody systems. The algorithms use standard finite-element and assumed modes models for the individual body deformation.

The Newton–Euler Operator Factorization of the mass matrix, $\mathcal{M} = \mathcal{H}\Phi M_m \Phi^* \mathcal{H}^*$, forms the basis for recursive algorithms such as those for the inverse dynamics, the computation of the mass matrix, and the composite body forward dynamics algorithm for the flexible multibody system. Subsequently, the articulated body forward dynamics algorithm is developed, which, in contrast to the composite body forward dynamics algorithm, does not require the explicit computation of the mass matrix. The key sequence of steps involved in the development of the articulated body forward dynamics algorithm is: (a) the development of an alternative Innovations Operator Factorization of the mass matrix, $\mathcal{M} = [I + \mathcal{H}\Phi K]D[I + \mathcal{H}\Phi K]^*$, (b) formation of the inverse of the factor $[I + \mathcal{H}\Phi K]^{-1} = [I - \mathcal{H}\Psi K]$, and (c) the formation of the operator expression for the mass matrix inverse, $\mathcal{M}^{-1} = [I - \mathcal{H}\Psi K]^* D^{-1} [I - \mathcal{H}\Psi K]$. While the major focus in this publication is on the dynamics of flexible multibody systems with serial topology, the extension of the algorithms developed here to tree and closed chain topology systems is straightforward.

Based on the discussion in reference [3] for rigid multibody systems, forward dynamics algorithms such as the conjugate gradient and triangularization algorithms can be extended to flexible multibody systems in a straightforward manner from the operator description of the dynamics presented here. While the computational cost of the algorithms depends on factors such as the topology and the amount of flexibility in the multibody system, in general, it appears that in contrast with the rigid multibody case, the articulated body forward dynamics algorithm is the more efficient algorithm for flexible multibody systems containing even a small number of flexible bodies. The variety of algorithms described here permits a user to choose the algorithm which is optimal for the multibody system at hand. The availability of a number of algorithms is even more important for real-time applications, where implementation on parallel processors or custom computing hardware is often necessary to maximize speed.

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Appendix A: Proofs of the Lemmas

At the operator level, the proofs of the lemmas in this publication are completely analogous to those for rigid multibody systems [1, 3].

Proof of Lemma 5.1: Using operators, we can rewrite Eq. (5.2) in the form

$$M_m = R - \mathcal{E}_\phi R \mathcal{E}_\phi^* \tag{A.1}$$

From Eq. (3.21) it follows that $\tilde{\Phi}\mathcal{E}_\Phi = \mathcal{E}_\Phi\tilde{\Phi} = \tilde{\Phi} - I = \tilde{\Phi}$. Multiplying Eq. (A.1) from the left and right by $\tilde{\Phi}$ and $\tilde{\Phi}^*$ respectively leads to

$$\tilde{\Phi}M_m\tilde{\Phi}^* = \tilde{\Phi}R\tilde{\Phi}^* - \tilde{\Phi}\mathcal{E}_\Phi R\mathcal{E}_\Phi^*\tilde{\Phi}^* = (\tilde{\Phi} + I)R(\tilde{\Phi} + I)^* - \tilde{\Phi}R\tilde{\Phi}^* = R + \tilde{\Phi}R + R\tilde{\Phi}^*$$

■

Proof of Lemma 6.1: It is easy to verify that $\bar{\tau}P\bar{\tau}^* = \bar{\tau}P$. As a consequence, the recursion for $P(\cdot)$ in Eq. (6.1) can be rewritten in the form

$$M_m = P - \mathcal{E}_\Psi P\mathcal{E}_\Psi^* = P - \mathcal{E}_\Psi P\mathcal{E}_\Phi^* = P - \mathcal{E}_\Phi P\mathcal{E}_\Phi^* + KDK^* \quad (\text{A.2})$$

Pre- and post-multiplying the above by $\tilde{\Phi}$ and $\tilde{\Phi}^*$ respectively then leads to

$$\tilde{\Phi}M_m\tilde{\Phi}^* = P + \tilde{\Phi}P + P\tilde{\Phi}^* + \tilde{\Phi}KDK^*\tilde{\Phi}^*$$

Hence,

$$\begin{aligned} \Rightarrow \mathcal{M} &= \mathcal{H}\tilde{\Phi}M_m\tilde{\Phi}^*\mathcal{H}^* = \mathcal{H}[P + \tilde{\Phi}P + P\tilde{\Phi}^* + \tilde{\Phi}KDK^*\tilde{\Phi}^*]\mathcal{H}^* \\ &= D + \mathcal{H}\tilde{\Phi}KD + DK^*\tilde{\Phi}^*\mathcal{H}^* + \mathcal{H}\tilde{\Phi}KDK^*\tilde{\Phi}^*\mathcal{H}^* = [I + \mathcal{H}\tilde{\Phi}K]D[I + \mathcal{H}\tilde{\Phi}K]^* \end{aligned}$$

■

Proof of Lemma 6.2: Using a standard matrix identity we have that

$$[I + \mathcal{H}\tilde{\Phi}K]^{-1} = I - \mathcal{H}\tilde{\Phi}[I + K\mathcal{H}\tilde{\Phi}]^{-1}K \quad (\text{A.3})$$

Note that

$$\Psi^{-1} = I - \mathcal{E}_\Psi = (I - \mathcal{E}_\Phi) + \mathcal{E}_\Phi G\mathcal{H} = \tilde{\Phi}^{-1} + K\mathcal{H} \quad (\text{A.4})$$

from which it follows that

$$\Psi^{-1}\tilde{\Phi} = I + K\mathcal{H}\tilde{\Phi}$$

Using this with Eq. (A.3) it follows that

$$[I + \mathcal{H}\tilde{\Phi}K]^{-1} = I - \mathcal{H}\tilde{\Phi}[\Psi^{-1}\tilde{\Phi}]^{-1}K = I - \mathcal{H}\Psi K$$

■

Proof of Lemma 7.1: From Eq. (3.44) and Eq. (3.45), the expression for the generalized accelerations $\dot{\chi}$ is given by

$$\begin{aligned}\dot{\chi} &= \mathcal{M}^{-1}(T - \mathcal{C}) \\ &= [I - \mathcal{H}\Psi K]^* D^{-1} [I - \mathcal{H}\Psi K] [T - \mathcal{H}\Phi [M_m \Phi^* a_m + b_m + K_m \vartheta]]\end{aligned}\quad (\text{A.5})$$

