NIAR Report 91-27

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COMPUTATION OF THREE-DIMENSIONAL FLOWS
USING TWO STREAM FUNCTIONS

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ABSTRACT

An approach to compute three dimensional flows using two stream functions is presented. The method generates a boundary fitted grid as part of its solution. Commonly used two steps for computing the flow fields: (1) boundary fitted grid generation, and (2) solution of Navier-Stokes equations on the generated grid, are combined into a single step in the present approach. The method presented can be used to compute directly 3-D viscous flows, or the potential flow approximation of this method can be used to generate grid for other algorithms to compute 3-D viscous flows.

The independent variables used are χ, a spatial coordinate, and ξ and η, values of stream functions along two sets of suitably chosen intersecting stream surfaces. The dependent variables used are the streamwise velocity, and two functions that describe the stream surfaces. Since for a three dimensional flow there is no unique way to define two sets of intersecting stream surfaces to cover the given flow, in the present study three different types of two sets of intersecting stream surfaces are considered. First is presented the metric of the (χ, ξ, η) curvilinear coordinate system associated with each type. Next equations for the steady state transport of mass, momentum, and energy are presented in terms of the metric of the (χ, ξ, η) coordinate system. Also included are the inviscid and the parabolized approximations to the general transport equations.

I. INTRODUCTION

Since the introduction of stream function by Lagrange (1) for two dimensional plane flows and by Stokes (2) for axisymmetric flows, the use of a stream function to study two dimensional flows has been extensive. Computation of two-dimensional incompressible potential flows using the stream function as the dependent variable and the space coordinates as the independent variables is well known and can be found in almost any introductory fluid mechanics book. The stream function has also been used in the computation of viscous flows. In the recent past, the stream function, along with the vorticity, has been used extensively to compute two-dimensional incompressible viscous flows. Patankar and Spalding (3) have used the stream function to construct the cross-stream coordinate for the computation of two-dimensional compressible 'parabolic' (boundary layer) type flows. Kwon and Pletcher (4) have used the stream function and the axial velocity as the dependent variables to compute two-dimensional incompressible separated channel flow. These are a few examples of the use of the stream function for the computation of viscous flows. More recently, streamlines of the incompressible potential flow corresponding to a given geometry have been used to construct boundary-fitted grid systems for the computation of viscous flows. The streamlines needed for the grid generation have been calculated by various methods. For example, Ghia et al. (5) generated the grid by the use of conformal mapping; Meyder (6), and Ferrel and Adamczyk (7), by solving the potential equation. A survey of the use of streamlines to generate a grid is included in the review article on grid generation by Thompson et al. (8).

The corresponding development for three dimensional flows, that is the use of two stream functions to study three dimensional flows, so far has been limited. Several authors in the past have introduced two stream function to describe three dimensional flows. Among the pioneering works are the works of Clebsch (9), Prasil (10), Maeder and Wood (11), and Yih (12). For the two dimensional plane flows it follows from the continuity equation

$$\frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} = 0 \tag{1.1}$$

that $u^1 dx^2 - u^2 dx^1$ is an exact differential of a function of x^1 and x^2, calling this function

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Ψ, we have

$$u^1 = \frac{\partial \Psi}{\partial x^2}, \quad \text{and} \quad u^2 = -\frac{\partial \Psi}{\partial x^1} \tag{1.2}$$

This is the approach Lagrange (1) used to introduce the stream function for two dimensional plane flows; and, later, by a similar approach Stokes (2) introduced the stream function for axisymmetric flows. One can also introduce the stream function for two dimensional flows by the following, slightly different, approach. A general solution of the continuity equation (1.1) is given by an arbitrary, as yet undetermined, function Ψ such that u^1 and u^2 are related to Ψ by (1.2). This approach to introduce the stream function for two dimensional flows can be immediatly extended to three dimensional flows. The extension is based on a theorem by Jacobi (quoted in Clebsch),which for our purposes can be stated as follows: The equation,

$$\frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \cdots + \frac{\partial u^n}{\partial x^n} = 0 \tag{1.3}$$

has a general solution given by $(n\text{-}1)$ arbitrary, as yet undetermined, functions

$$\Psi^1(x^1, x^2, \ldots, x^n), \ \Psi^2(x^1, x^2, \ldots, x^n), \ \cdots, \ \Psi^{n-1}(x^1, x^2, \ldots, x^n) \tag{1.4}$$

with u^i given by its cofactor in the matrix

$$\begin{pmatrix} u^1 & u^2 & \cdots & u^n \\ \frac{\partial \Psi^1}{\partial x^1} & \frac{\partial \Psi^1}{\partial x^2} & \cdots & \frac{\partial \Psi^1}{\partial x^n} \\ \frac{\partial \Psi^2}{\partial x^1} & \frac{\partial \Psi^2}{\partial x^2} & \cdots & \frac{\partial \Psi^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Psi^{n-1}}{\partial x^1} & \frac{\partial \Psi^{n-1}}{\partial x^2} & \cdots & \frac{\partial \Psi^{n-1}}{\partial x^n} \end{pmatrix} \tag{1.5}$$

Following this line of approach Clebsch (9) introduced stream functions for three dimensional flows. We denote the two arbitrary functions for the case of three dimensional flows by Ξ and H (instead of Ψ^1 and Ψ^2) and obtain from (1.5) for the velocity vector $\mathbf{v}$

$$\mathbf{v} = grad\,\Xi \times grad\,H \tag{1.6}$$

From (1.6) we note that $\mathbf{v}$ is normal to $grad\,\Xi$, thus the surface defined by Ξ equal to a constant contains streamlines and Ξ is appropriately called a stream function. We will

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denote by ζ the value of the function Ξ along a given stream surface. With a similar discussion of H, we write our stream function equations as,

$$\xi = \Xi(x, y, z); \qquad \eta = H(x, y, z) \qquad (1.7a, b)$$

In Eqs.(1.7) the space coordinates are denoted by (x, y, z) to establish continuity with what follows latter. Another approach to introduce stream functions for three dimensional flows, is presented by Yih (12). Following Yih we integrate the equations of a stream line

$$\frac{dx^1}{u^1} = \frac{dx^2}{u^2} = \frac{dx^3}{u^3} \qquad (1.8)$$

and obtain (1.7) as the integral surfaces of (1.8). Relation (1.6) is then obtained from the argument that since Ξ and H describe stream surfaces, their gradients are normal to the velocity vector. The preceding discussion, for the sake of simplicity, is given for the case of incompressible flows. The case of compressible flows follows similarly with u^i in the preceding discussion replaced by ρu^i.

The present work takes as its starting point the existence of two stream functions that will describe three dimensional steady flows. From that point on we develop techniques for computing three dimensional flows using two stream functions. The stream surfaces in the present work are defined parametrically by equations such as

$$x = \chi, \qquad y = Y(\chi, \xi, \eta), \quad \text{and} \quad z = Z(\chi, \xi, \eta), \qquad (1.9a, b, c)$$

For a given value of x Eqs. (1.7a,b) and Eqs. (1.9b,c) are inverse relations of each other. In the present work, an important compliment to the use of two stream functions to describe three dimensional flows is the choice of the independent and the dependent variables used to describe the flow. As discussed in detail in the next section, the independent variables used to describe the flow are χ, ξ, and η; and the the dependent variables, U, the streamwise velocity, and Y, and Z. With these variables we studied in Ref. 13 two dimensional (plane and axisymmetric) parabolized viscous flows, and in Ref. 14 parabolized three dimensional flows through straight rectangular ducts. For such simple flows equations for U and Y, for two dimensional flows, and for U, Y and Z for three dimensional flows, can be easily

obtained by partitioning the flow into a number of appropriately defined stream tubes, and then applying conservation principles directly to the flow through the individual stream tubes. In Refs. 15 and 16 we studied two and three dimensional potential flows in term of these variables. In these studies equations for Y and Z were obtained by projecting the streamline motion on to the x-y and x-z planes. In Ref. 16 equations for Y and Z were also derived for the three dimensional potential flows by setting, with the help of differential geometry, the vorticity around a closed contour drawn on a stream surface equal to zero.

In the present paper we present, with the help of tensor calculus, a general theory for studying three dimensional viscous flows using the aforementioned variables. The present work is restricted to steady flows.

