IN-18

p 47

NASA Contractor Report 189568

MODES OF INTERCONNECTED LATTICE TRUSSES USING CONTINUUM MODELS, PART I

(NASA-CR-189568) MODES OF INTERCONNECTED LATTICE TRUSSES USING CONTINUUM MODELS, PART 1 Interim Report (DYNACS Engineering Co.) 49 p CSCL 22B N92-17045
Unclas
G3/18 0068715

A. V. Balakrishnan

DYNACS ENGINEERING CORPORATION
Palm Harbor, Florida

Contract NAS1-19158
December 1991

National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

MODES OF INTERCONNECTED LATTICE TRUSSES USING CONTINUUM MODELS, PART I

A. V. Balakrishnan

Abstract

This paper is Part I of a two part report and represents a continuing systematic attempt to explore the use of continuum models — in contrast to the Finite Element Models currently universally in use — to develop feedback control laws for stability enhancement of structures, particularly large structures, for deployment in space. We shall show that for the control objective, continuum models do offer unique advantages.

It must be admitted of course that developing continuum models for arbitrary structures is no easy task. In this paper we take advantage of the special nature of current Large Space Structures — typified by the NASA-LaRC Evolutionary Model which will be our main concern — which consists of interconnected orthogonal lattice trusses each with identical bays. Using an equivalent one-dimensional Timoshenko beam model, we develop an almost complete continuum model for the Evolutionary structure. We do this in stages, beginning only with the main bus as flexible and then going on to make all the appendages also flexible — except only for the antenna structure.

Based on these models we proceed to develop formulas for mode frequencies and shapes. These are shown to be the roots of the determinant of a matrix of small dimension compared with mode calculations using Finite Element Models, even though the matrix involves transcendental functions. The formulas allow us to study asymptotic properties of the modes and how they evolve as we increase the number of bodies which are treated as flexible — as we shall see the asymptotics in fact become simpler.

MODES OF INTERCONNECTED LATTICE TRUSSES USING CONTINUUM MODELS, PART I

A. V. Balakrishnan

Summary

Continuum models are constructed for interconnected beam-like lattice trusses typified by the NASA-LaRC Phase Zero Evolutionary Model. For the main bus as well as the appendages we use equivalent one-dimensional Timoshenko beam models leaving only the antenna structure as lumped. The dynamic equation is cast as an abstract wave equation in a Hilbert space with a mass-inertia operator, a stiffness operator and a control operator. One novel feature is the introduction of "linkage conditions" to take care of interconnection of trusses. Formulas are developed for modes and mode shapes — they take the form of roots of determinants of matrices, albeit involving transcendental functions. One immediate use of the formulas involves the study of asymptotic modes.

1. Introduction

This paper is Part I of a two part report and represents a continuing systematic attempt [1-5] to explore the use of continuum models — in contrast to the Finite Element Models currently universally in use — to develop feedback control laws for stability enhancement of structures, particularly large structures, for deployment in space. We shall show that for the control objective, continuum models do offer unique advantages.

It must be admitted of course that developing continuum models for arbitrary structures is no easy task. Attempts are beginning in this direction, nevertheless — see [6]. In this paper we take advantage of the special nature of current Large Space Structures — typified by the NASA-LaRC Evolutionary Model [9] which will be our main concern — which consists of interconnected orthogonal lattice trusses each with identical bays. For beam-like lattice trusses, an equivalent one-dimensional Timoshenko beam model has been developed in [7]. Using this approximation, we develop an almost complete continuum model for the Evolutionary structure. We do this in stages, beginning only with the main bus as flexible and then going on to make all the appendages also flexible — except only for the antenna structure.

Based on these models we proceed to develop formulas for mode frequencies and shapes. These are shown to be the roots of the determinant of a matrix of small dimension compared with mode calculations using Finite Element Models, even though the matrix involves transcendental functions. The formulas allow us to study asymptotic properties of the modes and how they evolve as we increase the number of bodies which are treated as flexible — as we shall see the asymptotics in fact become simpler.

Our treatment is substantially different from extant approaches to modal analysis, e.g., [8].

We begin in Section 2 with a brief description of the NASA-LaRC Zero Phase Evolutionary Model. In Section 3 we describe the one-dimensional equivalent Timoshenko beam model of a lattice truss, following Noor et al. [7]. In Section 4 we develop continuum models of the Evolutionary Model in three stages: First we model only the bus as flexible; in the second case we model the bus as well as the laser tower as flexible; and finally the bus and the tower as well as the appendages are modelled as flexible with only the antenna as rigid. In Section 5 we develop formulas for the mode frequencies and shapes for all the three cases. A study of the asymptotic modes and mode shapes is presented in Section 6 drawing on the formulas in Section 5. The closing section, Section 7, contains some conclusions based on the study.

In Part II of this paper we shall present results of numerical computations of modes and mode shapes based on the formulas herein; and compare them with extant calculations based on Finite Element Models.

2. The NASA-Langley "Evolutionary Model" Structure

A schematic of the Evolutionary Model, consisting of a long truss bus and several appendages with varying degrees of flexibility, is shown in Figure 1. The main truss bus structure has 62 bays, each being a 10-inch cubical bay. The vertical appendage (Laser Tower) is a truss with 11 bays. There are four horizontal bay appendages each with 10 bays (to which suspension cables are attached). There are 4 bays on the reflector tower. The reflector has eight 0.25-inch thick (aluminum) ribs which taper in width from 2 inches to 1 inch over their 96-inch length. For more details see [9]. The relative positions of the appendges are schematized in Figure 2: s_2, s_5 locate the horizontal appendages, s_T denotes the tower truss; the antenna is at L; 0, s_T, s_4 are co-located

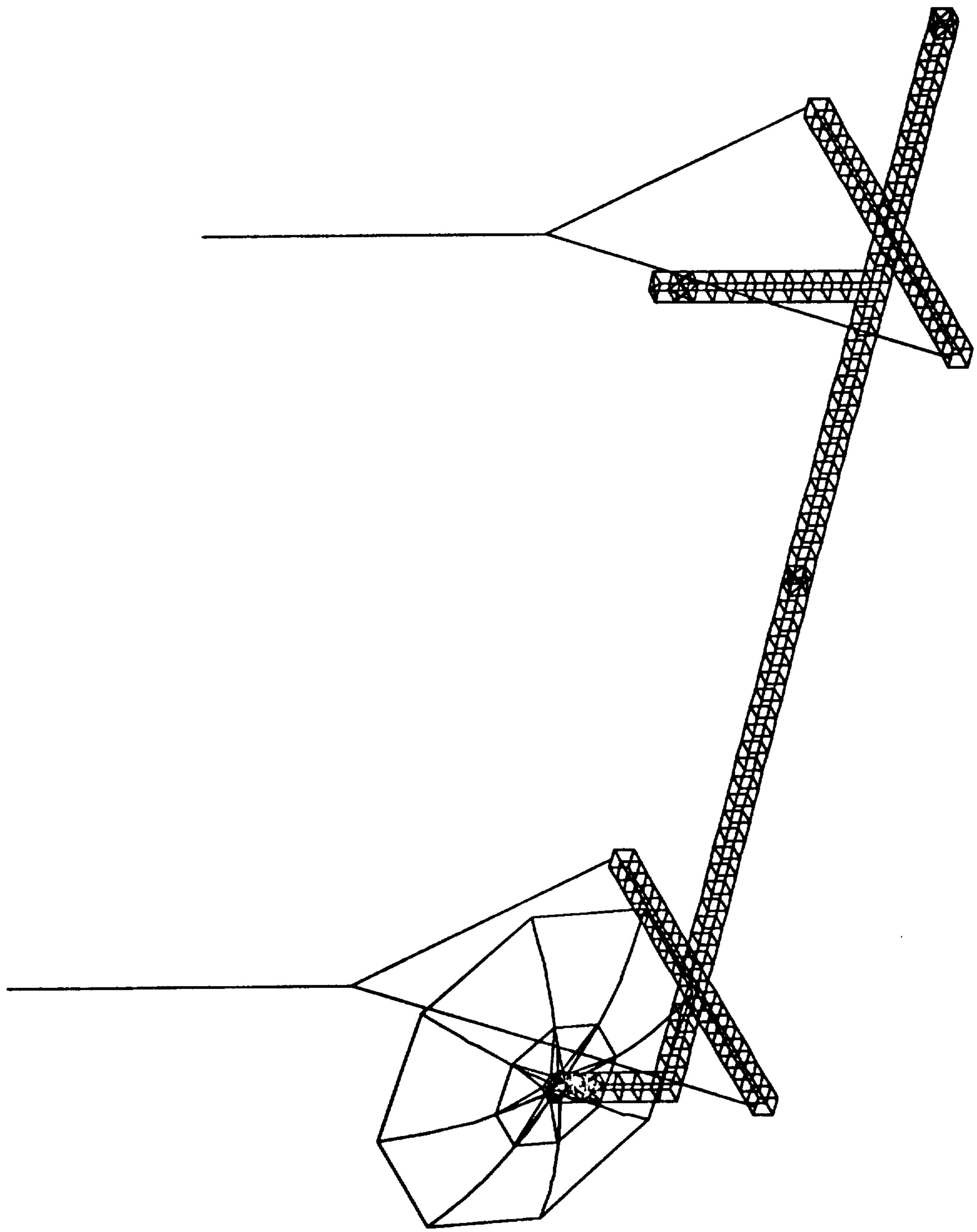

Figure 1

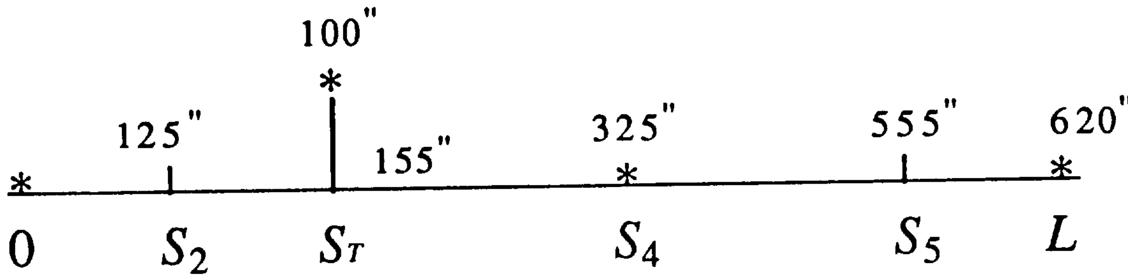

Figure 2

3. An Equivalent 1-D Timoshenko Beam Model of a Lattice Truss

Here we follow Noor et al. [7] in their technique for constructing an equivalent one-dimensional continuum model of a Lattice Truss as an anisotropic Timoshenko beam.