From Eq. (A.4) we have that

$$[I - \mathcal{H}\Psi K] \mathcal{H}\Phi = \mathcal{H}\Psi [\Psi^{-1} - K\mathcal{H}] \Phi = \mathcal{H}\Psi \quad (\text{A.6})$$

Thus Eq. (A.5) can be written as

$$\dot{\chi} = [I - \mathcal{H}\Psi K]^* D^{-1} [T - \mathcal{H}\Psi [KT + M_m \Phi^* a_m + b_m + K_m \vartheta]] \quad (\text{A.7})$$

From Eq. (A.2) it follows that

$$M_m = P - \mathcal{E}_\Psi P \mathcal{E}_\Phi^* \quad \Rightarrow \quad \Psi M_m \Phi^* = \Psi P + P \tilde{\Phi}^* \quad (\text{A.8})$$

and so Eq. (A.7) simplifies to

$$\dot{\chi} = [I - \mathcal{H}\Psi K]^* D^{-1} [T - \mathcal{H}\Psi [KT + P a_m + b_m + K_m \vartheta] - \mathcal{H} P \tilde{\Phi}^* a_m] \quad (\text{A.9})$$

From Eq. (A.4) we have that

$$[I - \mathcal{H}\Psi K]^* D^{-1} \mathcal{H} P \tilde{\Phi}^* = [I - \mathcal{H}\Psi K]^* K^* \tilde{\Phi}^* = K^* \Psi^* [\Psi^{-*} - K\mathcal{H}]^* \tilde{\Phi}^* = K^* \Psi^* \quad (\text{A.10})$$

Using this in Eq. (A.9) leads to the result. ■

Appendix B: Expressions for $M_m(k)$, $a_m(k)$ and $b_m(k)$

The modal spatial displacement influence vector $\Pi^j(k)$ for node j_k has the structure:

$$\Pi^j(k) = \begin{pmatrix} \lambda^j(k) \\ \gamma^j(k) \end{pmatrix} \in \mathcal{R}^{6 \times n_m(k)} \quad (\text{B.1})$$

The components of the vectors $\lambda^j(k) \in \mathcal{R}^{3 \times n_m(k)}$ and $\gamma^j(k) \in \mathcal{R}^{3 \times n_m(k)}$ are the *modal slope displacement influence vector* and the *modal translational displacement influence vector* re-

spectively for node j_k . They define the contribution of the various modes to the slope (or differential change in orientation) and translational deformation for the j_k^{th} node on the k^{th} body. Define

$$\delta_\omega(j_k) \triangleq \lambda^j(k)\dot{\eta}(k) \in \mathcal{R}^3, \quad \delta_v(j_k) \triangleq \gamma^j(k)\dot{\eta}(k) \in \mathcal{R}^3, \quad \text{and} \quad \delta_l(j_k) \triangleq \gamma^j(k)\eta(k) \in \mathcal{R}^3 \quad (\text{B.2})$$

Note that

$$l(k, j_k) = l_0(j_k) + \delta_l(j_k)$$

where $l_0(j_k)$ denotes the undeformed vector from frame \mathcal{F}_k to node j_k . Recall from Eq. (3.1) that

$$M_s(j_k) = \begin{pmatrix} \mathcal{J}(j_k) & m(j_k)\tilde{p}(j_k) \\ -m(j_k)\tilde{p}(j_k) & m(j_k)I \end{pmatrix} \in \mathcal{R}^{6 \times 6} \quad (\text{B.3})$$

B.1 Modal Integrals for the Individual Bodies

Defined below are a set of modal integrals for the k^{th} body which simplify the computation of the modal mass matrix $M_m(k)$ and the bias vector $b_m(k)$. These modal integrals can be computed as a part of the finite-element structural analysis of the individual bodies.

$$\begin{aligned} m(k) &\triangleq \sum_{j=1_k}^{n_s(k)} m(j_k) \\ p_0^k &\triangleq [1/m(k)] \sum_{j=1_k}^{n_s(k)} m(j_k)[p(j_k) + l_0(k, j_k)] \in \mathcal{R}^3 \\ p_1^k(r) &\triangleq [1/m(k)] \sum_{j=1_k}^{n_s(k)} m(j_k)\gamma_r^j(k) \in \mathcal{R}^3 \\ E^k(r) &\triangleq \sum_{j=1_k}^{n_s(k)} m(j_k)[\gamma_r^j(k) - \tilde{p}(j_k)\lambda_r^j(k)] \in \mathcal{R}^3 \\ F_0^k(r) &\triangleq \sum_{j=1_k}^{n_s(k)} \mathcal{J}(j_k)\lambda_r^j(k) + m(j_k)[\tilde{l}_0(k, j_k) + \tilde{p}(j_k)]\gamma_r^j(k) - m(j_k)\tilde{l}_0(k, j_k)\tilde{p}(j_k)\lambda_r^j(k) \in \mathcal{R}^3 \\ F_1^k(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} m(j_k)\tilde{\gamma}_s^j(k)[\gamma_r^j(k) - \tilde{p}(j_k)\lambda_r^j(k)] \in \mathcal{R}^3 \end{aligned}$$