II. INDEPENDENT AND DEPENDENT VARIABLES

In this section we introduce the independent and the dependent variables. The independent variables are χ, ξ, and η. Variable χ is a spatial coordinate along the main flow direction. Variables ξ and η are the values of stream functions along suitably chosen two sets of intersecting stream surfaces. A stream surface along which the stream function ξ is constant is referred to as a ξ=const. surface or as a χ-η surface. Similar nomenclature is used for stream surfaces along which η is constant. As discussed in the earlier studies (as, for example, in Maeder and Wood (11), and Yih (12)), in general, for three dimensional flows there is no unique way to define two sets of interacting stream surfaces. For a given flow there are numerous choices for the Ξ and H stream surfaces that will cover the given flow. However, to take advantage of the (χ, ξ, η) choice of the independent variables the general shapes of the Ξ and H stream surfaces are to be selected to facilitate the imposition of the required boundary conditions. In the present paper we present three different combinations of two basic types of Ξ and H stream surfaces that should cover many flows of practical interest. The two basic types of stream surfaces considered are: plane stream surfaces and cylinderical stream surfaces. The plane stream surfaces are not necessarily flat. The boundaries of plane stream surfaces intersect the flow boundaries. The cylinderical stream surfaces are not necessarily straight circular cylinders. The cylinderical stream surfaces are nested within each other. From these two basic types of stream surfaces we form three different combinations, each consisting of two sets of intersecting stream surfaces, to model three different types of flows. These three different types of flows are named: (i) Plane Flows, (ii) Axial Flows, and (iii) Circulating Flows. 'Plane' flows are modeled with one set of ξ=constant plane stream surfaces and one set of η=constant plane stream surfaces as shown in Fig.1a. This type of modeling is proposed for studying flows that are bounded by flat boundaries. 'Axial' flows are modeled with one set of ξ=constant cylinderical stream surfaces and one set of η=constant plane stream surfaces such that one edge of all the η=constant stream surfaces meet in the axis of the flow as shown in Fig.1b. This type of modeling is proposed for studying flows that are bounded by curved boundaries. 'Circulating' flows are modeled with one set of ξ=constant cylinderical
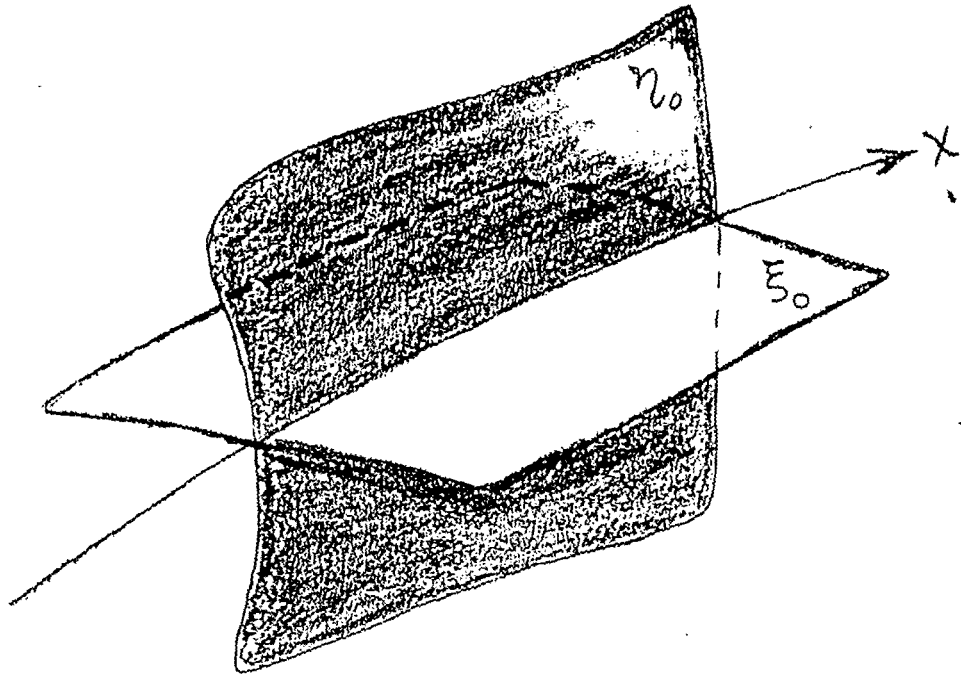
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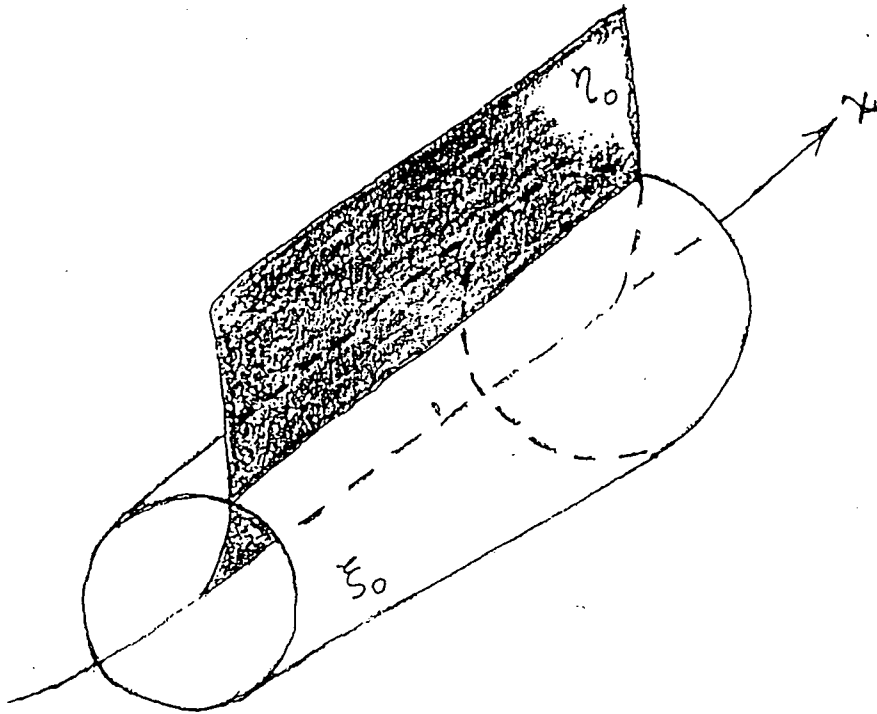Fig. 1a. General shape of a ξ=const. stream surface intersecting a η=const. stream surface for Plane Flows.

Fig. 1b. General shape of a ξ=const. stream surface intersecting a η=const. stream surface for Axial Flows.

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stream surfaces and one set of η=constant plane stream surfaces as shown in Fig.1c. The intersection of the ξ=constant stream surfaces and η=constant stream surfaces are closed curves. This type of modeling is proposed for the study of circulating flows.

The stream surfaces are defined parametrically by the following equations:

$$\text{Plane Flows:} \quad x = \chi, \quad y = Y(\chi, \xi, \eta), \quad \text{and} \quad z = Z(\chi, \xi, \eta), \quad (2.1a)$$

$$\text{Axial Flows:} \quad x = \chi, \quad r = R(\chi, \xi, \eta), \quad \text{and} \quad \theta = \Theta(\chi, \xi, \eta), \quad (2.1b)$$

$$\text{Circulating Flows:} \quad x = X(\chi, \xi, \eta), \quad r = R(\chi, \xi, \eta), \quad \text{and} \quad \theta = \chi \quad (2.1c)$$

In Eqs.(2.1) x, y, and z are the rectangular Cartesian coordinates, and x, r, and θ are polar cylinderical coordinates. For a given value of η Eqs.(2.1) define χ-ξ stream surfaces with χ and ξ as the parameters. Similarly, for a given value of ξ these equations parametrically define χ-η stream surfaces with χ and η as the parameters.

Let $U(\chi, \xi, \eta)$ represent the streamwise velocity. The dependent variables used to describe the flow are:

$$\text{Plane Flows:} \quad U, \quad Y, \quad Z \quad (2.2a)$$

$$\text{Axial Flows:} \quad U, \quad R, \quad \Theta \quad (2.2b)$$

$$\text{Circulating Flows:} \quad U, \quad X, \quad R \quad (2.2c)$$

Let g_χ and g_ξ represent the coordinate vectors of the χ-ξ stream surfaces, and g_χ and g_η the coordinate vectors of the χ-η stream surfaces (these vectors are calculated in the next section). Since the coordinate vector g_χ is common to both the χ-ξ and the χ-η stream surfaces, it is tangent to the stream line defined by the intersection of the χ-ξ and the χ-η stream surfaces. Since U is the streamwise velocity, we note, for later use, that U is along g_χ.

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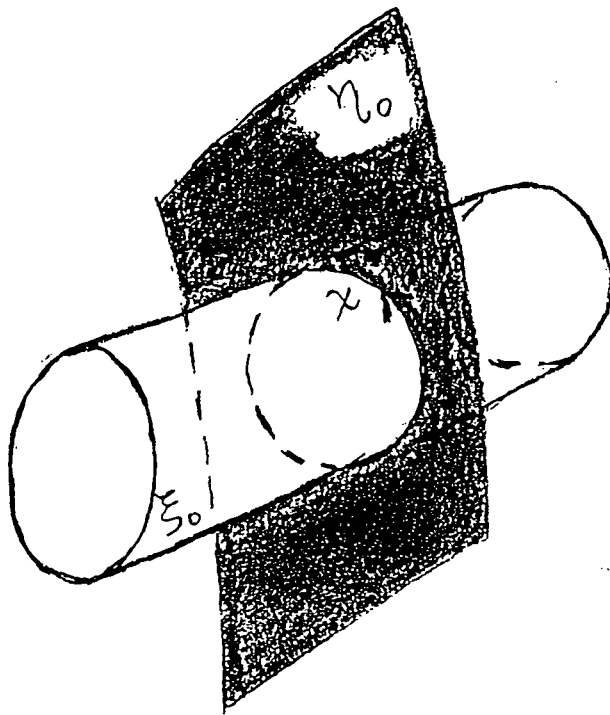
Fig. 1c. General shape of a $\xi=$const. stream surface intersecting a $\eta=$const. stream surface for Circulating Flows.

III. METRICS OF THE STREAMWISE COORDINATE SYSTEMS

The independent variables χ, ξ, and η form a *streamwise curvilinear coordinate system*. It is called streamwise, since, by virtue of the χ coordinate lines it is aligned with the streamlines. In this section we present the metric of the streamwise curvilinear coordinate system associated with each of the three different types of flows introduced in the previous section.