The element properties of a generic truss are shown in Figure 3.

Let the truss axis be the x-axis and let the z-axis be the vertical, and the x-y plane be the horizontal, plane. Let u, v, w denote the displacement along the axes at the bay vertices. Let s parametrize the position along the bus axis, $0 < s < N\ell$, where N is the number of bays. Let $u(s)$, $\mathrm{v}(s)$, $w(s)$ denote the displacement at s for the equivalent Timoshenko beam, and let $\phi_1(s)$, $\phi_2(s)$, $\phi_3(s)$ be rotation angles about the x, y and z axes respectively, $0 \leq s \leq N\ell$. Then the Timoshenko variables are related to the node displacements by:

$$u(k\ell) = \frac{u\left(k\ell,\frac{-b}{2},\frac{-b}{2}\right) + u\left(k\ell,\frac{-b}{2},\frac{b}{2}\right) + u\left(k\ell,\frac{b}{2},\frac{-b}{2}\right) + u\left(k\ell,\frac{b}{2},\frac{b}{2}\right)}{4}$$

$$\mathrm{v}(k\ell) = \frac{\mathrm{v}\left(k\ell,\frac{-b}{2},\frac{-b}{2}\right) + \mathrm{v}\left(k\ell,\frac{-b}{2},\frac{b}{2}\right) + \mathrm{v}\left(k\ell,\frac{b}{2},\frac{-b}{2}\right) + \mathrm{v}\left(k\ell,\frac{b}{2},\frac{b}{2}\right)}{4}$$

$$w(k\ell) = \frac{w\left(k\ell,\frac{-b}{2},\frac{-b}{2}\right) + w\left(k\ell,\frac{-b}{2},\frac{b}{2}\right) + w\left(k\ell,\frac{b}{2},\frac{-b}{2}\right) + w\left(k\ell,\frac{b}{2},\frac{b}{2}\right)}{4}$$

$$\begin{aligned}\phi_1(k\ell) = \frac{1}{4b}\Big[& w\left(k\ell,\frac{b}{2},\frac{-b}{2}\right) + w\left(k\ell,\frac{b}{2},\frac{b}{2}\right) + \mathrm{v}\left(k\ell,\frac{-b}{2},\frac{-b}{2}\right) + \mathrm{v}\left(k\ell,\frac{b}{2},\frac{-b}{2}\right) \\ & - w\left(k\ell,\frac{b}{2},\frac{-b}{2}\right) - w\left(k\ell,\frac{-b}{2},\frac{b}{2}\right) - \mathrm{v}\left(k\ell,\frac{-b}{2},\frac{+b}{2}\right) - \mathrm{v}\left(k\ell,\frac{b}{2},\frac{b}{2}\right)\Big]\end{aligned}$$

$$\phi_2(k\ell) = \frac{1}{2b}\left[u\left(k\ell,\frac{-b}{2},\frac{b}{2}\right) + u\left(k\ell,\frac{b}{2},\frac{b}{2}\right) - u\left(k\ell,\frac{-b}{2},\frac{-b}{2}\right) - u\left(k\ell,\frac{b}{2},\frac{-b}{2}\right)\right]$$

$$\phi_3(k\ell) = \frac{1}{2b}\left[u\left(k\ell,\frac{b}{2},\frac{-b}{2}\right) + u\left(k\ell,\frac{b}{2},\frac{b}{2}\right) - u\left(k\ell,\frac{-b}{2},\frac{-b}{2}\right) - u\left(k\ell,\frac{-b}{2},\frac{b}{2}\right)\right].$$

TABLE: Element Properties

	Battens	Longitudinal Bars	Diagonal Bars	Cross Bracing in Battens
Length L	b	ℓ	d	δ
Sectional Area A	A_b	A_ℓ	A_d	A_δ
Elastic Modulus E	E_b	E_ℓ	E_d	E_δ
Mass Density ρ	ρ_b	ρ_ℓ	ρ_d	ρ_δ
Element Mass = ρAL	m_b	m_ℓ	m_d	m_δ
Element Stiffness = EA/L	S_b	S_ℓ	S_d	S_δ

Figure 3

The anisotropic Timoshenko equations between nodes (discontinuities) are, introducing now the time variable t, so that

$$f(s, t) = \begin{vmatrix} u(s, t) \\ v(s, t) \\ w(s, t) \\ \phi_1(s, t) \\ \phi_2(s, t) \\ \phi_3(s, t) \end{vmatrix},$$

$$M_0 \frac{\partial^2 f}{\partial t^2} - A_2 \frac{\partial^2 f}{\partial s^2} + A_1 \frac{\partial f}{\partial s} + A_0 f(t, s) = 0, \qquad s_i < s < s_{i+1} \tag{3.1}$$

where s_i represent nodes, with the convention:

$s_1 = 0:$	sensor/actuator
$s_2:$	appendage
$s_3 = s_T$	tower/sensor/actuator
$s_4:$	sensor/actuator
$s_5:$	appendage
$s_6 = L:$	antenna/sensor/actuator

$$A_2 = \begin{vmatrix} C_1 & 0 \\ 0 & C_3 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} 0 & C_2 \\ -C_2^* & 0 \end{vmatrix}$$

$$A_0 = \text{Diag. } [\, 0\,,\ 0\,,\ 0\,,\ 0\,,\ c_{55}\,,\ c_{44}\,]$$

$$M_0 = \begin{vmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & m_{56} \\ 0 & 0 & 0 & 0 & m_{56} & m_{66} \end{vmatrix}$$

where

$$C_1 = \begin{vmatrix} c_{11} & c_{14} & c_{15} \\ c_{14} & c_{44} & c_{45} \\ c_{15} & c_{45} & c_{55} \end{vmatrix}$$

$$C_2 = \begin{vmatrix} 0 & -c_{15} & c_{14} \\ 0 & -c_{45} & c_{44} \\ 0 & -c_{55} & c_{45} \end{vmatrix}$$

$$C_3 = \begin{vmatrix} c_{66} & c_{36} & c_{26} \\ c_{36} & c_{33} & c_{23} \\ c_{26} & c_{23} & c_{22} \end{vmatrix} .$$

The mass coefficients m_{ij} in the Timoshenko equation are given in terms of the bay parameters by:

$$m_{11} = m_{22} = m_{33} = \frac{4m_b + 4m_\ell + 4m_d + m_\delta}{\ell}$$

$$m_{44} = 2m_{55} = 2m_{66} = \frac{\ell(8m_b + 12m_\ell + 8m_d + m_\delta)}{6\mu^2}$$

$$m_{56} = -\frac{\ell m_\delta}{12\mu^2}$$

The stiffness (flexibility) c_{ij} are given by:

$$c_{11} = 4\ell S_\ell + \frac{4\ell S_b S_d \mu^2}{S_d + S_b(\ell + \mu^2)}$$

$$c_{44} = \frac{c_{14}}{\mu} = c_{55} = -\frac{c_{15}}{\mu} = \frac{2\ell S_b S_d}{S_d + S_b(\ell + \mu^2)}$$

$$c_{22} = c_{33} = \frac{\ell^3 S_\ell}{\mu^2} + \frac{\ell^3 S_b S_d}{4(S_d + S_b(\ell + \mu^2))}$$

$$c_{23} = \frac{-\ell^3 S_b S_d}{4(S_d + S_b(\ell + \mu^2))}$$

$$c_{66} = 2c_{26} = -2c_{36} = \frac{\ell^3 S_b S_d}{\mu^2(S_d + S_b(\ell + \mu^2))}$$

where

$$\mu = \frac{\ell}{b} .$$

Evolutionary Model Parameters

For the evolutionary model, the coefficients specialize to:

$$m_{11} = m_{22} = m_{33} = 1.076 \times 10^{-3} \text{ sluglet/inch}$$

$$m_{44} = 48.31 \times 10^{-3} \text{ sluglet-inch}$$

$$m_{55} = m_{66} = 24.15 \times 10^{-3} \text{ sluglet-inch}$$

$$c_{11} = 62.45 \times 10^{5} \text{ lb}$$

$$c_{22} = c_{33} = 7.06 \times 10^{5} \text{ lb}$$

$$c_{44} = 353.14 \times 10^{5} \text{ lb-inch}^2$$

$$c_{55} = c_{66} = 1540.46 \times 10^{5} \text{ lb-inch}^2 .$$

4. Continuum Models of the Evolutionary Structure

We develop now (flexible) continuum models of the evolutionary structures at levels of increasing complexity:

i) Bus only as flexible

ii) Bus and tower as flexible

iii) All (bus, tower and appendages) as flexible, only reflector lumped.

In all cases we shall obtain the generic model dynamics as an abstract wave equation in a Hilbert space:

$$M\ddot{x}(t) + Ax(t) + Bu(t) = 0$$

where

$$x(\cdot) \in \text{ Hilbert Space } \mathcal{H} .$$

M is the mass-inertia operator: M is a self-adjoint and positive definite linear bounded operator on $\mathcal{H}$ onto $\mathcal{H}$ with bounded inverse

A is the stiffness operator: closed-linear operator with domain dense in $\mathcal{H}$: self-adjoint and nonnegative definite with compact resolvent

B is the control operator: B maps finite-dimensional Euclidean space into $\mathcal{H}$

$u(\cdot)$ denotes the control (input) .