$$\begin{aligned}
G^k(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} [\lambda_r^j(k)]^* \mathcal{J}(j_k) \lambda_s^j(k) + m(j_k) [\lambda_r^j(k)]^* \tilde{p}(j_k) \gamma_s^j(k) + m(j_k) [\lambda_s^j(k)]^* \tilde{p}(j_k) \gamma_r^j(k) + \\
&\quad m(j_k) [\gamma_r^j(k)]^* \gamma_s^j(k) \in \mathcal{R}^1 \\
\mathcal{J}_0^k &\triangleq - \sum_{j=1_k}^{n_s(k)} \mathcal{J}(j_k) - m(j_k) [\tilde{l}_0(k, j_k) \tilde{l}_0(k, j_k) + \tilde{p}(j_k) \tilde{l}_0(k, j_k) + \tilde{l}_0(k, j_k) \tilde{p}(j_k)] \in \mathcal{R}^{3 \times 3} \\
\mathcal{J}_1^k(r) &\triangleq - \sum_{j=1_k}^{n_s(k)} m(j_k) \tilde{\gamma}_r^j(k) [\tilde{l}_0(k, j_k) + \tilde{p}(j_k)] \in \mathcal{R}^{3 \times 3} \\
\mathcal{J}_2^k(r, s) &\triangleq - \sum_{j=1_k}^{n_s(k)} m(j_k) \tilde{\gamma}_r^j(k) \tilde{\gamma}_s^j(k) \in \mathcal{R}^{3 \times 3} \\
S_1^k(r) &\triangleq \sum_{j=1_k}^{n_s(k)} [m(j_k) \tilde{p}(j_k) \lambda_r^j(k)]^x \tilde{l}_0(k, j_k) - \mathcal{J}(j_k) \tilde{\lambda}_r^j(k) \in \mathcal{R}^{3 \times 3} \\
S_2^k(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} [m(j_k) \tilde{p}(j_k) \lambda_r^j(k)]^x \tilde{\gamma}_s^j(k) \in \mathcal{R}^{3 \times 3} \\
K_1^j(r) &\triangleq \sum_{j=1_k}^{n_s(k)} 2 \tilde{l}_0(k, j_k) [m(j_k) \tilde{p}(j_k) \lambda_r^j(k)]^x - \mathcal{J}(j_k) \tilde{\lambda}_r^j(k) + \tilde{\lambda}_r^j(k) \mathcal{J}(j_k) \in \mathcal{R}^{3 \times 3} \\
&= 2[S_1^k(r)]^* - \sum_{j=1_k}^{n_s(k)} [\tilde{\lambda}_r^j(k) \mathcal{J}(j_k) + \mathcal{J}(j_k) \tilde{\lambda}_r^j(k)] \\
K_2^j(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} 2 \tilde{\gamma}_s^j(k) [m(j_k) \tilde{p}(j_k) \lambda_r^j(k)]^x = 2[S_2^k(r, s)]^* \in \mathcal{R}^{3 \times 3} \\
R_1^k(r) &\triangleq \sum_{j=1_k}^{n_s(k)} \mathcal{J}(j_k) \lambda_r^j(k) \in \mathcal{R}^3 \\
R_2^k(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} [\tilde{\lambda}_r^j(k) \mathcal{J}(j_k) - m(j_k) \tilde{l}_0(k, j_k) \tilde{\lambda}_r^j(k) \tilde{p}(j_k)] \lambda_s^j(k) \in \mathcal{R}^3 \\
R_3^k(q, r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} -m(j_k) \tilde{\gamma}_q^j(k) \tilde{\lambda}_r^j(k) \tilde{p}(j_k) \lambda_s^j(k) \in \mathcal{R}^3 \\
W_1^k(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} \tilde{\lambda}_r^j(k) m(j_k) \tilde{p}(j_k) \gamma_s^j(k) \in \mathcal{R}^3
\end{aligned}$$

$$\begin{aligned}
W_2^k(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} \tilde{\lambda}_r^j(k) \mathcal{J}(j_k) \lambda_s^j(k) \in \mathcal{R}^3 \\
L^k(r, s) &\triangleq -[1/m(k)] \sum_{j=1_k}^{n_s(k)} m(j_k) \tilde{\lambda}_r^j(k) \tilde{p}(j_k) \lambda_s^j(k) \in \mathcal{R}^3 \\
T_1^k(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} [m(j_k) \tilde{\gamma}_r^j(k) \tilde{p}(j_k) + \mathcal{J}(j_k) \tilde{\lambda}_r^j(k)] \lambda_s^j(k) \in \mathcal{R}^3 \\
T_2^k(r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} [m(j_k) \tilde{p}(j_k) \tilde{\gamma}_r^j(k) - \tilde{\lambda}_r^j(k) \mathcal{J}(j_k)] \lambda_s^j(k) \in \mathcal{R}^3 \\
T_3^k(q, r, s) &\triangleq \sum_{j=1_k}^{n_s(k)} [\lambda_q^j(k)]^* [m(j_k) \tilde{\gamma}_r^j(k) \tilde{p}(j_k) + \mathcal{J}(j_k) \tilde{\lambda}_r^j(k)] \lambda_s^j(k) \in \mathcal{R}^1
\end{aligned} \tag{B.4}$$

Note that

$$G^k(r, s) = G^k(s, r) \quad \text{and} \quad \mathcal{J}_2^k(r, s) = \mathcal{J}_2^k(s, r)$$

Also define,

$$\begin{aligned}
p(k) &\triangleq p_0^k + \sum_{s=1}^{n_m(k)} p_1(s) \eta(s) \in \mathcal{R}^3 \\
F^k(r) &\triangleq F_0^k(r) + \sum_{s=1}^{n_m(k)} F_1^k(r, s) \eta(s) \in \mathcal{R}^3 \\
N^k(r) &\triangleq [\mathcal{J}_1^k(r) + \sum_{s=1}^{n_m(k)} \mathcal{J}_2^k(r, s) \eta(s)]^* \in \mathcal{R}^{3 \times 3} \\
\mathcal{J}(k) &\triangleq \mathcal{J}_0^k + \sum_{r=1}^{n_m(k)} [\mathcal{J}_1^k(r) + \{\mathcal{J}_1^k(r)\}^*] \eta(r) + \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} \mathcal{J}_2^k(r, s) \eta(r) \eta(s) \in \mathcal{R}^{3 \times 3} \\
S^k(r) &\triangleq S_1^k(r) + \sum_{s=1}^{n_m(k)} S_2^k(r, s) \eta(s) \in \mathcal{R}^{3 \times 3} \\
K^k(r) &\triangleq K_1^k(r) + \sum_{s=1}^{n_m(k)} K_2^k(r, s) \eta(s) \in \mathcal{R}^{3 \times 3} \\
R^k(r, s) &\triangleq R_2^k(r, s) + \sum_{q=1}^{n_m(k)} R_3^k(q, r, s) \eta(q) \in \mathcal{R}^3
\end{aligned} \tag{B.5}$$