We start with a brief summary of the summation and the tensor notation used in this paper. Latin letters i, j, k, and m are used for free and dummy indices. Whenever the same Latin letter i, j, k, or m appears in a product, once as a subscript and once as a superscript, it is understood that this means a sum over all terms; thus, for example,

$$u^i g_{ij} = \sum_{i=1}^{3} u^i g_{ij} = u^\chi g_{\chi j} + u^\xi g_{\xi j} + u^\eta g_{\eta j}. \tag{3.1}$$

Indices χ, ξ, and η represent definite directions; *summation is never intended on index χ, ξ, or η no matter how they appear*. A subscript or a superscript proceeded by a comma denotes ordinary partial derivative. Thus, for example,

$$Y_{,\xi} = \frac{\partial Y}{\partial \xi} \quad , \quad \text{and} \quad g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}. \tag{3.2}$$

We now return to the calculation of metrics of the streamwise curvilinear coordinate systems. Covariant base vectors $\mathbf{g}_i$ of the (χ, ξ, η) coordinate system are calculated from the transformation formula,

$$\mathbf{g}_i = \frac{\partial \hat{x}^j}{\partial x^i} \hat{\mathbf{g}}_j \tag{3.3}$$

where x^i represent the coordinate lines of the (χ, ξ, η) system and $\hat{x}^j$ are the coordinate lines of the (x, y, z) system for the Plane Flows, and the coordinate lines of the (x, r, θ) coordinate system for the Axial and the Circulating Flows. In Eq.(3.3) $\hat{\mathbf{g}}_j$ are the base vectors of the $\hat{x}^j$ system and are for the (x, y, z) system

$$\hat{\mathbf{g}}_x = (1, 0, 0); \quad \hat{\mathbf{g}}_y = (0, 1, 0); \quad \hat{\mathbf{g}}_z = (0, 0, 1) \tag{3.4}$$

and for the (x, r, θ) system

$$\hat{\mathbf{g}}_x = (1, 0, 0); \quad \hat{\mathbf{g}}_r = (0, cos\Theta, sin\Theta); \quad \hat{\mathbf{g}}_\theta = (0, -Rsin\Theta, Rcos\Theta) \tag{3.5}$$

Carrying out the transformation (3.3), as shown in Appendix A.1, we obtain for the covariant base vectors of the Plane Flow (χ, ξ, η) coordinate system

$$g_\chi = (1, Y_{,\chi}, Z_{,\chi}); \quad g_\xi = (0, Y_{,\xi}, Z_{,\xi}); \quad g_\eta = (0, Y_{,\eta}, Z_{,\eta}) \tag{3.6}$$

All these three covariant base vectors are expressed by the single equation,

$$g_i = (\delta_{i\chi}, Y_{,i}, Z_{,i}) \tag{3.7}$$

where the Kronecker delta, $\delta_{i\chi}$, is equal to one if $i = \chi$ or zero otherwise. The covariant components of the metric tensor, g_{ij}, defined by $g_{ij} = g_i \cdot g_j$ (where the dot between the vectors indicates scaler product) are given by:

$$\boxed{g_{ij} = g_i \cdot g_j = \delta_{i\chi}\delta_{j\chi} + Y_{,i}Y_{,j} + Z_{,i}Z_{,j}} \tag{3.8}$$

(Another derivation of g_{ij} is given in Appendix A.3)

The determinant, g, of the matrix of the metric coefficients is

$$g = (g_\chi \cdot g_\xi \times g_\eta)^2 = (Y_{,\xi} Z_{,\eta} - Y_{,\eta} Z_{,\xi})^2. \tag{3.9}$$

where $\times$ denotes the cross product. Since $\sqrt{g}$ appears often later in the transport equations, we denote $\sqrt{g}$ by D, thus

$$\boxed{D = (Y_{,\xi} Z_{,\eta} - Y_{,\eta} Z_{,\xi}).} \tag{3.10}$$

To facilitate the representation of the contravariant components of the metric tensor we introduce $Y^{,i}$, the contravariant components of the gradient of Y, (also shown in the equation are $Y_{,i}$, the covariant components of the gradient of Y),

$$grad\, Y = Y^{,i}g_i = Y^{,\chi}g_\chi + Y^{,\xi}g_\xi + Y^{,\eta}g_\eta$$
$$= Y_{,i}g^i = Y_{,\chi}g^\chi + Y_{,\xi}g^\xi + Y_{,\eta}g^\eta \tag{3.11}$$

Gradient of Z is expressed similarly. Relations between $Y^{,i}$, $Y_{,i}$, $Z^{,i}$ and $Z_{,i}$ are given in Appendix A.2. The contravariant base vectors defined by,

$$g^k = \frac{g_i \times g_j}{D}$$

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are expressed in terms of $Y^{,i}$ and $Z^{,i}$ and are,

$$\mathbf{g}^i = (g_\chi^i, Y^{,i}, Z^{,i}) \qquad \text{where} \tag{3.12}$$

$$g_\chi^i = \delta_\chi^i - Y_{,\chi} Y^{,i} - Z_{,\chi} Z^{,i} \tag{3.13}$$

The Kronecker delta, δ_χ^i, is equal to one if $i = \chi$ or zero otherwise. The contravariant components of the metric tensor, g^{ij}, are

$$\boxed{g^{ij} = g_\chi^i g_\chi^j + Y^{,i} Y^{,j} + Z^{,i} Z^{,j}} \tag{3.14}$$

Expressions for the contravariant base vectors and the contravariant components of the metric tensor in terms of $Y_{,i}$ and $Z_{,i}$ are given in Appendix A.2.

Corresponding results for the Axial and Circulating flows are:

AXIAL FLOWS:

$$\mathbf{g}_i = (\delta_{i\chi}, R_{,i}\cos\Theta - \Theta_{,i}R\sin\Theta, R_{,i}\sin\Theta + \Theta_{,i}R\cos\Theta) \tag{3.15}$$

$$\boxed{g_{ij} = \delta_{i\chi}\delta_{j\chi} + R_{,i}R_{,j} + R^2\Theta_{,i}\Theta_{,j}} \tag{3.16}$$

$$\boxed{D = R(R_{,\xi}\,\Theta_{,\eta} - R_{,\eta}\,\Theta_{,\xi})} \tag{3.17}$$

$$\mathbf{g}^i = (g_\chi^i, R^{,i}\cos\Theta - \Theta^{,i}R\sin\Theta, R^{,i}\sin\Theta + \Theta^{,i}R\cos\Theta) \qquad \text{where} \tag{3.18}$$

$$g_\chi^i = \delta_\chi^i - R_{,\chi}R^{,i} - R^2\Theta_{,\chi}\Theta^{,i} \tag{3.19}$$

$$\boxed{g^{ij} = g_\chi^i g_\chi^j + R^{,i}R^{,j} + R^2\Theta^{,i}\Theta^{,j}} \tag{3.20}$$

CIRCULATING FLOWS:

$$\mathbf{g}_i = (X_{,i}, R_{,i}\cos\Theta - \delta_{i\chi}R\sin\Theta, R_{,i}\sin\Theta + \delta_{i\chi}R\cos\Theta) \tag{3.21}$$

$$\boxed{g_{ij} = X_{,i}X_{,j} + R_{,i}R_{,j} + \delta_{i\chi}\delta_{j\chi}R^2} \tag{3.22}$$

$$\boxed{D = R(X_{,\xi}\,R_{,\eta} - X_{,\eta}\,R_{,\xi})} \tag{3.23}$$

$$\mathbf{g}^i = (X^{,i}, R^{,i}\cos\Theta - g_\chi^i R\sin\Theta, R^{,i}\sin\Theta + g_\chi^i R\cos\Theta) \qquad \text{where} \tag{3.24}$$

$$g_\chi^i = (\delta_\chi^i - X_{,\chi}X^{,i} - R_{,\chi}R^{,i})/R^2 \tag{3.25}$$

$$g^{ij} = X^{,i}X^{,j} + R^{,i}R^{,j} + R^2 g_\chi^i g_\chi^j$$

(3.26)

Definitions of the Christoffel symbols Γ_{ij}^k and Γ_{ijk}, used in this study are,

$$\Gamma_{ij}^k = g^{km}\Gamma_{ijm},$$

(3.27a)

$$2\Gamma_{ijk} = g_{ik,j} + g_{jk,i} - g_{ij,k},$$

(3.27b)

and, in particular

$$2\Gamma_{\chi\chi i} = 2g_{\chi i,\chi} - g_{\chi\chi,i}, \quad \text{and} \quad 2\Gamma_{i\chi\chi} = g_{\chi\chi,i}.$$

(3.27c, d)

IV. TRANSPORT EQUATIONS

Equations for the transport of mass, momentum, and energy in a general curvilinear coordinate system have been derived previously and are given in books and review articles, among others, by Aris[17], Flugge[18], and Serrin[19]. In this section we adapt these equations to the streamwise curvilinear coordinate system. The transport equations are presented in terms of the metric of a streamwise curvilinear coordinate system; and, thus, are equally valid for all the three curvilinear coordinate systems introduced in the last section.