See [1] for the first development of such a model. Among the advantages of this generic formulation is the close similarity of FEM and truncated modal models — excepting only for dimension not necessarily finite! We begin with the first case:

Case 1: Bus Only as Flexible

In this model the tower, the appendages and the reflector are modelled as offset lumped masses, as are the controllers, and the bus represented by the equivalent 1-D anisotropic Timoshenko model. Let s_i denote the location of the lumped masses. It is convenient at this point to invoke the abstract or function space representation as in [1]. Our function space denoted $\mathcal{H}$ is taken as:

$$\mathcal{H} = L_2[0, L]^6 \times R^{6\times 6}$$

with elements denoted x:

$$x = \begin{vmatrix} f \\ b \end{vmatrix}$$

$$f(s) = \begin{vmatrix} u(s) \\ v(s) \\ w(s) \\ \phi_1(s) \\ \phi_2(s) \\ \phi_3(s) \end{vmatrix}, \qquad 0 < s < L$$

$$b = \begin{vmatrix} f(0) \\ f(s_2) \\ f(s_T) \\ f(s_4) \\ f(s_5) \\ f(L) \end{vmatrix}$$

and the norm in $\mathcal{H}$ is given by:

$$\|x\|^2 = \int_0^L \|f(s)\|^2 \, ds + \|b\|^2 . \tag{4.1}$$

We now define the operator A:

$$A \begin{vmatrix} f \\ b \end{vmatrix} = \begin{vmatrix} g \\ c \end{vmatrix} \tag{4.2}$$

where

$$g(s) = -A_2 f''(s) + A_1 f'(s) + A_0 f(s) , \qquad s_i < s < s_{i+1} , \; i = 1,...,5 .$$

$$c = A_b f = \begin{vmatrix} -L_1 f(0) - A_2 f'(0) \\ A_2(f'(s_2-) - f'(s_2+)) \\ \vdots \\ A_2(f(s_i-) - f'(s_i+)) \\ \vdots \\ L_1 f(L) + A_2 f'(L) \end{vmatrix} \tag{4.3}$$

where

$$L_1 = \begin{vmatrix} 0 & -C_2 \\ 0 & 0 \end{vmatrix} .$$

The domain of A consists of functions which are continuous and piecewise smooth: in fact are in $\mathcal{H}^2(s_i, s_{i+1})$, $i = 1, ..., 5$, and the first derivative is possibly discontinuous at $s = s_i$. A is then a closed linear operator with domain dense in $\mathcal{H}$ and is self-adjoint and nonnegative definite. Moreover for x in $\mathfrak{D}(A)$

$$[Ax, x] = \int_0^L \left[H \begin{vmatrix} f'(s) \\ f(s) \end{vmatrix}, \begin{vmatrix} f'(s) \\ f(s) \end{vmatrix} \right] ds \tag{4.4}$$

$$= \int_0^L \left[C_1 \begin{vmatrix} u'(s) \\ v'(s) - \phi_3(s) \\ w'(s) + \phi_2(s) \end{vmatrix}, \begin{vmatrix} u'(s) \\ v'(s) - \phi_3(s) \\ w'(s) + \phi_2(s) \end{vmatrix} \right] ds + \int_0^L \left[C_3 \begin{vmatrix} \phi_1'(s) \\ \phi_2'(s) \\ \phi_3'(s) \end{vmatrix}, \begin{vmatrix} \phi_1'(s) \\ \phi_2'(s) \\ \phi_3'(s) \end{vmatrix} \right] ds \tag{4.5}$$

where

$$H = \begin{vmatrix} C_1 & 0 & 0 & -C_2 \\ 0 & C_3 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A_0}} \\ -C_2^* & 0 & & \end{vmatrix}$$

and the potential energy of the beam

$$= \frac{[Ax, x]}{2} .$$

It is of course assumed that C_1 and C_3 are positive definite and nonsingular, and hence A is nonnegative definite.

We have thus obtained our "stiffness" operator. Next we need to define the mass/ moment operator M.

$$M \begin{vmatrix} f \\ b \end{vmatrix} = \begin{vmatrix} M_0 f \\ M_b b \end{vmatrix} . \tag{4.6}$$

We proceed now to define M_b. M_b is "diagonal":

$$M_b b = \begin{vmatrix} M_{b_1} b_1 \\ \vdots \\ M_{b_6} b_6 \end{vmatrix} \tag{4.7}$$

where

$$b = \begin{vmatrix} b_1 \\ \vdots \\ b_6 \end{vmatrix} , \qquad b_i \in R^6$$

and M_{b_i} are nonsingular, symmetric and positive definite.

Finally we define the control operator B. Figure 4 is a schematic of the Evolutionary structure showing the disposition of the force actuators and the corresponding axes along which they act. There are 8 actuators. Hence let U denote the 8×1 column vector:

$$U = \begin{vmatrix} u_1 \\ u_2 \\ \vdots \\ u_8 \end{vmatrix}$$

Then

$$Bu = x ; \qquad x = \begin{vmatrix} 0 \\ B_U U \end{vmatrix}$$

where

$$B_u U = \begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{vmatrix} = b$$

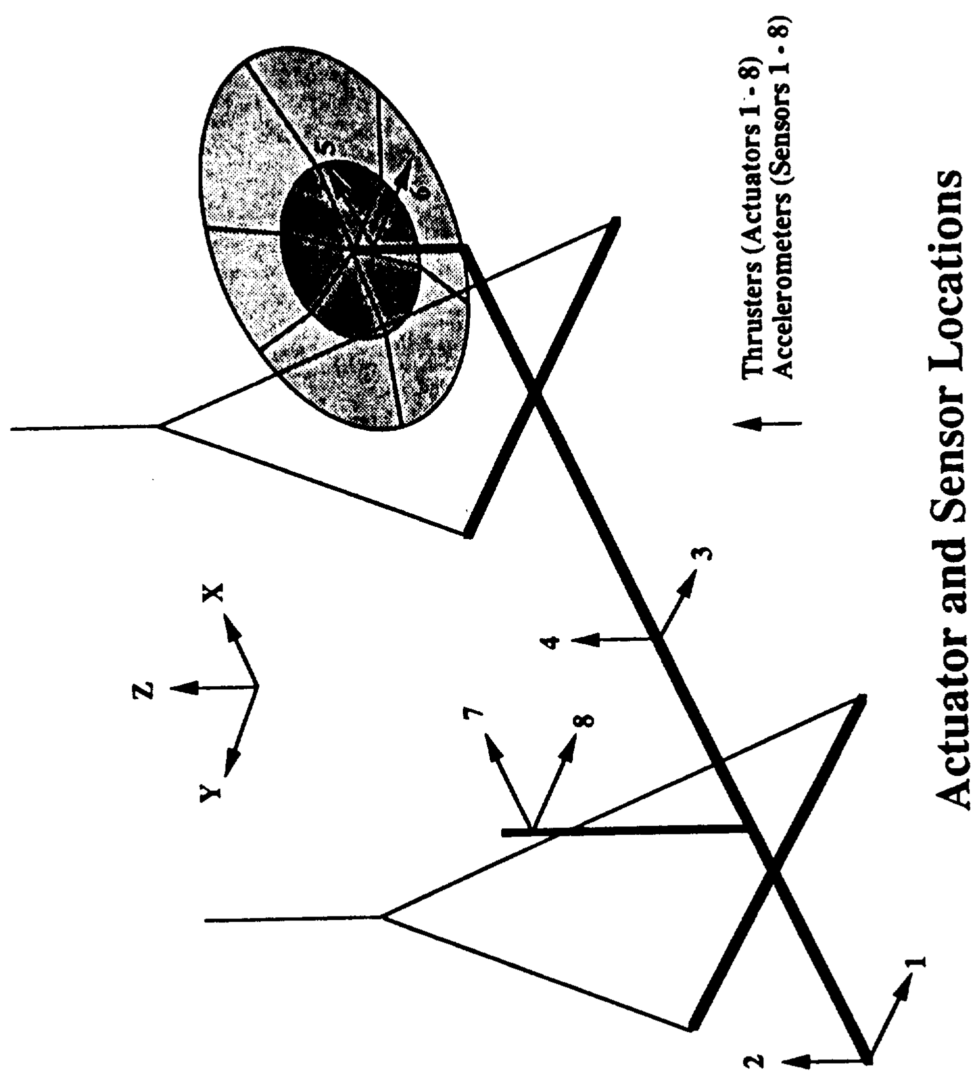

Figure 4
Schematic of Evolutionary Structure
Showing Disposition of Actuators/Sensors

$$b_1 = \begin{vmatrix} \begin{vmatrix} 0 \\ u_1 \\ u_2 \end{vmatrix} \\ \\ R_0 \times \begin{vmatrix} 0 \\ u_1 \\ u_2 \end{vmatrix} \end{vmatrix} \qquad b_2 = 0 \qquad b_3 = \begin{vmatrix} \begin{vmatrix} u_3 \\ u_4 \\ 0 \end{vmatrix} \\ \\ R_T \times \begin{vmatrix} u_3 \\ u_4 \\ 0 \end{vmatrix} \end{vmatrix}$$

$$b_4 = \begin{vmatrix} \begin{vmatrix} 0 \\ u_5 \\ u_6 \end{vmatrix} \\ \\ R_0 \times \begin{vmatrix} 0 \\ u_5 \\ u_6 \end{vmatrix} \end{vmatrix} \qquad b_5 = 0 \qquad b_6 = \begin{vmatrix} \begin{vmatrix} u_7 \\ u_8 \\ 0 \end{vmatrix} \\ \\ R_L \times \begin{vmatrix} u_7 \\ u_8 \\ 0 \end{vmatrix} \end{vmatrix}$$

where, with r denoting the position vector, $r(s)$ denoting position vector along bus axis:

$$R_0 = r(\text{controller}) - r(0)$$

$$R_T = r(\text{controller on tower}) - r(s_T)$$

$$R_I = r(\text{controller at } s = s_4) - r(s_4)$$

$$R_L = r(\text{controller at } s = L) - r(L) .$$

We note that the numerical values are:

$$R_0 = 0$$

$$R_T = \begin{vmatrix} 0 \\ 0 \\ 100 \end{vmatrix}$$

$$R_I = 0$$

$$R_L = \begin{vmatrix} 0 \\ 0 \\ 40 \end{vmatrix} .$$

Hence

$$B^*x = B_U^*b = \begin{vmatrix} v(0) \\ w(0) \\ u(s_T) + 100\phi_2(s_T) \\ v(s_T) - 100\phi_1(s_T) \\ v(s_4) \\ w(s_4) \\ u(L) + 40\phi_2(L) \\ v(L) - 40\phi_1(L) \end{vmatrix} .$$

Case 2: Bus and Tower as Flexible

For this case and the next it is convenient to change notation slightly. We use $f(x, y, z)$ in place of $f(s)$, so that

$$f(s) = f(s, 0, 0)$$

denotes the displacement vector along the axis of the bus and

$$f_T(s) = f(s_T, 0, s) , \qquad 0 < s < L_T$$

will denote the displacement vector along the axis of the tower truss in the equivalent 1-D Timoshenko model, with L_T denoting the length of the tower. Since the tower truss axis is now the z-axis, we redefine the tower truss coefficient matrices using a subscript:

$$A_{2,T} = \begin{vmatrix} C_{1,T} & 0 \\ 0 & C_{3,T} \end{vmatrix}$$

$$A_{1,T} = \begin{vmatrix} 0 & C_{2,T} \\ -C_{2,T}^* & 0 \end{vmatrix}$$

$$A_{0,T} = \text{Diag.} [0 , 0 , 0 , c_{44} , c_{55} , 0]$$

$$M_{0,T} = \begin{vmatrix} m_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{66} & m_{56} & 0 \\ 0 & 0 & 0 & m_{56} & m_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{44} \end{vmatrix}$$

$$L_{1,T} = \begin{vmatrix} 0 & -C_{2,T} \\ 0 & 0 \end{vmatrix}$$

$$C_{1,T} = \begin{vmatrix} c_{55} & c_{45} & c_{15} \\ c_{45} & c_{44} & c_{14} \\ c_{15} & c_{14} & c_{11} \end{vmatrix}$$

$$C_{2,T} = \begin{vmatrix} c_{45} & -c_{55} & 0 \\ c_{44} & -c_{45} & 0 \\ c_{14} & -c_{15} & 0 \end{vmatrix}$$

$$C_{3,T} = \begin{vmatrix} c_{22} & c_{23} & c_{26} \\ c_{23} & c_{33} & c_{36} \\ c_{26} & c_{36} & c_{66} \end{vmatrix} .$$