B.2 Modal Mass Matrix

We have from Eq. (3.25) that the modal mass matrix of the k^{th} body is given by

$$\begin{aligned}
 M_m(k) &= \begin{pmatrix} \Pi^*(k) \\ B(k) \end{pmatrix} M_s(k) [\Pi(k), B^*(k)] = \begin{pmatrix} \Pi^*(k)M_s(k)\Pi(k) & \Pi^*(k)M_s(k)B^*(k) \\ B(k)M_s(k)\Pi(k) & B(k)M_s(k)B^*(k) \end{pmatrix} \\
 &= \begin{pmatrix} M_m^{ff}(k) & M_m^{fr}(k) \\ M_m^{rf}(k) & M_m^{rr}(k) \end{pmatrix} \in \mathcal{R}^{\bar{\mathcal{N}}(k) \times \bar{\mathcal{N}}(k)} \quad (\text{B.6})
 \end{aligned}$$

Define the matrices:

$$\begin{aligned}
 p_1^k &\triangleq [p_1^k(1), \dots, p_1^k(n_m(k))] \in \mathcal{R}^{3 \times n_m(k)} \\
 F_0^k &\triangleq [F_0^k(1), \dots, F_0^k(n_m(k))] \in \mathcal{R}^{3 \times n_m(k)} \\
 F^k &\triangleq [F^k(1), \dots, F^k(n_m(k))] \in \mathcal{R}^{3 \times n_m(k)} \\
 E^k &\triangleq [E^k(1), \dots, E^k(n_m(k))] \in \mathcal{R}^{3 \times n_m(k)} \quad (\text{B.7})
 \end{aligned}$$

Also define the matrix $G^k \in \mathcal{R}^{n_m(k) \times n_m(k)}$ so that its $(r, s)^{th}$ element is given by the modal integral $G^k(r, s)$.

Using these matrices, and Eq. (B.6), it is easy to establish that

$$M_m^{ff}(k) = G^k, \quad M_m^{rf}(k) = \begin{pmatrix} F^k \\ E^k \end{pmatrix}, \quad \text{and} \quad M_m^{rr}(k) = \begin{pmatrix} \mathcal{J}(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & m(k)I \end{pmatrix}$$

Hence, in block partitioned form

$$M_m(k) = \begin{pmatrix} G^k & [F^k]^* & [E^k]^* \\ F^k & \mathcal{J}(k) & m(k)\tilde{p}(k) \\ E^k & -m(k)\tilde{p}(k) & m(k)I \end{pmatrix}$$

$$\begin{aligned}
&= \underbrace{\begin{pmatrix} G^k & [F_0^k]^* & [E^k]^* \\ \hline F_0^k & \mathcal{J}_0^k & m(k)\tilde{p}_0^k \\ \hline E^k & -m(k)\tilde{p}_0^k & m(k)I \end{pmatrix}}_{M_m^0(k)} \\
&+ \underbrace{\begin{pmatrix} 0 & [F_1^k\eta]^* & 0 \\ \hline F_1^k\eta & \sum_{r=1}^{n_m(k)} [\mathcal{J}_1^k(r) + [\mathcal{J}_1^k(r)]^*]\eta(r) & m(k)[p_1^k\eta(k)]^x \\ \hline 0 & -m(k)[p_1^k\eta(k)]^x & 0 \end{pmatrix}}_{M_m^1(k)} \\
&+ \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ \hline 0 & \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} \mathcal{J}_2^k(r,s)\eta(r)\eta(s) & 0 \\ \hline 0 & 0 & 0 \end{pmatrix}}_{M_m^2(k)} \tag{B.8}
\end{aligned}$$

The superscript $i = 0, 1, 2$ in $M_m^i(k)$ denotes the order of dependency of the terms on the deformation variables.

B.3 Expression for $a_m(k)$

In this section we derive explicit expressions for the Coriolis and centrifugal acceleration term $a_m(k)$. Since

$$\dot{\phi}(x, y) = \begin{pmatrix} 0 & \tilde{l}(x, y) \\ 0 & 0 \end{pmatrix}$$

it follows from Eq. (3.14) and Eq. (B.1) that

$$\begin{aligned} \dot{\Phi}(k+1, k) &= \begin{pmatrix} 0 & [\Pi^t(k+1)]^* \dot{\phi}(t_{k+1}, k) + [\dot{\Pi}^t(k+1)]^* \phi(t_{k+1}, k) \\ 0 & \dot{\phi}(k+1, k) \end{pmatrix} \\ &= \begin{pmatrix} 0 & [\dot{\lambda}^t(k+1)]^* & [\dot{\lambda}^t(k+1)]^* \tilde{l}(t_{k+1}, k) + [\dot{\gamma}^t(k+1)]^* + [\lambda^t(k+1)]^* \tilde{l}(t_{k+1}, k) \\ 0 & 0 & \tilde{l}(k+1, k) \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Recalling that the spatial velocity of frame \mathcal{F}_k is

$$V(k) = \begin{pmatrix} \omega(k) \\ v(k) \end{pmatrix}$$

where $\omega(k)$ and $v(k)$ denote the angular and linear velocity respectively of \mathcal{F}_k we have that

$$\dot{\Pi}^j(k) = \begin{pmatrix} \dot{\lambda}^j(k) \\ \dot{\gamma}^j(k) \end{pmatrix} = \begin{pmatrix} \tilde{\omega}(k) \lambda^j(k) \\ \tilde{\omega}(k) \gamma^j(k) \end{pmatrix}$$

And thus

$$\dot{\Phi}^*(k+1, k)V_m(k+1) = \begin{pmatrix} 0 \\ \hline \tilde{\omega}(k+1)\delta_\omega(t_{k+1}) \\ \hline -\tilde{l}(t_{k+1}, k)\tilde{\omega}(k+1)\delta_\omega(t_{k+1}) + \tilde{\omega}(k+1)\delta_v(t_{k+1}) \\ +\tilde{\delta}_\omega(t_{k+1})\dot{l}(t_{k+1}, k) + \tilde{\omega}(k+1)\dot{l}(k+1, k) \end{pmatrix}$$