As discussed in section II, the velocity vector at any point is directed along g_χ. Thus, the contravariant velocity components u^ξ and u^η are zero everywhere. To calculate u^χ we use the relation

$$U^2 = u^i u^j g_{ij} = u^\chi u^\chi g_{\chi\chi}. \qquad (4.1)$$

Thus, u^i, the contravariant velocity components are:

$$\boxed{u^\chi = U/\sqrt{g_{\chi\chi}}; \quad u^\xi = 0; \quad u^\eta = 0.} \qquad (4.2)$$

The covariant velocity components, u_i, are calculated from

$$u_i = g_{ij} u^j = g_{i\chi} u^\chi, \qquad (4.3a)$$

and are:

$$\boxed{u_\chi = \frac{g_{\chi\chi}}{\sqrt{g}_{\chi\chi}} U; \quad u_\xi = \frac{g_{\chi\xi}}{\sqrt{g}_{\chi\chi}} U; \quad u_\eta = \frac{g_{\chi\eta}}{\sqrt{g}_{\chi\chi}} U.} \qquad (4.3b)$$

The contravariant velocity components, $\hat{u}^j$, in the $\hat{x}^j$ coordinate system can now be calculated from

$$\hat{u}^j = \frac{\partial \hat{x}^j}{\partial x^i} u^i = \frac{\partial \hat{x}^j}{\partial \chi} u^\chi \qquad (4.4a)$$

and the physical velocity components, $\hat{u}(j)$, in the $\hat{x}^j$ coordinate system from

$$\hat{u}(j) = \sqrt{\hat{g}_{jj}}\, \hat{u}^j \qquad \text{(no sum over } j) \qquad (4.4b)$$

where $\hat{g}_{jj}$, the covariant components of the metric tensor of the $\hat{x}^j$ coordinate system, are for the (x, y, z) system

$$\hat{g}_{xx} = 1; \quad \hat{g}_{yy} = 1; \quad \hat{g}_{zz} = 1 \qquad (4.4c)$$

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and for the (x, r, θ) system

$$\hat{g}_{xx} = 1; \quad \hat{g}_{rr} = 1; \quad \hat{g}_{\theta\theta} = r^2 = R^2 \tag{4.4d}$$

From (4.4a,b,c,d) we obtain the physical velocity components for the (see Appendix B.1):

Plane Flows: $\quad \hat{u}(x) = \dfrac{U}{\sqrt{g_{xx}}}; \quad \hat{u}(y) = \dfrac{U Y_{,x}}{\sqrt{g_{xx}}}; \quad$ and $\quad \hat{u}(z) = \dfrac{U Z_{,x}}{\sqrt{g_{xx}}} \tag{4.4e}$

Axial Flows: $\quad \hat{u}(x) = \dfrac{U}{\sqrt{g_{xx}}}; \quad \hat{u}(r) = \dfrac{U R_{,x}}{\sqrt{g_{xx}}}; \quad$ and $\quad \hat{u}(\theta) = \dfrac{R U \Theta_{,x}}{\sqrt{g_{xx}}} \tag{4.4f}$

Circulating Flows: $\quad \hat{u}(x) = \dfrac{U X_{,x}}{\sqrt{g_{xx}}}; \quad \hat{u}(r) = \dfrac{U R_{,x}}{\sqrt{g_{xx}}}; \quad$ and $\quad \hat{u}(\theta) = \dfrac{R U}{\sqrt{g_{xx}}} \tag{4.4g}$

The expressions for the covariant derivatives $u^i|_j$ and $u_i|_j$ in terms of the Christoffel symbols used latter are (see Appendix B.2):

$$u^i|_j = u^i_{,j} + u^k \Gamma^i_{jk} = u^i_{,j} + u^x \Gamma^i_{jx}. \tag{4.5a}$$

$$u_i|_j = u^k_{,j} g_{ki} + u^k \Gamma_{kji} = u^x_{,j} g_{xi} + u^x \Gamma_{xji} \tag{4.5b}$$

$$u_i|_j = u_{i,j} - u^k \Gamma_{ijk} = u_{i,j} - u^x \Gamma_{ijx} \tag{4.5c}$$

In Eqs.(4.5) we have evaluated the sums over k using the fact that u^x is the only nonzero contravariant velocity component.

We next introduce expressions for the vorticity ω,

$$\omega^k = \epsilon^{ijk} u_j|_i \tag{4.6}$$

where ϵ^{ijk}, the permutation tensor, is equal to $+1/D$, $-1/D$, or 0 depending on whether i, j, k is a cyclic, an anticyclic, or an acyclic sequence. From (4.6) we obtain for the individual components (see Appendix B.3),

$$\boxed{\omega^x = (u_{\eta,\xi} - u_{\xi,\eta})/D} \tag{4.7a}$$

$$\boxed{\omega^\xi = (u_{x,\eta} - u_{\eta,x})/D} \tag{4.7b}$$

$$\boxed{\omega^\eta = (u_{\xi,x} - u_{x,\xi})/D} \tag{4.7c}$$

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We are now ready to develop the transport equations for a streamwise curvilinear coordinate system

CONTINUITY EQUATION:

$$\frac{\partial \rho}{\partial t} + div(\rho \mathbf{v}) = 0,\qquad(4.8)$$

where ρ is density. In streamwise computation of flow fields, u^χ is the only non-zero contravariant velocity component and the preceding equation becomes,

$$\frac{\partial \rho}{\partial t} + \frac{1}{D}\frac{\partial}{\partial \chi}(D\rho u^\chi) = 0.\qquad(4.9)$$

For steady flow $\rho_{,t} = 0$, and we obtain on integrating the remaining part of (4.9),

$$D\rho u^\chi = \text{a constant along } \chi.\qquad(4.10)$$

We select the numerical values of our stream functions such that the constant of integration is unity. The steady state continuity equation thus becomes,

$$\boxed{u^\chi = \frac{1}{\rho D}},\qquad(4.11a)$$

and with the help of (4.2),

$$U = \frac{\sqrt{g}_{\chi\chi}}{\rho D}.\qquad(4.11b)$$

Relationship of Eqs.(4.11b) and (4.4e) of this section with Eq.(1.6) of the introduction section is discussed in Appendix C.

MOMENTUM EQUATION:

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\frac{\partial \mathbf{v}}{\partial t} + \rho u^i \mathbf{v}_{,i} = \mathbf{F},\qquad(4.12)$$

where $\mathbf{F}$ is the force (surface and body) per unit volume. Once again, since u^χ is the only non-zero contravariant velocity component, the covariant and the contravariant momentum equations become, respectively,

$$\rho\frac{\partial}{\partial t}(u_i g^i) + \rho u^\chi u_i|_\chi g^i = F_i g^i.\qquad(4.13a)$$

$$\rho\frac{\partial}{\partial t}(u^\chi g_\chi) + \rho u^\chi u^i|_\chi g_i = F^i g_i.\qquad(4.13b)$$

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For steady flow Eqs.(4.13) become,

$$\text{Covariant:} \qquad \rho u^X u_i|_X = F_i \qquad\qquad (4.14a)$$

$$\text{Contravariant:} \qquad \rho u^X u^i|_X = F^i \qquad\qquad (4.14b)$$

Development of the covariant components of the momentum equation:

With the help of (4.5b) we rewrite Eq.(4.14a) as

$$\rho u^X (u^X_{,X} g_{Xi} + u^X \Gamma_{XXi}) = F_i . \qquad\qquad (4.15)$$

With the help of continuity equation (4.11a), (4.15) becomes

$$\frac{1}{D} \frac{\partial}{\partial \chi} \left(\frac{1}{\rho D} \right) g_{Xi} + \frac{1}{\rho D^2} \Gamma_{XXi} = F_i . \qquad\qquad (4.16)$$

Expressing Γ_{XXi} via (3.27c) we obtain after some algebra,

$$\boxed{\rho D g_{Xi,X} - g_{Xi}(\rho D)_{,X} - \frac{1}{2} \rho D g_{XX,i} = \rho^2 D^3 F_i} \qquad\qquad (4.17)$$

Another variation of momentum equation (4-14a) is obtained by using the expression for $u_i|_j$ given in (4.5c),

$$\rho u^X (u_{i,X} - u^X \Gamma_{iXX}) = F^i . \qquad\qquad (4.18)$$

Using (3.27d) to express Γ_{iXX} and that

$$u_{X,i} = (\sqrt{g_{XX}} U)_{,i} = \sqrt{g_{XX}} U_{,i} + \frac{1}{2} U g_{XX,i}/\sqrt{g_{XX}} = \sqrt{g_{XX}} U_{,i} + \frac{1}{2} u^X g_{XX,i} \qquad\qquad (4.19)$$

we obtain from (4.18),

$$\rho u^X (u_{i,X} - u_{X,i} + \sqrt{g_{XX}} U_{,i}) = F_i . \qquad\qquad (4.20)$$

With the help of the continuity equation and some rearrangement we obtain from (4.20)

$$\rho U U_{,i} = F_i - \frac{1}{D}(u_{i,X} - u_{X,i}) . \qquad\qquad (4.21a)$$

Writing out the individual components, and using the expressions for ω given in (4.7), we have,

$$\boxed{\rho U U_{,X} = F_X, \quad \rho U U_{,\xi} = F_\xi - \omega^\eta, \quad \rho U U_{,\eta} = F_\eta + \omega^\xi .} \qquad\qquad (4.21b,c,d)$$

Development of the contravariant components of the momentum equation:

With the help of (4.5a) we rewrite Eq.(4.14b) as

$$\rho u^{\chi}(u^i_{,\chi} + u^{\chi}\Gamma^i_{\chi\chi}) = F^i .$$ (4.22)

With the help of continuity equation (4.22) becomes

$$\boxed{\frac{\partial}{\partial\chi}\left(\frac{1}{\rho D}\right) = D\, F^{\chi}, \quad \Gamma^{\xi}_{\chi\chi} = \rho D^2 F^{\xi}, \quad \Gamma^{\eta}_{\chi\chi} = \rho D^2 F^{\eta} .}$$ (4.23a, b, c)