In the abstract version, the Hilbert Space $\mathcal{H}$ now is given by

$$\mathcal{H} = L_2[0, L]^6 \times L_2[0, L_T]^6 \times R^{6\times 6}$$

$$x = \begin{vmatrix} f(s,0,0), & 0<s<L \\ f(s_T,0,s), & 0<s<L_T \\ & b \end{vmatrix}$$

where

$$b = \begin{vmatrix} f(0, 0, 0) \\ f(s_2, 0, 0) \\ f(s_T, 0, L_T) \\ f(s_4, 0, 0) \\ f(s_5, 0, 0) \\ f(L, 0, 0) \end{vmatrix}$$

$$\|x\|^2 = \int_0^L |f(s,0,0)|^2\, ds + \int_0^{L_T} |f(s_T,0,s)|^2\, ds + \|b\|^2 .$$

The domain of A consists of functions

$$\left| \begin{matrix} f(s,0,0), & 0<s<L \\ f(s_T,0,s), & 0<s<L_T \end{matrix} \right|$$

where

$$f(s,0,0) \quad \text{and} \quad f'(s,0,0)$$

are absolutely continuous and $f'(s,0,0)$ has an L_2-derivative in the sub-intervals

$$0 < s < s_2 , \qquad s_2 < s < s_4 , \qquad s_4 < s < s_5 , \qquad s_5 < s < L ;$$

and

$$f(s_T,0,s) \quad \text{and} \quad f'(s_T,0,s)$$

are absolutely continuous with $f'(s_T,0,s)$ having an L_2-derivative in $0 < s < L_T$.

Moreover the following "linkage conditions" are satisfied:

i) $f(s,0,0)\big|_{s=s_T} = f(s_T,0,s)\big|_{s=0}$

ii) $L_{1,T} f(s_T,0,0) - A_{2,T} f_z(s_T,0,0) + A_2(f_x(s_T-,0,0) - f_x(s_T+,0,0) = 0$.

The stiffness operator A is now defined by

$$x = \left| \begin{matrix} f(\cdot,0,0) \\ f(s_T,0,\cdot) \\ b \end{matrix} \right|$$

$$Ax = \left| \begin{matrix} g(\cdot,0,0) \\ g(s_T,0,\cdot) \\ A_b f \end{matrix} \right|$$

where

$$g(s,0,0) = -A_2 f_{xx}(s,0,0) + A_1 f_x(s,0,0) + A_0 f(s,0,0)$$
$$0 < s < s_2, \quad s_2 < s < s_4, \quad s_4 < s < s_5, \quad s_5 < s < s_6 ,$$

$$g(s_T,0,s) = -A_{2,T} f_{zz}(s_T,0,s) + A_{1,T} f_z(s_T,0,s) + A_{0,T} f(s_T,0,s)$$
$$0 < s < L_T ,$$

$$A_b f = \begin{vmatrix} -L_1 f(0,0,0) - A_2 f_x(0,0,0) \\ A_2(f_x(s_2-,0,0) - f_x(s_2+,0,0)) \\ L_{1,T} f(s_T,0,L_T) + A_{2,T} f_z(s_T,0,L_T) \\ A_2(f_x(s_4-,0,0) - f_x(s_4+,0,0)) \\ A_2(f_x(s_5-,0,0) - f_x(s_5+,0,0)) \\ L_1 f(L,0,0) + A_2 f_x(L,0,0) \end{vmatrix} .$$

Thus defined, it is easy to verify that A is closed, self-adjoint and nonnegative definite and that

$$\frac{[Ax,\ x]}{2} = \text{Elastic Energy of Bus} + \text{Elastic Energy of Tower Truss} .$$

Finally the mass/moment operator M is defined by

$$Mx = \begin{vmatrix} M_0 f(\cdot,0,0) \\ M_{0,T}\, f(s_T,0,\cdot) \\ M_b b \end{vmatrix}$$

where

$$M_b b = \begin{vmatrix} M_{b_i} b_i \end{vmatrix}$$

and M_{b_i} are positive-definite and nonsingular.

Finally we define B the control operator. First we define

$$BU = x ; \qquad x = \begin{vmatrix} 0 \\ B_U U \end{vmatrix}$$

$$B_u U = \begin{vmatrix} \text{col.}[0,\ u_1,\ u_2,\ 0,\ 0,\ 0] \\ 0 \\ \text{col.}[u_3,\ u_4,\ 0,\ 0,\ 0,\ 0] \\ \text{col.}[0,\ u_5,\ u_6,\ 0,\ 0,\ 0] \\ 0 \\ \text{col.}\left[u_7,\ u_8,\ 0,\ R_L \times \begin{vmatrix} u_7 \\ u_8 \\ 0 \end{vmatrix}\right] \end{vmatrix} = b$$

$$B^*U = B_u^*b = \begin{vmatrix} v(0,0,0) \\ w(0,0,0) \\ u(s_T, 0, L_T) \\ v(s_T, 0, L_T) \\ v(s_4, 0, 0) \\ w(s_4, 0, 0) \\ u(L,0,0) + 40\phi_2(L,0,0) \\ v(L,0,0) - 40\phi_1(L,0,0) \end{vmatrix}$$

Case 3: Bus, Tower and Appendages Flexible

We now generalize to the case where the main bus, the laser tower and the horizontal appendages are modelled as flexible lattice trusses — or more precisely, their 1-D Timoshenko beam equivalents. We shall be briefer in our descriptions since we will follow the pattern already set in Case 2.

Thus let the subscript s_2 denote the coefficient matrices for the appendages at $s = s_2$ and similarly the subscript s_5 for the appendage at $s = s_5$. Then

$$A_{2,s_2} = \begin{vmatrix} C_{1,s_2} & 0 \\ 0 & C_{3,s_2} \end{vmatrix}$$

$$A_{1,s_2} = \begin{vmatrix} 0 & C_{2,s_2} \\ -C_{2,s_2}^* & 0 \end{vmatrix}$$

$$A_{0,s_2} = \text{Diag.}\,[\,0\,,\,0\,,\,0\,,\,c_{55}\,,\,0\,,\,c_{44}\,]$$

$$L_{1,s_2} = \begin{vmatrix} 0 & -C_{2,s_2} \\ 0 & 0 \end{vmatrix}$$

$$M_{0,s_2} = \begin{vmatrix} m_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{55} & 0 & m_{56} \\ 0 & 0 & 0 & 0 & m_{44} & 0 \\ 0 & 0 & 0 & m_{56} & 0 & m_{66} \end{vmatrix}$$

$$C_{1,S_2} = \begin{vmatrix} c_{44} & c_{14} & c_{45} \\ c_{14} & c_{11} & c_{15} \\ c_{45} & c_{15} & c_{55} \end{vmatrix}$$

$$C_{2,S_2} = \begin{vmatrix} -c_{45} & 0 & c_{44} \\ -c_{15} & 0 & c_{14} \\ -c_{55} & 0 & c_{45} \end{vmatrix}$$

$$C_{3,S_2} = \begin{vmatrix} c_{33} & c_{36} & c_{23} \\ c_{36} & c_{66} & c_{26} \\ c_{23} & c_{26} & c_{22} \end{vmatrix}$$

$$A_{2,S_5} = A_{2,S_2}$$

$$A_{1,S_5} = A_{1,S_2}$$

$$A_{0,S_5} = A_{0,S_2}$$

$$M_{0,S_5} = M_{0,S_2}$$

$$L_{1,S_5} = L_{1,S_2} .$$

The appendage displacement vectors are then

$$f(s_2, s, 0) , \qquad -\ell_1 < s < \ell_1$$

$$f(s_5, s, 0) , \qquad -\ell_2 < s < \ell_2$$

for the evolutionary truss $\ell_1 = \ell_2$. Thus let

$$f = \begin{vmatrix} f(s_2,s,0), & -\ell_1 < s < \ell_1 \\ f(s_T,s,0), & 0 < s < L_T \\ f(s_5,s,0), & -\ell_2 < s < \ell_2 \\ f(s,0,0), & 0 < s < L \end{vmatrix}$$

$$b = \begin{vmatrix} f(0,0,0) \\ f(s_2, -\ell_1, 0) \\ f(s_2, +\ell_1, 0) \\ f(s_T, 0, L_T) \\ f(s_4, 0, 0) \\ f(s_5, -\ell_2, 0) \\ f(s_5, +\ell_2, 0) \\ f(L, 0, 0) \end{vmatrix}$$

$$A_b f = \begin{vmatrix} -L_1 f(0,0,0) - A_2 f_x(0,0,0) \\ -L_{1,S_2} f(s_2, -\ell_1, 0) - A_{2,S_2} f_y(s_2, -\ell_1, 0) \\ L_{1,S_2} f(s_2, +\ell_1, 0) + A_{2,S_2} f_y(s_2, +\ell_1, 0) \\ L_{1,T} f(s_T, 0, L_T) + A_{2,T} f_z(s_T, 0, L_T) \\ A_2(f_x(s_4-, 0, 0) - f_x(s_4+, 0, 0)) \\ -L_{1,S_5} f(s_5, -\ell_2, 0) - A_{2,S_5} f_y(s_5, -\ell_2, 0) \\ L_{1,S_5} f(s_5, +\ell_2, 0) + A_{2,S_5} f_y(s_5, +\ell_2, 0) \\ L_1 f(L, 0, 0) + A_2 f_x(L, 0, 0) \end{vmatrix}$$

Plus Linkage Conditions:

(1) $f(s, 0, 0)\big|_{s=s_T} = f(s_T, 0, s)\big|_{s=0}$

$$-L_{1,T} f(s_T, 0, 0) - A_{2,T} f_z(s_T, 0, 0) + A_2(f_x(s_T-, 0, 0) - f_x(s_T+, 0, 0)) = 0$$

(2) $f(s, 0, 0)\big|_{s=s_2} = f(s_2, s, 0)\big|_{s=0}$

$$A_2(f_x(s_2-, 0, 0) - f_x(s_2+, 0, 0)) + A_{2,S_2}(f_y(s_2, 0-, 0) - f_y(s_2, 0+, 0)) = 0$$

(3) $f(s, 0, 0)\big|_{s=s_5} = f(s_5, s, 0)\big|_{s=0}$

$$A_2(f_x(s_5-, 0, 0) - f_x(s_5+, 0, 0)) + A_{2,S_5}(f_y(s_5, 0-, 0) - f_y(s_5, 0+, 0)) = 0 .$$

Remark

If suspension "ends" are treated as "free-free," then remove

$$f(s_2, -\ell_1, 0)$$

$$f(s_2, +\ell_1, 0)$$

from b and instead take:

$$L_{1,S_1} f(s_2, -\ell_1, 0) + A_{2,S_1} f_y(s_2, -\ell_1, 0) = 0$$

$$L_{1,S_1} f(s_2, +\ell_1, 0) + A_{2,S_1} f_y(s_2, +\ell_1, 0) = 0$$

and similarly for the other suspension beam.