The vector above has been partitioned so that the term on the top corresponds to modal accelerations, the term in the middle to the angular acceleration and the term at the bottom to the linear acceleration of the body. Also

$$\dot{l}(k+1, j_{k+1}) = \tilde{\omega}(k+1)l(k+1, j_{k+1}) + \delta_v(j_{k+1})$$

and

$$\begin{aligned} \dot{l}(t_{k+1}, k) &= \tilde{\omega}(t_{k+1})l(t_{k+1}, k) + \Delta_v(k) - \delta_v(d_k) + \tilde{\Delta}_\omega(k)l(\mathcal{O}_k, k) \\ &= [\tilde{\omega}(k+1) + \tilde{\delta}_\omega(t_{k+1})]l(t_{k+1}, k) + \Delta_v(k) - \delta_v(d_k) + \tilde{\Delta}_\omega(k)l(\mathcal{O}_k, k) \end{aligned}$$

where

$$\Delta_v(k) = \begin{pmatrix} \Delta_\omega(k) \\ \Delta_v(k) \end{pmatrix} = H^*(k)\beta(k) \quad (\text{B.9})$$

Thus

$$\begin{aligned} \dot{l}(k+1, k) &= \dot{l}(k+1, t_{k+1}) + \dot{l}(t_{k+1}, k) \\ &= \tilde{\omega}(k+1)l(k+1, k) + \delta_v(t_{k+1}) + \tilde{\delta}_\omega(t_{k+1})l(t_{k+1}, k) + \Delta_v(k) \\ &\quad - \delta_v(d_k) + \tilde{\Delta}_\omega(k)l(\mathcal{O}_k, k) \end{aligned}$$

Also

$$\dot{\mathcal{H}}^*(k) \triangleq \begin{pmatrix} 0 & 0 \\ -\dot{\Pi}^d(k) & \dot{H}_{\mathcal{F}}^*(k) \end{pmatrix}$$

and

$$\begin{aligned} \dot{H}^*(k) &= \begin{pmatrix} \tilde{\omega}(\mathcal{O}_k) & 0 \\ 0 & \tilde{\omega}(\mathcal{O}_k) \end{pmatrix} H^*(k) \\ \dot{l}(\mathcal{O}_k, k) &= \tilde{\omega}(\mathcal{O}_k) l(\mathcal{O}_k, k) - \delta_v(d_k) \end{aligned}$$

Thus we have that

$$\dot{\mathcal{H}}^*(k)\chi(k) = \begin{pmatrix} 0 \\ \hline \tilde{\omega}(\mathcal{O}_k)\Delta_\omega(k) - \tilde{\omega}(k)\delta_\omega(d_k) \\ \hline \tilde{\omega}(\mathcal{O}_k)\Delta_v(k) - \tilde{\omega}(k)\delta_v(d_k) + \tilde{\Delta}_\omega(k)\dot{l}(\mathcal{O}_k, k) - \tilde{l}(\mathcal{O}_k, k)\tilde{\omega}(\mathcal{O}_k)\Delta_\omega(k) \end{pmatrix}$$

From Eq. (3.27) and the above expressions it follows that

$$a_m(k) = \frac{d\Phi^*(k+1, k)}{dt} V_m(k+1) + \frac{d\mathcal{H}^*(k)}{dt} \chi(k) = \begin{pmatrix} 0 \\ \hline a_{mR}(k) \end{pmatrix} \quad (\text{B.10})$$

where

$$a_{mR}(k) \triangleq \begin{pmatrix} \tilde{\omega}(k+1)\delta_\omega(t_{k+1}) + \tilde{\omega}(k)\Delta_\omega(k) - \tilde{\omega}(\mathcal{O}_k)\delta_\omega(d_k) \\ \hline \tilde{\omega}(k+1)[\tilde{\omega}(k+1)l(k+1, k) + 2\delta_v(t_{k+1})] + [\tilde{\omega}(t_{k+1}) + \tilde{\omega}(\mathcal{O}_k)][v(k) - v(\mathcal{O}_k^+)] \\ + [\tilde{\omega}(k+1) + \tilde{\omega}(t_{k+1})]\tilde{\delta}_\omega(t_{k+1})l(t_k, k-1) \\ - [2\tilde{\Delta}_\omega(k) - \delta_\omega(d_k)]\delta_v(d_k) \end{pmatrix}$$

$$\begin{aligned}
&= \left(\frac{\tilde{\omega}(k)\Delta_\omega(k)}{\tilde{\omega}(k+1)\tilde{\omega}(k+1)l_0(k+1, k) + [\tilde{\omega}(k+1) + \tilde{\omega}(k)][v(k) - v(\mathcal{O}_k^+)]} \right) \\
&\quad \underbrace{\hspace{10em}}_{a_{mR}^0(k)} \\
&+ \left(\frac{\tilde{\omega}(k+1)\delta_\omega(t_{k+1}) - \tilde{\omega}(k)\delta_\omega(d_k)}{\tilde{\omega}(k+1)[\tilde{\omega}(k+1)[\delta_l(t_{k+1}) - \delta_l(d_k)] + 2\delta_v(t_{k+1})] + [\tilde{\delta}_\omega(t_{k+1}) - \tilde{\delta}_\omega(d_k)][v(k) - v(\mathcal{O}_k^+)] + 2\tilde{\omega}(k+1)\tilde{\delta}_\omega(t_{k+1})l_0(t_{k+1}, k) - 2\tilde{\Delta}_\omega(k)\delta_v(d_k)} \right) \\
&\quad \underbrace{\hspace{10em}}_{a_{mR}^1(k)} \\
&+ \left(\frac{0}{2\tilde{\omega}(k+1)\tilde{\delta}_\omega(t_{k+1})\delta_l(t_{k+1}) + \tilde{\delta}_\omega(t_{k+1})\tilde{\delta}_\omega(t_{k+1})l_0(t_k, k-1) + \delta_\omega(d_k)\delta_v(d_k)} \right) \\
&\quad \underbrace{\hspace{10em}}_{a_{mR}^2(k)} \\
&+ \left(\frac{0}{\tilde{\delta}_\omega(t_{k+1})\tilde{\delta}_\omega(t_{k+1})\delta_l(t_k, k-1)} \right) \\
&\quad \underbrace{\hspace{10em}}_{a_{mR}^3(k)} \tag{B.11}
\end{aligned}$$

In the above, $a_{mR}^0(k)$ denotes the deformation independent part of the Coriolis acceleration, while $a_{mR}^1(k)$, $a_{mR}^2(k)$ and $a_{mR}^3(k)$ denote the parts whose dependency on the deformation is up to first, second and third order respectively.