ENERGY EQUATION:

$$\frac{\partial(\rho e)}{\partial t} + div(\rho e\mathbf{v}) = -div\mathbf{q} + div(\mathbf{T}.\mathbf{v}) + \rho\mathbf{v}.\mathbf{b} ,$$ (4.24)

where e is the energy (sum of internal, kinetic, and potential) per unit mass, $\mathbf{q}$ the heat flux, $\mathbf{T}$ the stress tensor, and $\mathbf{b}$ the body force per unit mass. Once again using the fact that u^{χ} is the only non-zero velocity component, we obtain from (4.24) for steady flow,

$$\boxed{\frac{1}{D}\frac{\partial}{\partial\chi}(D\rho e u^{\chi}) = -\frac{1}{D}\frac{\partial}{\partial x^i}(Dq^i) + \frac{1}{D}\frac{\partial}{\partial x^i}(DT^{ij}u_j) + \rho u^{\chi}b_{\chi} .}$$ (4.25)

CALCULATION OF F_i:

We separate the pressure, the viscous, and the body force contributions to F_i and write

$$F_i = -p_{,i} + f_i + \rho b_i$$ (4.26)

where b_i is the body force, and the viscous contribution,

$$f_k = g_{ki}f^i = g_{ki}\tau^{ij}|_j = g_{ki}\left[\frac{1}{D}\frac{\partial}{\partial x^j}\left(D\tau^{ij}\right) + \tau^{jm}\Gamma^i_{jm}\right]$$ (4.27)

with the viscous stress tensor τ^{ij} given by

$$\tau^{ij} = -\frac{2}{3}\mu u^m|_m\, g^{ij} + \mu(g^{jm}\,u^i|_m + g^{im}\,u^j|m) .$$ (4.28)

The stress tensor T^{ij} is given by

$$T^{ij} = -pg^{ij} + \tau^{ij}$$ (4.29)

V. POTENTIAL FLOW

In this section we discuss the inviscid approximation to the transport equations presented in the preceding section. The main purpose of this discussion is to illustrate the procedure for obtaining transport equations for a particular type of flow (for example, Plane, Axial, or Circulating) from the general transport equations.

For inviscid flow and neglecting body forces,

$$F_i = -p_{,i} \,, \tag{5.1}$$

and the momentum equation (4.21b) becomes,

$$\rho U U_{,X} = -p_{,X}. \tag{5.2}$$

We further assume that the flow is isentropic, and thus (see Appendix D.1),

$$\frac{dp}{\rho} = d \left(\frac{\gamma}{\gamma - 1} \frac{p}{\rho} \right) \tag{5.3}$$

where γ is the ratio of specific heats. With the help of (5.3) we integrate (5.2) and obtain,

$$\frac{U^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = const. \tag{5.4}$$

With the help of the continuity equation (4.11b), we rewrite (5.4) as,

$$\frac{g_{XX}}{2\rho^2 D^2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = const. \tag{5.5}$$

From (5.3) and (5.5) we obtain,

$$p_{,i} = -\rho (g_{XX}/2\rho^2 D^2)_{,i} \tag{5.6}$$

From here on subscript i stands only for ξ and η. From Eqs.(4.17), (5.1), and (5.6) we obtain,

$$\rho D g_{Xi},_X - g_{Xi}(\rho D)_{,X} - \frac{1}{2}\rho D g_{XX},_i = \rho^3 D^3 (g_{XX}/2\rho^2 D^2)_{,i} \tag{5.7}$$

With a little algebra (5.7) can be rewritten as (see Appendix D.2),

$$\boxed{(g_{Xi}/\rho D)_{,X} - (g_{XX}/\rho D)_{,i} = 0.} \tag{5.8}$$

Equation (5.8) can also be obtained by setting the vorticity ω^i (see Eqs. 4.7b and 4.7c) equal to zero and using the continuity equation (4.11a) along with Eq (4.3a) to express u_i, the covariant components of the velocity.

To separate the effects of compressibility we rewrite (5.7) as,

$$g_{\chi i,\chi}D - g_{\chi i}D_{,\chi} - g_{\chi\chi,i}D + g_{\chi\chi}D_{,i} = \frac{D}{\rho}(g_{\chi i}\rho_{,\chi} - g_{\chi\chi}\rho_{,i}) \tag{5.9}$$

For incompressible flows the right hand side of equation (5.9) vanishes and we have,

$$g_{\chi i,\chi}D - g_{\chi i}D_{,\chi} - g_{\chi\chi,i}D + g_{\chi\chi}D_i = 0 \tag{5.10}$$

Equation (5.8) is our basic equation to describe isentropic flows. Upon substitution of appropriate expressions for $g_{\chi i}$, $g_{\chi\chi}$, and D from section III for a given type of flow, Eq. (5.8) will yield the streamwise isentropic flow equations for that particular flow. In the following we illustrate the procedure for obtaining isentropic flow equations for a particular type of flow from the general equation (5.8).

Incompressible Plane Flows: Upon substituting derivatives of g_{ij} and D from Eqs.(3.9) and (3.10) into (5.10) we get the following equation for the χ-ξ stream surfaces,

$$D(Y_{,\chi\chi}Y_{,\xi} + Z_{,\chi\chi}Z_{,\xi} - Y_{,\chi\xi}Y_{,\chi} - Z_{,\chi\xi}Z_{,\chi}) - g_{\chi\xi}(Y_{,\chi\xi}Z_{,\eta} + Y_{,\xi}Z_{,\chi\eta} - Y_{,\chi\eta}Z_{,\xi} + Y_{,\eta}Z_{,\chi\xi})$$

$$+g_{\chi\chi}(Y_{,\xi\xi}Z_{,\eta} + Y_{,\xi}Z_{,\xi\eta} - Y_{,\xi\eta}Z_{,\xi} + Y_{,\eta}Z_{,\xi\xi}) = 0, \tag{5.11a}$$

and the following equation for the χ-η stream surfaces,

$$D(Y_{,\chi\chi}Y_{,\eta} + Z_{,\chi\chi}Z_{,\eta} - Y_{,\chi\eta}Y_{,\chi} - Z_{,\chi\eta}Z_{,\chi}) - g_{\chi\eta}(Y_{,\chi\xi}Z_{,\eta} + Y_{,\xi}Z_{,\chi\eta} - Y_{,\chi\eta}Z_{,\xi} - Y_{,\eta}Z_{,\chi\xi})$$

$$+g_{\chi\chi}(Y_{,\xi\eta}Z_{,\eta} + Y_{,\xi}Z_{,\eta\eta} - Y_{,\eta\eta}Z_{,\xi} - Y_{,\eta}Z_{,\xi\eta}) = 0, \tag{5.11b}$$

where D, $g_{\chi\xi}$, and $g_{\chi\eta}$ are given by Eqs.(3.9) and (3.10). We have two equations for two unknowns Y and Z. These equations were solved numerically in Ref. 16 for (i) flow through a rectangular diffuser with an offset and a change in the aspect ratio, and (ii) flow through a duct whose cross-section changes from a square to a rhombus. For a two-dimensional flow in the χ-ξ plane

$$Y_{,\eta} = Z_{,\chi} = Z_{,\xi} = 0, \quad \text{and} \quad Z_{,\eta} = 1$$

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and Eq.(5.11a) reduces to,

$$Y_{,\xi}^2 Y_{,\chi\chi} - 2Y_{,\chi}Y_{,\xi}Y_{,\chi\xi} + (1 + Y_{,\chi}^2)Y_{,\xi\xi} = 0 \qquad (5.12)$$

This equation was solved for flow past an infinite cylinder placed symmetrically between two parallel plates in Ref. 15.

Equations for axial and circulating flows are obtained in a similar fashion. Solution of the axial flow equations for flow through superelliptic transitional ducts is given in a forthcoming article. We next present one special case each of axial and circulating flow.

AXIAL FLOW SPECIAL CASE: For axisymmetric flow in χ-ξ,

$$\Theta_{,\xi} = \Theta_{,\chi} = R_{,\eta} = 0, \quad \text{and} \quad \Theta_{,\eta} = 1 \qquad (5.13a)$$

and we have from Eqs.(3.14) and (3.15)

$$g_{\chi\chi} = 1 + R_{,\chi}^2, \quad g_{\chi\xi} = R_{,\chi}R_{,\xi}, \quad \text{and} \quad D = RR_{,\xi} \qquad (5.13b)$$

With the help of Eqs.(5.13), Eq.(5.9) becomes,

$$R_{,\xi}^2 R_{,\chi\chi} - 2R_{,\chi}R_\xi R_{,\chi\xi} + (1 + R_{,\chi}^2)R_{,\xi\xi} + R_{,\xi}^2/R = (R_{,\xi}/\rho)[R_{,\chi}R_{,\xi}\rho_{,\chi} - (1 + R_{,\chi}^2)\rho_\xi] \qquad (5.14)$$

Equation (5.14) can be used to study isentropic flow through nozzles.