Finally:

$$x = \begin{vmatrix} f \\ b \end{vmatrix}$$

$$\mathcal{H} = L_2[-\ell_1, \ell_1]^6 \times L_2[0, L_T]^6 \times L_2[-\ell_2, \ell_2]^6 \times L_2[0, L]^6 \times R^{6\times 8}$$

$$Ax = y$$

$$x = \begin{vmatrix} f \\ b \end{vmatrix} \qquad y = \begin{vmatrix} g \\ c \end{vmatrix}$$

where

$$g = \begin{vmatrix} g(s_2, \cdot, 0) \\ g(s_T, 0, \cdot) \\ g(s_5, \cdot, 0) \\ g(\cdot, 0, 0) \end{vmatrix}$$

$$g(s_2, s, 0) = -A_{2,S_2} f_{yy}(s_2, s, 0) + A_{1,S_2} f_y(s_2, s, 0) + A_{0,S_2} f(s_2, s, 0)$$
$$-\ell_1 < s < \ell_1 ,$$

$$g(s_T, s, 0) = -A_{2,T} f_{zz}(s_T, s, 0) + A_{1,T} f_z(s_T, s, 0) + A_{0,T} f(s_T, s, 0)$$
$$0 < s < L ,$$

$$g(s_5, s, 0) = -A_{2,S_5} f_{yy}(s_5, s, 0) + A_{1,S_5} f_y(s_5, s, 0) + A_{0,S_5} f(s_5, s, 0)$$
$$-\ell_2 < s < \ell_2 ,$$

$$g(s, 0, 0) = -A_2 f_{xx}(s, 0, 0) + A_1 f_x(s, 0, 0) + A_0 f(s, 0, 0)$$

$$c = A_b f \; ; \; f \text{ subject to linkage conditions .}$$

Then A is self-adjoint and nonnegative definite and

$$\frac{[Ax, x]}{2} = [\text{Sum of Elastic Energy of Tower, Suspensions and Main Beam}] .$$

Next, the mass/inertia operator M is defined by

$$Mx = \begin{vmatrix} M_{0,S_2} f(s_2, \cdot, 0) \\ M_{0,T} f(s_T, 0, \cdot) \\ M_{0,S_5} f(s_5, \cdot, 0) \\ M_0 f(\cdot, 0, 0) \\ M_b b \end{vmatrix}$$

where again

$$M_b b = \begin{vmatrix} M_{b_1} b_1 \\ \vdots \\ M_{b_8} b_8 \end{vmatrix}$$

where M_{b_i} are mass/inertia matrices.

Finally we define the control operator B.

$$x = \begin{vmatrix} f \\ b \end{vmatrix}$$

$$BU = \begin{vmatrix} 0 \\ B_U U \end{vmatrix}$$

$$B_U U = \text{col.} [\, b_1 , b_2 , b_3 , b_4 , b_5 , b_6 , b_7 , b_8 \,]$$

$$b_1 = \text{col.} [\, 0 , u_1 , u_2 , 0 , 0 , 0 \,]$$

$$b_2 = 0 = b_3$$

$$b_4 = \text{col.} [\, u_3 , u_4 , 0 , 0 , 0 , 0 \,]$$

$$b_5 = \text{col.} [\, 0 , u_5 , u_6 , 0 , 0 , 0 \,]$$

$$b_6 = 0 = b_7$$

$$b_8 = \begin{vmatrix} \begin{vmatrix} u_7 \\ u_8 \\ 0 \end{vmatrix} \\ \\ R_L \times \begin{vmatrix} u_7 \\ u_8 \\ 0 \end{vmatrix} \end{vmatrix} .$$

Hence:

$$B^* x = B_u^* b = \begin{vmatrix} v(0,0,0) \\ w(0,0,0) \\ u(s_T, 0, L_T) \\ v(s_T, 0, L_T) \\ v(s_4, 0, 0) \\ w(s_4, 0, 0) \\ u(L,0,0) + 40\phi_2(L,0,0) \\ v(L,0,0) - 40\phi_1(L,0,0) \end{vmatrix}$$

file: june91y

5. Mode Formulas

In this section we develop formulas for modes and mode shapes to find the modes we need to solve the eigenvalue problem:

$$Ax = \omega^2 Mx . \tag{5.1}$$

Letting

$$x = \begin{vmatrix} f \\ b \end{vmatrix}$$

we begin with Case 1.

Case 1: Bus Only Flexible

In this case (5.1) translates into:

$$x = \begin{vmatrix} f \\ b \end{vmatrix}$$

$$-A_2 f''(s) + A_1 f'(s) + A_0 f(s) = \omega^2 M_0 f(s) , \qquad s_i < s < s_{i+1} , \tag{5.2}$$

$$A_b f = \omega^2 M_b b \tag{5.3}$$

where the second equation can be expanded as:

$$\left.\begin{aligned} -L_1 f(0) - A_2 f'(0) &= M_{b,0} f(0) \\ A_2(f'(s_i-) - f'(s_i+)) &= M_{b,i} f(s_i) , \qquad i = 2,\ldots,5 \\ L_1 f(L) + A_2 f'(L) &= M_{b,L} f(L) \end{aligned}\right\} \tag{5.3a}$$

For combining (5.2) and (5.3a) let

$$\mathcal{A}(\omega) = \begin{vmatrix} 0 & I \\ A_2^{-1}(A_0 - \omega^2 M_0) & A_2^{-1} A_1 \end{vmatrix} \tag{5.4}$$

(a 12×12 matrix). Let

$$e^{\mathcal{A}(\omega)s} = \begin{vmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{vmatrix} \tag{5.5}$$

where, of course,

$$P_{21}(s) = P_{11}'(s)$$

$$P_{22}(s) = P_{12}'(s) .$$

Let

$$\mathcal{L}_1 = \mathcal{A}_2^{-1}(\omega^2 M_{b,0} - L_1)$$

$$\mathcal{L}_i = \omega^2 A_2^{-1} M_{b,i} , \qquad i = 2, \ldots, 5$$

$$\mathcal{L}_6 = \mathcal{A}_2^{-1}(\omega^2 M_{b,L} - L_1) .$$

Then (4.9a) yields:

$$\begin{vmatrix} f(s_{i+1}) \\ f'(s_{i+1}-) \end{vmatrix} = e^{\mathcal{A}(\omega)(s_{i+1}-s_i)} \begin{vmatrix} f(s_i) \\ f'(s_i-) - \mathcal{L}_i f(s_i) \end{vmatrix} \tag{5.6}$$

with the convention that

$$f'(s_1-) = f'(0-) = 0$$

and condition (4.9a) requires that

$$f'(L) = \mathcal{L}_6 f(L) .$$

But we can write

$$f'(L) - \mathcal{L}_6 f(L) = D(\omega) f(0)$$

where

$$D(\omega) = \begin{vmatrix} A_2^{-1}(L_1 - \omega^2 M_{b,L}) & I_6 \end{vmatrix} \cdot e^{\mathcal{A}(\omega)(L-s_5)} \begin{vmatrix} I_6 & 0 \\ -\omega^2 A_2^{-1} M_{b,5} & I_6 \end{vmatrix}$$

$$\cdot\, e^{\mathcal{A}(\omega)(s_5-s_4)} \begin{vmatrix} I_6 & 0 \\ -\omega^2 A_2^{-1} M_{b,4} & I_6 \end{vmatrix} \cdot e^{\mathcal{A}(\omega)(s_4-s_3)} \begin{vmatrix} I_6 & 0 \\ -\omega^2 A_2^{-1} M_{b,3} & I_6 \end{vmatrix}$$

$$\cdot\, e^{\mathcal{A}(\omega)(s_3-s_2)} \begin{vmatrix} I_6 & 0 \\ -\omega^2 A_2^{-1} M_{b,2} & I_6 \end{vmatrix} \cdot e^{\mathcal{A}(\omega)s_2} \begin{vmatrix} I_6 \\ A_2^{-1}(L_1 - \omega^2 M_{b,0}) \end{vmatrix} \tag{5.7}$$

where I_6 is the 6×6 Identity matrix. Thus the mode frequencies are determined from

$$|D(\omega)|_{\mathrm{Det}} = 0 \tag{5.8}$$

and the mode shapes from the corresponding eigenvector $f(0)$:

$$D(\omega)f(0) = 0 \tag{5.9}$$

the corresponding $f(s_i)$ being determined from (5.6). Or, more explicitly

$$f(s) = | I_6 \quad 0 | \, e^{A(\omega)(s-s_i)} \, T_i \cdot e^{A(\omega)\Delta_{i-1}} \, T_{i-1}$$

$$\cdots e^{A(\omega)\Delta_1} \begin{vmatrix} f(0) \\ A_2^{-1}(-L_1 - \omega^2 M_{b,0}) f(0) \end{vmatrix}, \qquad s < s_i \tag{5.6a}$$

where

$$T_i = \begin{vmatrix} I_6 & 0 \\ -\omega^2 A_2^{-1} M_{b,i} & I_6 \end{vmatrix}, \qquad i = 2,3,4,5, \quad \Delta_i = s_{i+1} - s_i \,.$$