B.4 Expression for $b_m(k)$

We have from Eq. (3.30) that

$$a(j_k) = \frac{d[\Pi^j(k), \phi^*(k, j_k)]}{dt} V_m(k) = \dot{\Pi}^j(k)\dot{\eta}(k) + \dot{\phi}^*(k, j_k)V(k)$$

Since,

$$l(k, j_k) = \tilde{\omega}(k)l(k, j_k) + \delta_v(j_k)$$

it follows that

$$a(j_k) = \left(\frac{\tilde{\omega}(k)\delta_\omega(j_k)}{\tilde{\omega}(k)[\tilde{\omega}(k)l(k, j_k) + 2\delta_v(j_k)]} \right)$$

Also from Eq. (3.33) we have that

$$b(j_k) = \left(\frac{\tilde{\omega}(j_k)\mathcal{J}(j_k)\omega(j_k)}{m(j_k)\tilde{\omega}(j_k)\tilde{\omega}(j_k)p(j_k)} \right)$$

Thus,

$$\begin{aligned} & b(j_k) + M_s(j_k)a(j_k) \\ &= \left(\frac{\tilde{\omega}(j_k)\mathcal{J}(j_k)\omega(j_k) + \mathcal{J}(j_k)\tilde{\omega}(k)\delta_\omega(j_k) + m(j_k)\tilde{p}(j_k)\tilde{\omega}(k)[\tilde{\omega}(k)l(k, j_k) + 2\delta_v(j_k)]}{m(j_k)\left\{ -\tilde{p}(j_k)\tilde{\omega}(k)\delta_\omega(j_k) + \tilde{\delta}_\omega(j_k)\tilde{\delta}_\omega(j_k)p(j_k) + \tilde{\delta}_\omega(j_k)\tilde{\omega}(k)p(j_k) \right.} \right. \\ & \quad \left. \left. + \tilde{\omega}(k)[\tilde{\omega}(k)\{l(k, j_k) + p(j_k)\} + \tilde{\delta}_\omega(j_k)p(j_k) + 2\delta_v(j_k)] \right\}} \right) \end{aligned} \tag{B.12}$$

From Eq. (3.39) we write

$$b_m(k) = \begin{pmatrix} \Pi^*(k) \\ B(k) \end{pmatrix} [b(k) + M_s(k)a(k)] \triangleq \begin{pmatrix} b_\eta^k(1) \\ \vdots \\ b_\eta^k(n_m(k)) \\ \hline b_\omega^k \\ \hline b_\nu^k \end{pmatrix} \quad (\text{B.13})$$

We develop expressions for $b_\eta^k(r)$, b_ω^k and b_ν^k in Eq. (B.13) below. From Eq. (B.12) and Eq. (B.13) we have that

$$\begin{aligned} b_\eta^k(r) &= \sum_{j=1_k}^{n_s(k)} -\omega^*(k) \tilde{\lambda}_r^j(k) \mathcal{J}(j_k) \omega(k) - \omega^*(k) \tilde{\lambda}_r^j(k) \mathcal{J}(j_k) \delta_\omega(j_k) \\ &\quad - \delta_\omega(j_k) \tilde{\lambda}_r^j(k) \mathcal{J}(j_k) \omega(k) \\ &\quad - \delta_\omega(j_k) \tilde{\lambda}_r^j(k) \mathcal{J}(j_k) \delta_\omega(j_k) - [\lambda_r^j(k)]^* \mathcal{J}(j_k) \tilde{\delta}_\omega(j_k) \omega(k) \\ &\quad + \omega^*(k) [m(j_k) \tilde{p}(j_k) \lambda_r^j(k)]^\times [-\tilde{l}(k, j_k) \omega(k) + 2\delta_\nu(j_k)] \\ &\quad + m(j_k) [\gamma_r^j(k)]^* \tilde{p}(j_k) \tilde{\delta}_\omega(j_k) \omega(k) \\ &\quad - m(j_k) \omega^*(k) \tilde{\gamma}_r^j(k) [-\{\tilde{l}(k, j_k) + \tilde{p}(j_k)\} \omega(k) + 2\delta_\nu(j_k)] \\ &\quad + m(j_k) [\gamma_r^j(k)]^* \{\tilde{\delta}_\omega(j_k) \tilde{\delta}_\omega(j_k) p(j_k) + \tilde{\delta}_\omega(j_k) \omega(k) p(j_k) + \tilde{\omega}(k) \tilde{\delta}_\omega(j_k) p(j_k)\} \\ &= \sum_{j=1_k}^{n_s(k)} -\omega^*(k) \tilde{\lambda}_r^j(k) \mathcal{J}(j_k) \omega(k) \end{aligned} \quad (\text{B.14})$$

$$-\omega^*(k) [m(j_k) \tilde{p}(j_k) \lambda_r^j(k)]^\times \tilde{l}(k, j_k) \omega(k) \quad (\text{B.15})$$

$$+\omega^*(k) m(j_k) \tilde{\gamma}_r^j(k) \{\tilde{l}(k, j_k) + \tilde{p}(j_k)\} \omega(k) \quad (\text{B.16})$$

$$+2\omega^*(k) [m(j_k) \tilde{p}(j_k) \lambda_r^j(k)]^\times \delta_\nu(j_k) \quad (\text{B.17})$$

$$-2\omega^*(k) m(j_k) \tilde{\gamma}_r^j(k) \delta_\nu(j_k) \quad (\text{B.18})$$

$$-[\lambda_r^j(k)]^* \mathcal{J}(j_k) \tilde{\delta}_\omega(j_k) \omega(k) \quad (\text{B.19})$$

$$+m(j_k)[\gamma_r^j(k)]^* \tilde{p}(j_k) \tilde{\delta}_\omega(j_k) \omega(k) \quad (\text{B.20})$$

$$+\delta_\omega(j_k)^* \mathcal{J}(j_k) \tilde{\lambda}_r^j(k) \omega(k) \quad (\text{B.21})$$

$$-\delta_\omega(j_k)^* \tilde{\lambda}_r^j(k) \mathcal{J}(j_k) \omega(k) \quad (\text{B.22})$$

$$+m(j_k) \delta_\omega(j_k)^* \tilde{\gamma}_r^j(k) \tilde{p}(j_k) \delta_\omega(j_k) \quad (\text{B.23})$$

$$+\delta_\omega(j_k)^* \mathcal{J}(j_k) \tilde{\lambda}_r^j(k) \delta_\omega(j_k) \quad (\text{B.24})$$

$$+m(j_k)[\gamma_r^j(k)]^* \tilde{\delta}_\omega(j_k) \tilde{\omega}(k) p(j_k) \quad (\text{B.25})$$

$$+m(j_k)[\gamma_r^j(k)]^* \tilde{\omega}(k) \tilde{\delta}_\omega(j_k) p(j_k) \quad (\text{B.26})$$