CIRCULATING FLOW SPECIAL CASE: For two-dimensional circulating flows in χ-R

$$X_{,\chi} = X_{,\xi} = R_{,\eta} = 0, \quad \text{and} \quad X_{,\eta} = 1 \qquad (5.15a)$$

and we have from Eqs.(3.22) and (3.23)

$$g_{\chi\chi} = R^2 + R_{,\chi}^2, \quad g_{\chi\xi} = R_{,\chi}R_{,\xi}, \quad \text{and} \quad D = -RR_{,\xi} \qquad (5.15b)$$

With the help of Eqs.(5.15), Eq.(5.10) becomes,

$$R_{,\xi}^2 R_{,\chi\chi} - 2R_{,\chi}R_{,\xi}R_{,\chi\xi} + (R^2 + R_{,\chi}^2)R_{,\xi\xi} - RR_{,\xi}^2 = 0 \qquad (5.16)$$

We note that the free line vortex given by

$$R = const.\ e^\xi \qquad (5.17)$$

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is a solution of Eq.(5.16). To see the relation between (5.17) and the more familiar form of the line vortex equation, we obtain from Eqs.(4.4g) for the solution given in (5.17)

$$\hat{u}(x) = 0, \quad \hat{u}(r) = 0, \quad \text{and} \quad \hat{u}(\theta) = U \tag{5.18}$$

For compressible flows ξ and η are related to the mass flow rate with the SI units of $\sqrt{kg/s}$. For flows with constant density we rewrite ρD for the circulating flow as

$$\rho D = R\left(X_{,(\xi/\sqrt{\rho})}\, R_{,(\eta/\sqrt{\rho})} - X_{,(\eta/\sqrt{\rho})}\, R_{,(\xi/\sqrt{\rho})} \right) \tag{5.19}$$

For such flows we absorb $\sqrt{\rho}$ into the definitins of ξ and η, which now become related to the volume flow rate with the SI units of $\sqrt{m^3/s}$. Thus, for flows with constant density the continuity equation (4.11) becomes in terms of the newly defined ξ and η,

$$U = \frac{\sqrt{g_{XX}}}{D} \tag{5.20}$$

By the use of (5-15b) and (5-17) we obtain from (5.20)

$$U = \frac{\sqrt{g_{XX}}}{D} = -\frac{R}{RR_{,\xi}} = -\frac{1}{const.R} \tag{5.21}$$

From (5.18) and (5.21) we obtain the familiar form of the line vortex equation.

VI. PARABOLIZED FLOWS

In this section we present an approximation to the viscous forces which for two dimensional flows reduces to the well known boundary layer approximation. For lack of a better name we have called it parabolized flow approximation.

We assume that all contributions to the viscous forces that arise from the field variation of the curvature of the coordinate system (that is, from the terms involving Christoffel symbols) are negligible. Thus, (a)

$$f^i = \tau^{ij}|_j = \frac{1}{D}\frac{\partial}{\partial x^j}\left(D\tau^{ij}\right),\tag{6.1}$$

and, (b) $u^i|_j$ are equal to $u^i_{,j}$ which have only one non zero component $u^\chi_{,j}$. Approximation (b) leads to that

$$\tau^{\chi\chi} = (-2/3)\mu g^{\chi\chi} u^\chi_{,\chi} + \mu g^{\chi m} u^\chi_{,m}, \quad \tau^{\chi\xi} = \tau^{\xi\chi} = \mu g^{\xi m} u^\chi_{,m},$$

$$\tau^{\chi\eta} = \tau^{\eta\chi} = \mu g^{\eta m} u^\chi_{,m}, \quad \tau^{\xi\xi} = (-2/3)\mu g^{\xi\xi} u^\chi_{,\chi}, \quad \tau^{\eta\eta} = (-2/3)\mu g^{\eta\eta} u^\chi_{,\chi},\tag{6.2}$$

and that $\tau^{\xi\eta}$ is negligible. We also assume that the contributions of the cross-stream velocity derivative terms $u^\chi_{,\xi}$ and $u^\chi_{,\eta}$ to the stress dominate over the contribution of the $u^\chi_{,\chi}$ terms. Thus, the only non-negligible terms of the stress tensor we are left with are,

$$\tau^{\chi\chi} = \mu(g^{\chi\xi} u^\chi_{,\xi} + g^{\chi\eta} u^\chi_{,\eta})\tag{6.3a}$$

$$\tau^{\chi\xi} = \tau^{\xi\chi} = \mu(g^{\xi\xi} u^\chi_{,\xi} + g^{\xi\eta} u^\chi_{,\eta})\tag{6.3b}$$

$$\tau^{\chi\eta} = \tau^{\eta\chi} = \mu(g^{\xi\eta} u^\chi_{,\xi} + g^{\eta\eta} u^\chi_{,\eta})\tag{6.3c}$$

We further assume that in Eq.(6.1) the contribution of the term involving the derivative with respect to χ is negligible, thus

$$f^i = \frac{1}{D}\frac{\partial}{\partial \xi}\left(D\tau^{i\xi}\right) + \frac{1}{D}\frac{\partial}{\partial \eta}\left(D\tau^{i\eta}\right)\tag{6.4}$$

From (6.3) and (6.4) we have that

$$f^\chi = \frac{1}{D}\frac{\partial}{\partial \xi}\left[\mu D\left(g^{\xi\xi} u^\chi_{,\xi} + g^{\xi\eta} u^\chi_{,\eta}\right)\right] + \frac{1}{D}\frac{\partial}{\partial \eta}\left[\mu D\left(g^{\eta\xi} u^\chi_{,\xi} + g^{\eta\eta} u^\chi_{,\eta}\right)\right]\tag{6.5}$$

and that f^ξ and f^η are negligible. For the covariant component of the viscous force we now have

$$f_i = g_{ij}f^j = g_{i\chi}f^\chi, \tag{6.6a}$$

which leads to

$$f_\chi = g_{\chi\chi}f^\chi, \quad f_\xi = g_{\xi\chi}f^\chi, \quad f_\eta = g_{\eta\chi}f^\chi. \tag{6.6b, c, d}$$

We conclude this section by deriving the viscous force term used in the boundary layer approximation for two dimensional flows from the viscous force approximation given by Eq. (6.5). From equation (6.5) we obtain for two dimensional plane flows in (x,y) with $\chi = x$, $\xi = y$, $g^{yy} = 1$, $D = 1$, and the physical velocity along x, $u = \sqrt{g_{xx}}\, u^x = u^x$,

$$f^x = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{6.7a}$$

and for axisymmetric flows in (x,r) with $\chi = x$, $\xi = r$, $g^{rr} = 1$, $D = r$, and the physical velocity along x, $u = \sqrt{g_{xx}}\, u^x = u^x$,

$$f^x = \frac{1}{r}\frac{\partial}{\partial r}\left(\mu r\frac{\partial u}{\partial r}\right). \tag{6.7b}$$

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VII. CONCLUDING REMARKS

A theory to compute three-dimensional flows using two stream functions is presented. The values of two stream functions along with a spatial coordinate are used as independent variables. Since the value of a stream function is constant along the solid boundaries, this choice of variables makes it easy to satisfy the boundary conditions. The dependent variables employed are the streamwise velocity and two functions that define the stream surfaces. Thus, the method generates a boundary fitted grid that is aligned with the flow streamlines as part of its solution.

For three-dimensional flows so far there is no general theory to define two sets of intersecting stream surfaces to cover a given flow. In the present work we have presented three different combinations, each consisting of two sets of intersecting stream surfaces, which should cover many flows of practical interest. However, a general theory regarding the selection of two sets of intersecting stream surfaces that will cover a given flow for the most efficient computation of the flow is needed.

APPENDIX A

A.1 In this appendix we derive the covariant base vectors of the streamwise curvilinear coordinate system, x^i, for the three different types of flows introduced in section II.

PLANE FLOWS:

Base vectors, $\hat{g}_i$, of the cartesian coordinate system $\hat{x}^i = (x, y, z)$

$$\hat{g}_x = (1,0,0); \quad \hat{g}_y = (0,1,0); \quad \hat{g}_z = (0,0,1) \qquad (A.1.1)$$

are transformed by

$$g_i = \frac{\partial \hat{x}^j}{\partial x^i}\hat{g}_j \qquad (A.1.2)$$

to obtain the base vectors, g_i, of the $x^i = (\chi, \xi, \eta)$ coordinate system. From Eqs. (A.1.1) and (A.1.2) we obtain,

$$g_i = \frac{\partial x}{\partial x^i}(1,0,0) + \frac{\partial y}{\partial x^i}(0,1,0) + \frac{\partial z}{\partial x^i}(0,0,1) = (\frac{\partial x}{\partial x^i}, \frac{\partial y}{\partial x^i}, \frac{\partial z}{\partial x^i}) \qquad (A.1.3)$$

Now using the transformation equations

$$x = \chi, \qquad y = Y(\chi, \xi, \eta), \quad \text{and} \quad z = Z(\chi, \xi, \eta), \qquad (A.1.4)$$

we obtain Eqs.(3.6) from Eqs.(A.1.3).

AXIAL FLOWS:

Base vectors, $\hat{g}_i$, of the polar cylindrical coordinate system $\hat{x}^i = (x, r, \theta)$

$$\hat{g}_x = (1,0,0); \quad \hat{g}_r = (0, \cos\theta, \sin\theta); \quad \hat{g}_\theta = (0, -r\sin\theta, r\cos\theta) \qquad (A.1.5)$$

are transformed by Eq.(A.1.2) to obtain the base vectors, g_i, of the $x^i = (\chi, \xi, \eta)$ coordinate system,

$$g_i = \frac{\partial x}{\partial x^i}(1,0,0) + \frac{\partial r}{\partial x^i}(0, \cos\theta, \sin\theta) + \frac{\partial \theta}{\partial x^i}(0, -r\sin\theta, r\cos\theta)$$

$$= (\frac{\partial x}{\partial x^i}, \frac{\partial r}{\partial x^i}\cos\theta - \frac{\partial \theta}{\partial x^i}r\sin\theta, \frac{\partial r}{\partial x^i}\sin\theta - \frac{\partial \theta}{\partial x^i}r\cos\theta). \qquad (A.1.6)$$

Now using the transformation equations

$$x = \chi, \qquad r = R(\chi, \xi, \eta), \quad \text{and} \quad \theta = \Theta(\chi, \xi, \eta), \qquad (A.1.7)$$

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we obtain Eqs.(3.15) from Eqs.(A.1.6).