We have thus "reduced" a mode determination problem to finding the zeros of a transcendental function

$$|D(\omega)|_{\mathrm{Det}} = 0 \,.$$

The crucial calculation is that of the matrix exponential $e^{A(\omega)(s_{i+1}-s_i)}$. We note that we can "expand" $D(\omega)$ as:

$$D(\omega) = \sum_1^6 \omega^{2k} D_k$$

since

$$D(0) = 0 \,.$$

Pure Modes

The evolutionary model trusses are actually isometric:

$$c_{ij} = 0\,, \qquad i \neq j$$

$$m_{ij} = 0\,, \qquad i \neq j$$

$$M_{b_0} = M_{b_4} = \text{Diag.}\,(\,.05\,,\ .05\,,\ .05\,,\ 1.9\,,\ 0.95\,,\ 0.95\,)$$

$$M_{s_2} = M_{s_5} = \left|\begin{array}{ccc|ccc} 0.28 & 0 & 0 & 0 & 0.71 & 0 \\ 0 & 0.28 & 0 & -0.71 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.71 & 0 & 1538 & 0 & 0 \\ -0.71 & 0 & 0 & 0 & 53.9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1494 \end{array}\right|$$

$$M_{s_T} = \left|\begin{array}{ccc|ccc} 0.18 & 0 & 0 & 0 & 13.23 & 0 \\ 0 & 0.18 & 0 & -13.23 & 0 & 0 \\ 0 & 0 & 0.18 & 0 & 0 & 0 \\ 0 & 13.23 & 0 & 1132 & 0 & 0 \\ -13.23 & 0 & 0 & 0 & 1132 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7.3 \end{array}\right|$$

$$M_{b_6} = M_L = \left|\begin{array}{cccccc} 0.38 & 0 & 0 & 0 & 22 & 0 \\ 0 & 0.38 & 0 & -22 & 0 & -0.91 \\ 0 & 0 & 0.38 & 0 & 0.91 & 0 \\ 0 & 22 & 0 & 1511 & 0 & 120 \\ -22 & 0 & -0.91 & 0 & 1459 & 0 \\ 0 & 0.91 & 0 & 120 & 0 & 229 \end{array}\right|$$

$$M_0 = \text{Diag.}(\,1.08\times10^{-3},\ 1.08\times10^{-3},\ 1.08\times10^{-3},\ 48.3\times10^{-3},\ 24.15\times10^{-3},\ 24.15\times10^{-3})$$

$$A_2 = \text{Diag.}(\,62.45\times10^{5},\ 7.06\times10^{5},\ 7.06\times10^{5},\ 353.1\times10^{5},\ 1540\times10^{5},\ 1540\times10^{5})\,.$$

The inertia matrices M_{b_i} are nearly diagonal. If we retain only the diagonal terms, we can easily see that there are "pure" modes: a pure "axial" mode in which

$$f(s) = a(s) = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

and pure "torsion" mode:

$$f(s) = a(s) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

and we can calculate the corresponding mode frequencies (and shapes). Thus let

$$e_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} .$$

Then we have

$$D(\omega)e_1 = d(\omega)e_1$$

where

$$d(\omega) = \begin{vmatrix} \frac{-\omega^2 m_{b,L}}{c_{11}} & 1 \end{vmatrix} \cdot \begin{vmatrix} P_{11}(\Delta_5) & P_{12}(\Delta_5) \\ P_{21}(\Delta_5) & P_{22}(\Delta_5) \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{-\omega^2 m_{b,5}}{c_{11}} & 1 \end{vmatrix}$$

$$\cdots \begin{vmatrix} P_{11}(\Delta_i) & P_{12}(\Delta_i) \\ P_{21}(\Delta_i) & P_{22}(\Delta_i) \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{-\omega^2 m_{b,i}}{c_{11}} & 1 \end{vmatrix}$$

$$\cdots \begin{vmatrix} P_{11}(\Delta_1) & P_{12}(\Delta_1) \\ P_{21}(\Delta_1) & P_{22}(\Delta_1) \end{vmatrix} \cdot \begin{vmatrix} 1 \\ \frac{-\omega^2 m_{b,1}}{c_{11}} \end{vmatrix}$$

where

$$\Delta_i = s_{i+1} - s_i , \qquad i = 1, ..., 5$$

$$P_{11}(s) = \cos \lambda s$$

$$P_{12}(s) = \frac{\sin \lambda s}{\lambda}$$

$$P_{21}(s) = -\lambda \sin \lambda s$$

$$P_{22}(s) = \cos \lambda s$$

where

$$\lambda = \omega\sqrt{m_{11}/c_{11}}$$

and

$$m_{b_i} = \text{the } 1\text{-}1 \text{ entry in } M_{b,i}$$

and the mode shape

$$a(s) = \begin{vmatrix} 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} P_{11}(s-s_i) & P_{12}(s-s_i) \\ P_{21}(s-s_i) & P_{22}(s-s_i) \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{-\omega^2 m_{b,i}}{c_{11}} & 1 \end{vmatrix}$$

$$\cdots \begin{vmatrix} P_{11}(\Delta_1) & P_{12}(\Delta_1) \\ P_{21}(\Delta_1) & P_{22}(\Delta_1) \end{vmatrix} \cdot \begin{vmatrix} 1 \\ \frac{-\omega^2 m_{b,1}}{c_{11}} \end{vmatrix} a(0) , \qquad s_i \le s \le s_{i+1} .$$

We list below the first few modes corresponding to

$$d(\omega) = 0 .$$

Pure Axial Modes (Hz)

29.1

82.45

116.35

187.4

218.7

281.9

370.3

610.9

Pure Torsion Modes

Here

$$f(s) = a(s)e_4$$

where

$$e_4 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

$$D(\omega)e_4 = ds(\omega)e_4$$

$$d(\omega) = \begin{vmatrix} \frac{-\omega^2 m_{b,4,L}}{c_{44}} & 1 \end{vmatrix} \cdot \begin{vmatrix} P_{11}(\Delta_5) & P_{12}(\Delta_5) \\ P_{21}(\Delta_5) & P_{22}(\Delta_5) \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{-\omega^2 m_{b,4,5}}{c_{44}} & 1 \end{vmatrix}$$

$$\cdots \begin{vmatrix} P_{11}(\Delta_i) & P_{12}(\Delta_i) \\ P_{21}(\Delta_i) & P_{22}(\Delta_i) \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ \frac{-\omega^2 m_{b,4,i}}{c_{44}} & 1 \end{vmatrix}$$

$$\cdots \begin{vmatrix} P_{11}(\Delta_1) & P_{12}(\Delta_1) \\ P_{21}(\Delta_1) & P_{22}(\Delta_1) \end{vmatrix} \cdot \begin{vmatrix} 1 \\ \frac{-\omega^2 m_{b,4,1}}{c_{44}} \end{vmatrix}$$

where

$$m_{b,4,i} = 4\times4 \text{ terms in } M_{b,i}$$

$$\lambda = \omega\sqrt{m_{44}/c_{44}}$$

$$P_{11}(s) = \cos \lambda s$$

$$P_{12}(s) = \frac{\sin \lambda s}{\lambda}$$

$$P_{21}(s) = -\lambda \sin \lambda s$$

$$P_{22}(s) = \cos \lambda s \; .$$

The first few modes are:

Pure Torsion Modes (Hz)
1.2
4.29
6.79
30.6
41.6
67.46
94.53
127.4
163
186.9
208
232
249
292
317

Case 2: Bus and Tower Flexible

For this case (5.1) yields:

$$D(\omega)f = 0$$

where

$$f = \begin{vmatrix} f_s(s_T, 0, s)\big|_{s=0} \\ \\ f(0, 0, 0) \end{vmatrix}_{12\times 1}$$

$$D(\omega) = \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}$$

$$D_{11} = \begin{vmatrix} A_{2,T}^{-1}(L_{1,T} - \omega^2 M_{b,L}) & I \end{vmatrix} \cdot e^{\mathcal{A}_T(\omega)L_T} \begin{vmatrix} 0 \\ I \end{vmatrix}$$

$$D_{12} = \begin{vmatrix} A_{2,T}^{-1}(L_{1,T} - \omega^2 M_{b,L}) & I \end{vmatrix} \cdot e^{\mathcal{A}_T(\omega)L_T} \begin{vmatrix} I & 0 \\ 0 & 0 \end{vmatrix} \cdot e^{\mathcal{A}(\omega)(S_T - S_2)}T_2$$
$$\cdot\, e^{\mathcal{A}(\omega)S_2} \begin{vmatrix} I \\ A_2^{-1}(-L_2 - \omega^2 M_{b,1}) \end{vmatrix} .$$

$$D_{21} = \begin{vmatrix} A_2^{-1}(L_1 - \omega^2 M_{b,L}) & I \end{vmatrix} \cdot (e^{\mathcal{A}(\omega)(L - S_5)}T_5) \cdot (e^{\mathcal{A}(\omega)(S_5 - S_4)}T_4)$$
$$\cdot\, e^{\mathcal{A}(\omega)(S_4 - S_T)} \begin{vmatrix} 0 \\ A_2^{-1}A_{2,T} \end{vmatrix}$$

$$D_{22} = \begin{vmatrix} A_2^{-1}(L_1 - \omega^2 M_{b,L}) & I \end{vmatrix} \cdot (e^{\mathcal{A}(\omega)(L - S_5)}T_5) \cdot (e^{\mathcal{A}(\omega)(S_5 - S_4)}T_4)$$
$$\cdot \left(e^{\mathcal{A}(\omega)(S_4 - S_T)} \begin{vmatrix} I & 0 \\ A_2^{-1}L_{1,T} & I \end{vmatrix}\right) \cdot (e^{\mathcal{A}(\omega)(S_T - S_2)}T_2)e^{\mathcal{A}(\omega)S_2} \begin{vmatrix} I \\ A_2^{-1}(L_1 - \omega^2 M_{b,1} \end{vmatrix}$$

$$|D(\omega)| = \left|D_{11}D_{22} - D_{11}^{-1}D_{21}D_{11}D_{22}\right|$$

where

$$\mathcal{A}(\omega) = \begin{vmatrix} 0 & I \\ A_2^{-1}(A_0 - \omega^2 M_0) & A_2^{-1}A_1 \end{vmatrix}$$

$$T_i = \begin{vmatrix} I & 0 \\ -\omega^2 A_2^{-1}M_{b,i} & I \end{vmatrix}, \qquad i = 2, 3, 4, 5$$

$$\mathcal{A}_T(\omega) = \begin{vmatrix} 0 & I \\ A_{2,T}^{-1}(A_{0,T} - \omega^2 M_{0,T}) & A_{2,T}^{-1} A_{1,T} \end{vmatrix}$$