Using the modal integrals defined in Section B.1, the above terms can be expressed in the following manner:

$$\frac{1}{2}[B.17] + B.22 = -\omega^*(k) \sum_{s=1}^{n_m(k)} T_1^k(s, r) \dot{\eta}(s)$$

$$\frac{1}{2}[B.17] + B.19 + B.21 = -\omega^*(k) \sum_{s=1}^{n_m(k)} [T_2^k(s, r) + W_1^k(r, s) + W_2^k(s, r)] \dot{\eta}(s)$$

$$B.14 + B.15 = -\omega^*(k) S^k(r) \omega(k)$$

$$B.16 = -\omega^*(k) N^k(r) \omega(k) \quad (\text{B.27})$$

$$B.23 + B.24 = \sum_{q=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} T_3^k(q, r, s) \dot{\eta}(q) \dot{\eta}(s)$$

$$B.20 + B.25 = B.26$$

$$B.18 + B.20 + B.25 + B.26 = -2\omega^*(k) \sum_{s=1}^{n_m(k)} F_1^k(s, r) \dot{\eta}(s)$$

Using these, it follows that

$$\begin{aligned} b_\eta^k &= -\omega^*(k) [S^k(r) + N^k(r)] \omega(j_k) + \sum_{q=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} T_3^k(q, r, s) \dot{\eta}(q) \dot{\eta}(s) \\ &\quad - \omega^*(k) \sum_{j=s}^{n_m(k)} [T_1^k(s, r) + T_2^k(s, r) + W_1^k(r, s) + W_2^k(s, r) + 2F_1^k(s, r)] \dot{\eta}(s) \\ &= -\omega^*(k) [S^k(r) + N^k(r)] \omega(j_k) - \omega^*(k) \sum_{j=s}^{n_m(k)} Q^k(r, s) \dot{\eta}(s) + \sum_{q=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} T_3^k(q, r, s) \dot{\eta}(q) \dot{\eta}(s) \end{aligned}$$

(B.28)

where

$$Q^k(r, s) \triangleq T_1^k(s, r) + T_2^k(s, r) + W_1^k(r, s) + W_2^k(s, r) + 2F_1^k(s, r) \quad (\text{B.29})$$

Once again from Eq. (B.12) and Eq. (B.13) we have that

$$\begin{aligned} b_\omega^k &= \sum_{j=1_k}^{n_s(k)} \tilde{\omega}(j_k) \mathcal{J}(j_k) \omega(j_k) + \mathcal{J}(j_k) \tilde{\omega}(k) \delta_\omega(j_k) + m(j_k) \tilde{p}(j_k) \tilde{\omega}(k) [\tilde{\omega}(k) l(k, j_k) + 2\delta_v(j_k)] \\ &\quad + m(j_k) \tilde{l}(j_k) \left\{ -\tilde{p}(j_k) \tilde{\omega}(k) \delta_\omega(j_k) + \tilde{\omega}(k) [\tilde{\omega}(k) \{l(k, j_k) + p(j_k)\} + 2\delta_v(j_k)] \right. \\ &\quad \left. + \tilde{\delta}_\omega(j_k) \tilde{\delta}_\omega(j_k) p(j_k) + \tilde{\delta}_\omega(j_k) \tilde{\omega}(k) p(j_k) + \tilde{\omega}(k) \tilde{\delta}_\omega(j_k) p(j_k) \right\} \\ &= \sum_{j=1_k}^{n_s(k)} \tilde{\omega}(k) \left[\mathcal{J}(j_k) - m(j_k) (\tilde{p}(j_k) \tilde{l}(k, j_k) + \tilde{l}(j_k) \tilde{p}(k, j_k) + \tilde{l}(j_k) \tilde{l}(k, j_k)) \right] \omega(k) \quad (\text{B.30}) \end{aligned}$$

$$-2m(j_k) [\tilde{l}(j_k) + \tilde{p}(j_k)] \tilde{\delta}_v(j_k) \omega(k) \quad (\text{B.31})$$

$$-\mathcal{J}(j_k) \tilde{\delta}_\omega(j_k) \omega(k) \quad (\text{B.32})$$

$$+\tilde{\delta}_\omega(j_k) \mathcal{J}(k) \omega(k) \quad (\text{B.33})$$

$$+\tilde{l}(j_k) m(j_k) \tilde{p}(j_k) \tilde{\delta}_\omega(j_k) \omega(k) \quad (\text{B.34})$$

$$+\tilde{\omega}(k) \mathcal{J}(j_k) \delta_\omega(j_k) \quad (\text{B.35})$$

$$+\tilde{\delta}_\omega(j_k) \mathcal{J}(j_k) \delta_\omega(j_k) \quad (\text{B.36})$$

$$+\tilde{l}(j_k) m(j_k) \tilde{\delta}_\omega(j_k) \tilde{\delta}_\omega(j_k) p(j_k) \quad (\text{B.37})$$

$$+m(j_k) \tilde{l}(j_k) \tilde{\delta}_\omega(j_k) \tilde{\omega}(k) p(j_k) \quad (\text{B.38})$$

$$+m(j_k) \tilde{l}(j_k) \tilde{\omega}(k) \tilde{\delta}_\omega(j_k) p(j_k) \quad (\text{B.39})$$

Once again, using modal integrals, the above terms can be reexpressed in the following manner:

$$B.30 = \omega^*(k) \mathcal{J}(k) \omega(k)$$

$$B.31 = \left[2 \sum_{r=1}^{n_m(k)} N^k(r) \dot{\eta}(r) \right] \omega(k)$$

$$B.32 + B.33 + B.34 + B.38 + B.39 = \sum_{r=1}^{n_m(k)} K^k(r) \dot{\eta}(r) \omega(k)$$

$$B.35 = \tilde{\omega}(k) \sum_{r=1}^{n_m(k)} R_1^k(r) \dot{\eta}(r) \quad (\text{B.40})$$

$$B.36 + B.37 = \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} R^k(r, s) \dot{\eta}(r) \dot{\eta}(s)$$