CIRCULATING FLOWS:

Base vectors, $\hat{g}_i$, of the polar cylindrical coordinate system $\hat{x}^i = (x, r, \theta)$ given in (A.1.5) are transformed by Eq.(A.1.2) to obtain the base vectors, g_i, of the $x^i = (\chi, \xi, \eta)$ coordinate system, once again, given by Eq.(A.1.6). Now using the transformation equations

$$x = X(\chi, \xi, \eta), \qquad r = R(\chi, \xi, \eta), \quad \text{and} \quad \theta = \chi \qquad (A.1.8)$$

we obtain Eqs.(3.21) from Eqs.(A.1.6).

A.2 In this appendix we present the contravariant base vectors and the contravariant components of the the metric tensor in terms $Y_{,i}$ and $Z_{,i}$.

PLANE FLOWS:

$$g^\chi = \frac{g_\xi \times g_\eta}{D} = (1, 0, 0)$$
$$g^\xi = \frac{g_\eta \times g_\chi}{D} = \frac{(Y_{,\eta} Z_{,\chi} - Y_{,\chi} Z_{,\eta} \,,\, Z_{,\eta} \,,\, -Y_{,\eta})}{D} \qquad (A.2.1a, b, c)$$
$$g^\eta = \frac{g_\chi \times g_\xi}{D} = \frac{(Y_{,\chi} Z_{,\xi} - Y_{,\xi} Z_{,\chi} \,,\, -Z_{,\xi} \,,\, Y_{,\xi})}{D}$$

The contravariant components of the metric tensor, g^{ij}, defined by $g^{ij} = g^i . g^j$, are:

$$g^{\chi\chi} = 1 \qquad (A.2.2a)$$

$$g^{\chi\xi} = g^{\xi\chi} = (Y_{,\eta} Z_{,\chi} - Y_{,\chi} Z_{,\eta})/D \qquad (A.2.2b)$$

$$g^{\chi\eta} = g^{\eta\chi} = (Y_{,\chi} Z_{,\xi} - Y_{,\xi} Z_{,\chi})/D \qquad (A.2.2c)$$

$$g^{\xi\xi} = (g^{\chi\xi})^2 + (Y_{,\eta}^2 + Z_{,\eta}^2)/D^2 \qquad (A.2.2d)$$

$$g^{\xi\eta} = g^{\eta\xi} = g^{\chi\xi} g^{\chi\eta} - (Y_{,\xi} Y_{,\eta} + Z_{,\xi} Z_{,\eta})/D^2 \qquad (A.2.2e)$$

$$g^{\eta\eta} = (g^{\chi\eta})^2 + (Y_{,\xi}^2 + Z_{,\xi}^2)/D^2 \qquad (A.2.2f)$$

Relations between the covariant and the contravariant components of *gradY* and *gradZ* are obtained from

$$Y^{,i} = g^{ij} Y_{,j}, \qquad Z^{,i} = g^{ij} Z_{,j}, \qquad (A.2.2g)$$

with g^{ij} given in (A2.2), and are,

$$Y^{,X} = 0, \quad Y^{,\xi} = Z_{,\eta}/D, \quad Y^{,\eta} = -Z_{,\xi}/D \tag{A.2.2h}$$

$$Z^{,X} = 0, \quad Z^{,\xi} = -Y_{,\eta}/D, \quad Z^{,\eta} = Y_{,\xi}/D \tag{A.2.2i}$$

$$Y^{,i}Y_{,i} = Z^{,i}Z_{,i} = 1 \quad \text{and} \quad Y^{,i}Z_{,i} = Z^{,i}Y_{,i} = 0 \tag{A.2.2j}$$

AXIAL FLOWS:

$$\mathbf{g}^{X} = (1,0,0)$$

$$\mathbf{g}^{\xi} = \frac{1}{D}[R(R_{,\eta}\Theta_{,X} - R_{,X}\Theta_{,\eta}), (R_{,\eta}\sin\theta + \theta_{,\eta}R\cos\theta), (-R_{,\eta}\cos\theta + \Theta_{,\eta}R\sin\Theta)]$$

$$\mathbf{g}^{\eta} = \frac{1}{D}[R(R_{,X}\Theta_{,\xi} - R_{,\xi}\Theta_{,X}), -(R_{,\xi}\sin\theta + \theta_{,\xi}R\cos\theta), (R_{,\xi}\cos\theta - \Theta_{,\xi}R\sin\Theta)]$$

$$\tag{A.2.3a, b, c}$$

$$g^{XX} = 1 \tag{A.2.4a}$$

$$g^{X\xi} = g^{\xi X} = R(R_{,\eta}\Theta_{,X} - R_{,X}\Theta_{,\eta})/D \tag{A.2.4b}$$

$$g^{X\eta} = g^{\eta X} = R(R_{,X}\Theta_{,\xi} - R_{,\xi}\Theta_{,X})/D \tag{A.2.4c}$$

$$g^{\xi\xi} = (g^{X\xi})^2 + (R_{,\eta}^2 + R^2\Theta_{,\eta}^2)/D^2 \tag{A.2.4d}$$

$$g^{\xi\eta} = g^{\eta\xi} = g^{X\xi}g^{X\eta} - (R_{,\xi}R_{,\eta} + R^2\Theta_{,\xi}\Theta_{,\eta})/D^2 \tag{A.2.4e}$$

$$g^{\eta\eta} = (g^{X\eta})^2 + (R_{,\xi}^2 + R^2\Theta_{,\xi}^2)/D^2 \tag{A.2.4f}$$

$$R^{,X} = 0, \quad R^{,\xi} = R\Theta_{,\eta}/D \quad R^{,\eta} = -R\Theta_{,\xi}/D \tag{A.2.4g}$$

$$\Theta^{,X} = 0, \quad R\Theta^{,\xi} = -R_{,\eta}/D, \quad R\Theta^{,\eta} = R_{,\xi}/D \tag{A.2.4h}$$

$$R^{,i}R_{,i} = R^2\Theta^{,i}\Theta_{,i} = 1 \quad \text{and} \quad R^{,i}R\Theta_{,i} = R\Theta^{,i}R_{,i} = 0 \tag{A.2.4i}$$

CIRCULATING FLOWS:

$$\mathbf{g}^X = (0, -sin\Theta, cos\Theta)/R$$

$$\mathbf{g}^\xi = \frac{1}{D}[RR_{,\eta}, (R_{,\eta}X_{,\chi} - R_{,\chi}X_{,\eta})sin\Theta - X_{,\eta}Rcos\Theta, (X_{,\eta}R_{,\chi} - X_{,\chi}R_{,\eta})cos\Theta - RX_{,\eta}sin\Theta]$$

$$\mathbf{g}^\eta = \frac{1}{D}[RR_{,\xi}, (R_{,\chi}X_{,\xi} - R_{,\xi}X_{,\chi})sin\Theta + X_{,\xi}Rcos\Theta, (X_{,\chi}R_{,\xi} - X_{,\xi}R_{,\chi})cos\Theta + X_{,\xi}Rsin\Theta]$$

$$(A.2.5a,b,c)$$

$$g^{XX} = 1/R^2 \qquad (A.2.6a)$$

$$g^{X\xi} = g^{\xi X} = (X_{,\eta}R_{,\chi} - X_{,\chi}R_{,\eta})/RD \qquad (A.2.6b)$$

$$g^{X\eta} = g^{\eta X} = (X_{,\chi}R_{,\xi} - X_{,\xi}R_{,\chi})/RD \qquad (A.2.6c)$$

$$g^{\xi\xi} = R^2[(g^{X\xi})^2 + (X_{,\eta}^2 + R_{,\eta}^2)/D^2] \qquad (A.2.6d)$$

$$g^{\xi\eta} = g^{\eta\xi} = R^2[(g^{X\xi}g^{X\eta} - (X_{,\xi}X_{,\eta} + R_{,\xi}R_{,\eta})/D^2] \qquad (A.2.6e)$$

$$g^{\eta\eta} = R^2[(g^{X\eta})^2 + (X_{,\xi}^2 + R_{,\xi}^2)/D^2] \qquad (A.2.6f)$$

$$X^{,X} = 0, \quad X^{,\xi} = RR_{,\eta}/D, \quad X^{,\eta} = -RR_{,\xi}/D \qquad (A.2.6g)$$

$$R^{,X} = 0, \quad R^{,\xi} = -RX_{,\eta}/D, \quad R^{,\eta} = RX_{,\xi}/D \qquad (A.2.6h)$$

$$X^{,i}X_{,i} = R^{,i}R_{,i} = 1 \quad \text{and} \quad X^{,i}R_{,i} = R^{,i}X_{,i} = 0 \qquad (A.2.6i)$$

A.3 In this appendix we calculate g_{ij} from the transformation formula,

$$g_{ij} = \frac{\partial \hat{x}^m}{\partial x^i}\frac{\partial \hat{x}^n}{\partial x^j}\hat{g}_{mn} \qquad (A.3.1)$$

where $\hat{g}_{mn}$ are the covariant coefficients of the metric tensor of the (x,y,z) coordinate system for the Plane flows, and of the (x,r,θ) cordinate system for the Axial and the