$$v = \begin{vmatrix} f_z \\ f_0 \end{vmatrix}$$

$$D(\omega) f = 0$$

Mode Shapes: Tower

$$f(s_T, 0, z) = | I \quad 0 | \, e^{\mathcal{A}_T(\omega) z}$$

$$\times \left\{ \begin{vmatrix} 0 \\ f_z(s_T, 0, 0) \end{vmatrix} + \begin{vmatrix} I & 0 \\ 0 & 0 \end{vmatrix} e^{\mathcal{A}(\omega)\Delta_2} T_2 \cdot e^{\mathcal{A}(\omega)\Delta_1} \begin{vmatrix} f(0,0,0) \\ A_2^{-1}(-L_1 - \omega^2 M_{b,1}) f(0,0,0) \end{vmatrix} \right\}$$

$$0 \le z \le L_T \,.$$

Case 3: Bus, Tower and Appendages Flexible

For this case (5.1) reduces to:

$$D(\omega) f = 0$$

where

$$f = \begin{vmatrix} f_{y-}(s_2, 0, 0) \\ f_{y+}(s_2, 0, 0) \\ f_z(s_T, 0, 0) \\ f_{y-}(s_5, 0, 0) \\ f_{y+}(s_5, 0, 0) \\ f(0, 0, 0) \end{vmatrix}_{36 \times 1}$$

$$D(\omega) = \{D_{ij}\} \qquad i,j = 1, \ldots, 6$$

$$D_{11} = \left| -L_{1,S_2} - \omega^2 M_{S_2,-\ell_1} \quad -A_{2,S_2} \right| e^{A(\omega)_{S_2}(-\ell_1)} \left| \begin{matrix} 0 \\ I \end{matrix} \right|$$

$$D_{22} = \left| L_{1,S_2} - \omega^2 M_{S_2,\ell_1} \quad A_{2,S_2} \right| e^{A(\omega)_{S_2}\ell_1} \left| \begin{matrix} 0 \\ I \end{matrix} \right|$$

$$D_{33} = \left| L_{1,T} - \omega^2 M_{S_2,L_T} \quad A_{2,T} \right| e^{A(\omega)_T L_T} \left| \begin{matrix} 0 \\ I \end{matrix} \right|$$

$$D_{44} = \left| -L_{1,S_5} - \omega^2 M_{S_5,-\ell_1} \quad -A_{2,S_5} \right| e^{A(\omega)_{S_5}(-\ell_1)} \left| \begin{matrix} 0 \\ I \end{matrix} \right|$$

$$D_{55} = \left| L_{1,S_5} - \omega^2 M_{S_5,\ell_1} \quad A_{2,S_5} \right| e^{A(\omega)_{S_5}\ell_1} \left| \begin{matrix} 0 \\ I \end{matrix} \right|$$

$$D_{16} = \left| -L_{1,S_2} - \omega^2 I \quad -A_{2,S_2} \right| e^{A(\omega)_{S_2}(-\ell_1)} \cdot \left| \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \right| e^{A(\omega)\Delta_1} \left| \begin{matrix} I \\ -A_2^{-1}(L_1 + \omega^2 M_{b,0}) \end{matrix} \right|$$

$$D_{26} = \left| L_{1,S_2} - \omega^2 I \quad A_{2,S_2} \right| e^{A(\omega)_{S_2}\ell_1} \cdot \left| \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \right| e^{A(\omega)\Delta_1} \left| \begin{matrix} I \\ -A_2^{-1}(L_1 + \omega^2 M_{b,0}) \end{matrix} \right|$$

$$D_{36} = \left| L_{1,T} - \omega^2 I \quad A_{2,T} \right| e^{A(\omega)_T L_T} \cdot \left| \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \right| e^{A(\omega)(\Delta_2 + \Delta_1)} \left| \begin{matrix} I \\ -A_2^{-1}(L_1 + \omega^2 M_{b,0}) \end{matrix} \right|$$

$$D_{46} = \left| (-L_{1,S_5} - \omega^2 M_{S_5,-\ell_2}) \quad -A_{2,S_5} \right| e^{A(\omega)_{S_5}(-\ell_2)} \cdot \left| \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \right| e^{A(\omega)\Delta_4}\, T_4\, e^{A(\omega)(\Delta_3 + \Delta_2 + \Delta_1)}$$

$$\cdot \left| \begin{matrix} I \\ -A_2^{-1}(L_1 + \omega^2 M_{b,0}) \end{matrix} \right|$$

$$D_{56} = \left| (L_{1,S_5} - \omega^2 M_{S_5,\ell_2}) \quad A_{2,S_5} \right| e^{A(\omega)_{S_5}\ell_2} \cdot \left| \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \right| e^{A(\omega)\Delta_4}\, T_4\, e^{A(\omega)(\Delta_3 + \Delta_2 + \Delta_1)}$$

$$\cdot \left| \begin{matrix} I \\ -A_2^{-1}(L_1 + \omega^2 M_{b,0}) \end{matrix} \right|$$

$$D_{66} = \left| L_1 - \omega^2 M_{b,L} \quad A_2 \right| e^{\mathcal{A}(\omega)(\Delta_5+\Delta_4)}\, T_4\, e^{\mathcal{A}(\omega)\Delta_3} \cdot \left| \begin{array}{cc} I & 0 \\ -A_2^{-1} L_{1,T} & I \end{array} \right| e^{\mathcal{A}(\omega)(\Delta_2+\Delta_1)}$$

$$\cdot \left| \begin{array}{c} I \\ -A_2^{-1}(L_1 + \omega^2 M_{b,1}) \end{array} \right|$$

$$D_{65} = \left| L_1 - \omega^2 M_{b,L} \quad A_2 \right| e^{\mathcal{A}(\omega)\Delta_5} \left| \begin{array}{c} 0 \\ -A_2^{-1} A_{2,S_5} \end{array} \right|$$

$$D_{64} = \left| L_1 - \omega^2 M_{b,L} \quad A_2 \right| e^{\mathcal{A}(\omega)\Delta_5} \left| \begin{array}{c} 0 \\ A_2^{-1} A_{2,S_5} \end{array} \right|$$

$$D_{63} = \left| L_1 - \omega^2 M_{b,L} \quad A_2 \right| e^{\mathcal{A}(\omega)(\Delta_5+\Delta_4)}\, T_4\, e^{\mathcal{A}(\omega)\Delta_3} \cdot \left| \begin{array}{c} 0 \\ -A_2^{-1} A_{2,T} \end{array} \right|$$

$$D_{62} = \left| L_1 - \omega^2 M_{b,L} \quad A_2 \right| e^{\mathcal{A}(\omega)(\Delta_5+\Delta_4)}\, T_4\, e^{\mathcal{A}(\omega)\Delta_3} \cdot \left| \begin{array}{cc} I & 0 \\ -A_2^{-1} L_{1,T} & I \end{array} \right| e^{\mathcal{A}(\omega)\Delta_2}$$

$$\cdot \left| \begin{array}{c} I \\ -A_2^{-1} A_{2,S_2} \end{array} \right|$$

$$D_{61} = \left| L_1 - \omega^2 M_{b,L} \quad A_2 \right| e^{\mathcal{A}(\omega)(\Delta_5+\Delta_4)}\, T_4\, e^{\mathcal{A}(\omega)\Delta_3} \cdot \left| \begin{array}{cc} I & 0 \\ -A_2^{-1} L_{1,T} & I \end{array} \right| e^{\mathcal{A}(\omega)\Delta_2}$$

$$\cdot \left| \begin{array}{c} I \\ A_2^{-1} A_{2,S_2} \end{array} \right|$$

$$|D(\omega)| = \left| D_{11} \cdot D_{22} \cdots D_{55} \cdot (D_{66} - D_{61} D_{11}^{-1} D_{16} - \cdots - D_{65} D_{55}^{-1} D_{56}) \right|$$

6. Asymptotic Modes

In this section we use our mode formulas of Section 5 to study the asymptotic behavior of modes for all the three cases.

Case 1: Bus Only Flexible

For the asymptotic study we use the expansion

$$D(\omega) = \sum_{1}^{6} \omega^{2k} D_k .$$

For large ω therefore the roots of

$$|D(\omega)| = 0$$

are those of

$$|D_6| = 0$$

with increasing accuracy. Now

$$D_6 = A_2^{-1} M_{b,L} \; P_{12}(\Delta_5) \, A_2^{-1} M_{b,5} \; P_{12}(\Delta_4) \; \cdots \; P_{12}(\Delta_1) \, A_2^{-1} M_{b,1}$$

where

$$P_{12}(s) = \begin{vmatrix} I & 0 \end{vmatrix} e^{A(\omega)s} \begin{vmatrix} I \\ 0 \end{vmatrix} .$$

Hence

$$|D_6| = 0$$

if and only if

$$|P_{12}(\Delta_i)| = 0 , \qquad \text{for some} \quad i = 1, 2, 3, 4, 5 .$$

But these roots are recognized as the modes of a "clamped-clamped" beam. Thus asymptotically all the "clamped-clamped" modes correspond to every beam segment between nodes. In fact for large ω,

$$P_{12}(\Delta) = \sum_{1}^{6} \frac{\sin \lambda_k(\omega)\Delta}{\lambda_k(\omega)} e_k$$

where e_k are the unit vectors, $k = 1, \ldots, 6$,

$$\lambda_k(\omega) = \omega\sqrt{\gamma_k}$$

$$A_2^{-1} M_0 e_k = \gamma_k e_k \, .$$

The eigenvalues of $P_{12}(\Delta_i)$ are given by

$$\frac{\sin \sqrt{\gamma_k} \ \omega\Delta_i}{\sqrt{\gamma_k}} , \qquad k = 1, ..., 6$$

or, the modes are given by

$$\sqrt{\gamma_k} \ \omega\Delta_i = n\pi$$

$$\omega = \frac{n\pi}{\Delta_i} \cdot \frac{1}{\sqrt{\gamma_k}} , \qquad i = 1,...,5, \ k = 1,...,6 \, .$$

For the evolutionary model

$$\omega = \frac{n\pi}{\Delta_i} \sqrt{c_{kk}/m_{kk}}$$

or

$$\nu = \frac{\pi}{2\Delta_i} \sqrt{c_{kk}/m_{kk}} \ \text{Hz} \, .$$

For Δ_4 the largest segment, this yields for the axial mode:

$$\nu = (165.4)n \ \text{Hz}$$

and for the torsion mode

$$\nu = (58.7)n \qquad (\Delta_4)$$

$$= (77.1)n \qquad (\Delta_3) \, .$$

It is difficult to recognize these in the few modes we have calculated.