This results in the following expression

$$\begin{aligned} b_\omega^k &= \tilde{\omega}(k) \mathcal{J}(k) \omega(k) + \sum_{r=1}^{n_m(k)} [2N^k(r) + K^k(r)] \dot{\eta}(r) \omega(k) \\ &\quad + \tilde{\omega}(k) \sum_{r=1}^{n_m(k)} R_1^k(r) \dot{\eta}(r) + \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} R^k(r, s) \dot{\eta}(r) \dot{\eta}(s) \end{aligned} \quad (\text{B.41})$$

Using Eq. (B.12) and Eq. (B.13) it also follows that

$$\begin{aligned} b_v^k &= \sum_{j=1_k}^{n_s(k)} -m(j_k) \tilde{p}(j_k) \tilde{\omega}(k) \delta_\omega(j_k) + m(j_k) \tilde{\omega}(k) [\tilde{\omega}(k) \{l(k, j_k) + p(j_k)\} + 2\delta_v(j_k)] \\ &\quad + m(j_k) \tilde{\delta}_\omega(j_k) \tilde{\delta}_\omega(j_k) p(j_k) + m(j_k) \tilde{\delta}_\omega(j_k) \omega(k) p(j_k) + m(j_k) \tilde{\omega}(k) \tilde{\delta}_\omega(j_k) p(j_k) \\ &= \sum_{j=1_k}^{n_s(k)} -m(j_k) \tilde{p}(j_k) \tilde{\omega}(k) \delta_\omega(j_k) \end{aligned} \quad (\text{B.42})$$

$$+ m(j_k) \tilde{\omega}(k) \tilde{\omega}(k) \{l(k, j_k) + p(j_k)\} \quad (\text{B.43})$$

$$+ m(j_k) \tilde{\delta}_\omega(j_k) \tilde{\delta}_\omega(j_k) p(j_k) \quad (\text{B.44})$$

$$+ 2m(j_k) \tilde{\omega}(k) \delta_v(j_k) \quad (\text{B.45})$$

$$+ m(j_k) \tilde{\delta}_\omega(j_k) \tilde{\omega}(k) p(j_k) \quad (\text{B.46})$$

$$+ m(j_k) \tilde{\omega}(k) \tilde{\delta}_\omega(j_k) p(j_k) \quad (\text{B.47})$$

Using the modal integrals we have that

$$B.43 = m(k) \tilde{\omega}(k) \tilde{\omega}(k) p(k)$$

$$B.44 = m(k) \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} L(r, s) \dot{\eta}(r) \dot{\eta}(s)$$

$$B.42 + B.45 + B.46 + B.47 = 2\tilde{\omega}(k) \sum_{r=1}^{n_m(k)} E^k(r) \dot{\eta}(r) \quad (\text{B.48})$$

and thus

$$b_v^k = m(k)\tilde{\omega}(k)\tilde{\omega}(k)p(k) + 2\tilde{\omega}(k) \sum_{r=1}^{n_m(k)} E^k(r) \dot{\eta}(r) + m(k) \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} L(r,s) \dot{\eta}(r) \dot{\eta}(s) \quad (\text{B.49})$$

Putting together Eq. (B.28), Eq. (B.41) and Eq. (B.49) we have that

$$b_m(k) = \underbrace{\left(\begin{array}{c} -\omega^*(k) [S_1^k(1) + \mathcal{J}_1^k(1)] \omega(k) \\ \vdots \\ -\omega^*(k) [S_1^k(n_m(k)) + \mathcal{J}_1^k(n_m(k))] \omega(k) \\ \hline \tilde{\omega}(k) \mathcal{J}_0^k \omega(k) \\ \hline m(k) \tilde{\omega}(k) \tilde{\omega}(k) p_0(k) \end{array} \right)}_{b_m^0(k)} + \underbrace{\left(\begin{array}{c} -\omega^*(k) \sum_{j=s}^{n_m(k)} [Q^k(1,s) \dot{\eta}(s) + \{S_2^k(1,s) + \mathcal{J}_2^k(1,s)\} \eta(s)] \\ \vdots \\ -\omega^*(k) \sum_{j=s}^{n_m(k)} [Q^k(n_m(k),s) \dot{\eta}(s) + \{S_2^k(n_m(k),s) + \mathcal{J}_2^k(n_m(k),s)\} \eta(s)] \\ \hline \sum_{r=1}^{n_m(k)} [\tilde{\omega}(k) \{ \mathcal{J}_1^k(r) + [\mathcal{J}_1^k(r)]^* \} \eta(r) \omega(k) + \{2[\mathcal{J}_1^k(r)]^* + K^k(r)\} \dot{\eta}(r) \omega(k) \\ + \tilde{\omega}(k) R_1^k(r) \dot{\eta}(r)] \\ \hline \tilde{\omega}(k) [m(k) \tilde{\omega}(k) p_1^k \eta + 2E^k \dot{\eta}] \end{array} \right)}_{b_m^1(k)}$$

$$\begin{aligned}
& \left(\begin{array}{c} \sum_{q=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} T_3^k(q, 1, s) \dot{\eta}(q) \dot{\eta}(s) \\ \vdots \\ \sum_{q=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} T_3^k(q, n_m(k), s) \dot{\eta}(q) \dot{\eta}(s) \\ \hline \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} \left[\tilde{\omega}(k) \mathcal{J}_2^k(r, s) \omega(k) \eta(r) \eta(s) + \mathcal{J}_2^k(r, s) \omega(k) \dot{\eta}(r) \eta(s) \right. \\ \left. + R_2^k(r, s) \dot{\eta}(r) \dot{\eta}(s) \right] \\ \hline m(k) \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} L(r, s) \dot{\eta}(r) \dot{\eta}(s) \end{array} \right) \\
& \qquad \qquad \qquad \underbrace{\hspace{10em}}_{b_m^2(k)} \\
& + \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline \sum_{q=1}^{n_m(k)} \sum_{r=1}^{n_m(k)} \sum_{s=1}^{n_m(k)} R_3^k(q, r, s) \eta(q) \dot{\eta}(r) \dot{\eta}(s) \\ \hline 0 \end{array} \right) \\
& \qquad \qquad \qquad \underbrace{\hspace{10em}}_{b_m^3(k)}
\end{aligned} \tag{B.50}$$