Case 2: Bus and Tower Flexible

Here $|D(\omega)| = 0$ for large ω yields

$$|D_{11}(\omega)| \, |D_{22}(\omega)| = 0 \, .$$

Now

$$|D_{11}(\omega)| = 0$$

means the roots of

$$\left| I \quad 0 \right| e^{A_T(\omega) L_T} \left| \begin{matrix} I \\ 0 \end{matrix} \right| = 0$$

which are recognized as the "clamped-clamped" modes of the tower truss — as we should expect.

$$|D_{22}(\omega)| = 0$$

yields

$$\left| P_{12}(\Delta_5)\, A_2^{-1}\, M_{b,5}\, P_{12}(\Delta_4)\, A_2^{-1}\, M_{b,4} \right| = 0$$

$$\left| (P_{11}(\Delta_3) - P_{12}(\Delta_3) A_2^{-1} L_{1,T}) P_{12}(\Delta_2) A_2^{-1} M_{b,2} P_{12}(\Delta_1) + P_{12}(\Delta_3) P_{22}(\Delta_2) A_2^{-1} M_{b,2} P_{12}(\Delta_1) \right| = 0$$

The first relation is equivalent to:

$$|P_{12}(\Delta_5)| = 0 \sim \text{clamped-clamped modes of segment } \Delta_5$$

$$|P_{12}(\Delta_4)| = 0 \sim \text{clamped-clamped modes of segment } \Delta_4 \,.$$

The second relation yields

$$|P_{12}(\Delta_1)| = 0 \sim \text{clamped-clamped modes of segment } \Delta_1$$

and

$$\left| (P_{11}(\Delta_3) - P_{12}(\Delta_3) A_2^{-1} L_{1,T}) P_{12}(\Delta_2) + P_{12}(\Delta_3) P_{22}(\Delta_2) \right| = 0 \,.$$

Since asymptotically

$$P_{11}(\Delta) = \sum_1^6 \cos \lambda_k(\omega) \Delta e_k$$

$$P_{22}(\Delta) = \sum_1^6 \cos \lambda_k(\omega) \Delta e_k \,.$$

Now

$$A_2^{-1} L_{1,T} e_k = 0 \,, \qquad k = 1, 2, 3, 6$$

(corresponding to "displacement" modes about the tower axis) and the torsion mode and hence for $k = 1, 2, 3, 6$

$$(P_{11}(\Delta_3) - P_{12}(\Delta_3)A_2^{-1}L_{1,T})P_{12}(\Delta_2)e_k \; + \; P_{12}(\Delta_3)P_{22}(\Delta_2)e_k$$

$$= \left[\left(\cos \lambda_k(\omega)\Delta_3 \frac{\sin \lambda_k(\omega)\Delta_2}{\lambda_k(\omega)}\right) + \frac{\sin \lambda_k(\omega)\Delta_3}{\lambda_k(\omega)} \cos \lambda_k(\omega)\Delta_2\right] e_k$$

$$= \frac{\sin \lambda_k(\omega)(\Delta_3 + \Delta_2)}{\lambda_k(\omega)} e_k \, .$$

Hence we see asymptotically the clamped-clamped displacement modes of the segment $(\Delta_3 + \Delta_2)$. Hence we have the clamped-clamped displacement modes of segments:

$$\Delta_5$$
$$\Delta_4$$
$$\Delta_1$$
$$\Delta_2 + \Delta_3$$

but these are now recognized as the segments between lumped masses. We note that for the evolutionary truss

$$\Delta_2 \; + \; \Delta_3 \; = \; 205 \; < \; \Delta_4 \, .$$

Hence these modes are still too high.

Case 3: Bus, Tower and Appendges Flexible

Here

$$|D(\omega)| \; = \; 0$$

asymptotically

$$|D_{11}(\omega)| \; |D_{22}(\omega)| \; |D_{33}(\omega)| \; |D_{44}(\omega)| \; |D_{55}(\omega)| \; |D_{66}(\omega)| \; = \; 0$$

$$\left.\begin{aligned} |D_{11}(\omega)| &= 0 \\ |D_{22}(\omega)| &= 0 \end{aligned}\right\} \approx \text{ clamped-clamped modes of each appendage (length } \ell_1\text{) at } s_2$$

$$|D_{33}(\omega)| \; = \; 0 \;\; \sim \;\; \text{clamped-clamped modes of tower truss}$$

$$\left.\begin{aligned} |D_{44}(\omega)| &= 0 \\ |D_{55}(\omega)| &= 0 \end{aligned}\right\} \approx \text{ clamped-clamped modes of each appendage at } s_5$$

$$|D_{66}(\omega)| \; = \; 0 \;\; \Rightarrow \;\; \text{bus modes}$$

$$= (M_{b,L}\, P_{12}(\Delta_4+\Delta_5)\, A_2^{-1}\, M_{b,4}\,)$$
$$\cdot\ ((P_{11}(\Delta_3) - P_{12}(\Delta_3)A_2^{-1}L_{1,T})P_{12}(\Delta_2+\Delta_1) \ + \ P_{12}(\Delta_3)P_{22}(\Delta_2+\Delta_1) \ = \ 0$$

$$= \ |P_{12}(\Delta_4 + \Delta_5)| \ = \ 0 \quad \text{clamped-clamped modes of segment } (\Delta_4 + \Delta_5)\ .$$

As in Case 2, for $e_k = 1, 2, 3,$ the displacement modes, we have

$$A_2^{-1} L_{1,T} e_k \ = \ 0$$

and hence we obtain

$$\sin \lambda_k(\omega)(\Delta_1 + \Delta_2 + \Delta_3) \ = \ 0\ .$$

Or, we have the clamped-clamped modes of the segment $(\Delta_1 + \Delta_2 + \Delta_3)$. But

$$\Delta_4 \ + \ \Delta_5$$

$$\Delta_1 \ + \ \Delta_2 \ + \ \Delta_3$$

are now the segments between lumped masses. Moreover for the evolutionary truss:

$$\Delta_4 \ + \ \Delta_5 \ = \ 295$$

$$\Delta_1 \ + \ \Delta_2 \ + \ \Delta_3 \ = \ 330\ .$$

The clamped-clamped mode frequencies corresponding to the segment $\Delta_1 + \Delta_2 + \Delta_3$ are given by

$$\nu \ = \ \frac{n}{660}\sqrt{c_{11}/m_{11}}\ \text{Hz} \ \sim \ (\text{Axial}) \ = \ (20.5)n \ \text{ Hz}$$

and corresponding to the segment $\Delta_4 + \Delta_5$:

$$\nu \ = \ \frac{n}{590}\sqrt{c_{11}/m_{11}} \ = \ (22.9)n \ \text{ Hz}$$

which are now low enough to be found in the range of modes of practical interest!

7. Conclusions

It is feasible to construct continuum models of flexible multibodies if they take the form of large interconnected trusses with many bays where advantage can be taken of 1-D equivalent Timoshenko beam models. Using these models it is possible to construct formulas for modes where the matrix size is insignificant compared to the Finite Element version. However transcendental functions are involved. It is possible to make explicit use of the mode formulas to estimate asymptotic modes. Asymptotic modes would appear to be more realistic as the number of flexible parts which are modelled as continua increases. The asymptotic modes then are recognized as the clamped-clamped modes of beam segments between lumped masses.

References

[1] A. V. Balakrishnan. "A Mathematical Formulation of the SCOLE Control Problem," NASA CR172581. May 1985.

[2] A. V. Balakrishnan. "Compensator Design for Stability Enhancement with Co-located Controllers." *IEEE Transactions on Automatic Control,* September 1991.

[3] A. V. Balakrishnan. "Stochastic Regulator Theory for a Class of Abstract Wave Equations." *SIAM Journal on Control and Optimization,* November 1991.

[4] A. V. Balakrishnan. "Combined Structures-Controls Optimization of Lattice Trusses." *Computer Methods in Applied Mechanics and Engineering,* November 1991.

[5] A. V. Balakrishnan. "Attitude Error Response of Structures to Actuator/Sensor Noise." In: *Proceedings of American Control Conference, June 1991.*

[6] L. W. Taylor. "Distributed Parameter Modeling for the Control of Flexible Spacecraft." In: *Proceedings of the NASA-UCLA Workshop on Computational Techniques in Identification and Control of Flexible Flight Structures, Lake Arrowhead, November 1989.* New York-Los Angeles: OSI Publications. Pp. 87-114.

[7] A. K. Noor and W. C. Russell. "Anisotropic Continuum Models for Beam-Like Lattice Trusses," *Computer Methods in Applied Mechanics and Engineering,* Vol. 57 (1986), pp. 257-277.

[8] W. H. Wittrick and F. W. Williams. "A General Algorithm for Computing Natural Frequencies of Elastic Structures." *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 24, No. 3 (1971).

[9] W. K. Belvin, L. G. Horta and K. E. Elliott. "The LaRC CSI Phase-0 Evolutionary Model Test Bed: Design and Experimental Results." Paper presented at the 4th Annual NASA-DOD Conference on CSI Technology, November 5-7, 1990, Orlando, Florida.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	December 1991	Contractor Report

4. TITLE AND SUBTITLE Modes of Interconnected Lattice Trusses Using Continuum Models, Part I

5. FUNDING NUMBERS
C NAS1-19158
WU 590-14-51-01

6. AUTHOR(S) A. V. Balakrishnan

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
DYNACS Engineering Corp.
34650 U. S. Hwy 19 North
Suite 301
Palm Harbor, FL 34684

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665-5225

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
NASA CR-189568

11. SUPPLEMENTARY NOTES
Langley Technical Monitor: Suresh M. Joshi
Interim Report

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Unclassified - Unlimited
Subject Category 18

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

This paper is Part I of a two-part report and represents a continuing systematic attempt to explore the use of continuum models--in contrast to the Finite Element Models currently universally in use--to develop feedback control laws for stability enhancement of structures, particularly large structures, for deployment in space. We shall show that for the control objective, continuum models do offer unique advantages.

It must be admitted of course that developing continuum models for arbitrary structures is no easy task. In this paper we take advantage of the special nature of current Large Space Structures--typified by the NASA-LaRC Evolutionary Model which will be our main concern--which consists of interconnected orthogonal lattice trusses each with identical bays. Using an equivalent one-dimensional Timoshenko beam model, we develop an almost complete continuum model for the Evolutionary structure. We do this in stages, beginning only with the main bus as flexible and then going on to make all the appendages also flexible--except only for the antenna structure.

Based on these models we proceed to develop formulas for mode frequencies and shapes. These are shown to be the roots of the determinant of a matrix of small dimension compared with mode calculations using Finite Element Models, even though the matrix involves transcendental functions. The formulas allow us to study asymptotic properties of the modes and how they evolve as we increase the number of bodies which are treated as flexible--as we shall see the asymptotics in fact become simpler.

14. SUBJECT TERMS
Flexible Spacecraft, Distributed Parameter System

15. NUMBER OF PAGES
49

16. PRICE CODE
A03

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	