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Circulating flows. For all three flows (Plane, Axial, and Circulating), $\hat{g}_{mn} = 0$ if $m \neq n$ and (A.3.1) reduces to,

$$g_{ij} = \frac{\partial \hat{x}^1}{\partial x^i} \frac{\partial \hat{x}^1}{\partial x^j} \hat{g}_{11} + \frac{\partial \hat{x}^2}{\partial x^i} \frac{\partial \hat{x}^2}{\partial x^j} \hat{g}_{22} + \frac{\partial \hat{x}^3}{\partial x^i} \frac{\partial \hat{x}^3}{\partial x^j} \hat{g}_{33} \qquad (A.3.2)$$

PLANE FLOWS:

By the use of following transformation equations and $\hat{g}_{ii}$

$$x = \chi, \qquad y = Y(\chi, \xi, \eta), \qquad z = Z(\chi, \xi, \eta), \qquad (2.1a)$$

$$\hat{g}_{xx} = 1; \quad \hat{g}_{yy} = 1; \quad \hat{g}_{zz} = 1 \qquad (4.4c)$$

we obtain from (A.3.2),

$$g_{ij} = \delta_{i\chi}\delta_{j\chi} + Y_{,i}Y_{,j} + Z_{,i}Z_{,j} \qquad (3.8)$$

AXIAL FLOWS:

By the use of following transformation equations and $\hat{g}_{ii}$

$$x = \chi, \qquad r = R(\chi, \xi, \eta), \qquad \theta = \Theta(\chi, \xi, \eta), \qquad (2.1b)$$

$$\hat{g}_{xx} = 1; \quad \hat{g}_{rr} = 1; \quad \hat{g}_{\theta\theta} = r^2 = R^2 \qquad (4.4d)$$

we obtain from (A.3.2),

$$g_{ij} = \delta_{i\chi}\delta_{j\chi} + R_{,i}R_{,j} + R^2\Theta_{,i}\Theta_{,j} \qquad (3.16)$$

CIRCULATING FLOWS:

By the use of following transformation equations and $\hat{g}_{ii}$

$$x = X(\chi, \xi, \eta), \qquad r = R(\chi, \xi, \eta), \qquad \theta = \chi \qquad (2.1c)$$

$$\hat{g}_{xx} = 1; \quad \hat{g}_{rr} = 1; \quad \hat{g}_{\theta\theta} = r^2 = R^2 \qquad (4.4d)$$

we obtain from (A.3.2),

$$g_{ij} = X_{,i}X_{,j} + R_{,i}R_{,j} + \delta_{i\chi}\delta_{j\chi}R^2 \qquad (3.22)$$

APPENDIX B

B.1 Equations (4.4e)-(4.4g) are obtained from Eqs. (2.1), (4.4a), (4.4b) and (4.4c). Recall

$$\hat{u}(j) = \sqrt{\hat{g}_{jj}}\,\hat{u}^j = \sqrt{\hat{g}_{jj}}\,\frac{\partial \hat{x}^j}{\partial x^i}u^i = \sqrt{\hat{g}_{jj}}\,\frac{\partial \hat{x}^j}{\partial \chi}u^\chi \qquad \text{(no sum over } j) \qquad (4.4a,b)$$

and

$$\hat{g}_{xx} = 1; \quad \hat{g}_{yy} = 1; \quad \hat{g}_{zz} = 1 \qquad (4.4c)$$

$$\hat{g}_{xx} = 1; \quad \hat{g}_{rr} = 1; \quad \hat{g}_{\theta\theta} = r^2 = R^2 \qquad (4.4d)$$

The details are:

PLANE FLOWS:

$$\hat{u}(x) = \sqrt{\hat{g}_{xx}}\,\frac{\partial x}{\partial \chi}u^\chi = \frac{U}{\sqrt{g_{\chi\chi}}}$$

$$\hat{u}(y) = \sqrt{\hat{g}_{yy}}\,\frac{\partial y}{\partial \chi}u^\chi = \frac{Y_{,\chi}U}{\sqrt{g_{\chi\chi}}}$$

$$\hat{u}(z) = \sqrt{\hat{g}_{zz}}\,\frac{\partial z}{\partial \chi}u^\chi = \frac{Z_{,\chi}U}{\sqrt{g_{\chi\chi}}} \qquad (B.1.1a,b,c)$$

AXIAL FLOWS:

$$\hat{u}(x) = \sqrt{\hat{g}_{xx}}\,\frac{\partial x}{\partial \chi}u^\chi = \frac{U}{\sqrt{g_{\chi\chi}}}$$

$$\hat{u}(r) = \sqrt{\hat{g}_{rr}}\,\frac{\partial r}{\partial \chi}u^\chi = \frac{R_{,\chi}U}{\sqrt{g_{\chi\chi}}}$$

$$\hat{u}(\theta) = \sqrt{\hat{g}_{\theta\theta}}\,\frac{\partial \theta}{\partial \chi}u^\chi = \frac{R\Theta_{,\chi}U}{\sqrt{g_{\chi\chi}}} \qquad (B.1.2a,b,c)$$

CIRCULATING FLOWS:

$$\hat{u}(x) = \sqrt{\hat{g}_{xx}}\,\frac{\partial x}{\partial \chi}u^\chi = \frac{X_{,\chi}U}{\sqrt{g_{\chi\chi}}}$$

$$\hat{u}(r) = \sqrt{\hat{g}_{rr}}\,\frac{\partial r}{\partial \chi}u^\chi = \frac{R_{,\chi}U}{\sqrt{g_{\chi\chi}}}$$

$$\hat{u}(\theta) = \sqrt{\hat{g}_{\theta\theta}}\,\frac{\partial \theta}{\partial \chi}u^\chi = \frac{RU}{\sqrt{g_{\chi\chi}}} \qquad (B.1.3a,b,c)$$

B.2 In this appendix we derive the expressions for the covariant derivatives given in section IV. We start with the expression for $u^i|_j$, the i-contravariant component of the j-derivative of $\mathbf{v}$ (see Ref. 18 Eq.(5.11)),

$$u^i|_j = u^i_{,j} + u^k \Gamma^i_{jk} \qquad (B.2.1)$$

This is Eq.(4.5a). On multiplying (B.2.1) with g_{im} and using

$$u^i|_j g_{im} = (u^i g_{im})|_j = u_m|_j,$$

$$u^i_{,j} g_{im} = u^k_{,j} g_{km} \quad \text{and}$$

$$\Gamma^i_{jk} g_{im} = \Gamma_{jkm} = \Gamma_{kjm} \qquad (B.2.2a,b,c)$$

we obtain

$$u_m|_j = u^k_{,j} g_{km} + u^k \Gamma_{kjm} \qquad (B.2.3)$$

Equation (B.2.3) is Eq.(4.5b) with the free index i replaced by m. To obtain Eq.(4.5c) we start with Eq.(5.13) of Ref.18,

$$u_i|_j = u_{i,j} - u_k \Gamma^k_{ij} \qquad (B.2.3)$$

On multiplying the last term on the right hand side of (B.2.4) with $g_{kl}g^{kl} = 1$, and then using g^{kl} to raise the index on u_k and using g_{kl} to lower the index on Γ^k_{ij} we obtain Eq.(4.5c).

B.3 From

$$\omega^k = \epsilon^{ijk} u_j|_i \qquad (4.6)$$

we obtain for ω^η,

$$\omega^\eta = \epsilon^{ij\eta} u_j|_i = \epsilon^{\chi\xi\eta} u_\xi|_\chi + \epsilon^{\xi\chi\eta} u_\chi|_\xi = (u_\xi|_\chi - u_\chi|_\xi)/D \qquad (B.3.1a)$$

Similarily we obtain,

$$\omega^\xi = (u_\chi|_\eta - u_\eta|_\chi)/D \qquad (B.3.1b)$$

$$\omega^\chi = (u_\eta|_\xi - u_\xi|_\eta)/D \qquad (B.3.1c)$$

Now,

$$u_i|_j - u_j|_i = u_{i,j} - u^k\Gamma_{ijk} - u^{j,i} + u^k\Gamma_{jik} = u_{i,j} - u_{j,i} \qquad (B.3.2)$$

where we have used $\Gamma_{ijk} = \Gamma_{jik}$. Equations (B.3.1) and (B.3.2) lead to Eqs. (4.7)

APPENDIX C

From Eqs.(4.11b) and (4.4e) we obtain

$$\rho u(x) = \frac{1}{D}, \quad \rho u(y) = \frac{Y_{,x}}{D}, \quad \rho u(z) = \frac{Z_{,x}}{D}. \qquad (C.1)$$

For a fixed value of ξ, say ξ_0, equations

$$x = \chi, \quad y = Y(\chi, \xi_0, \eta), \quad z = Z(\chi, \xi_0, \eta), \qquad (C.2)$$

describe the stream surface Ξ. The gradient of Ξ is given by the cross product of the base vectors of Ξ, g_χ and g_η, thus,

$$grad\,\Xi = (1, Y_{,\chi}, Z_\chi) \times (1, Y_{,\eta}, Z_\eta) \qquad (C.3)$$

Similarly we have,

$$grad\,H = (1, Y_{,\xi}, Z_\xi) \times (1, Y_{,\chi}, Z_\chi) \qquad (C.4)$$

From(C.1),(C.3), and(C.4) we obtain,

$$\rho v = grad\,\Xi \times grad\,H \qquad (C.5)$$

Equation(C.5) is the compressible version of (1.6).

APPENDIX D

D.1 We start with the well known thermodynamic relation

$$T ds = dh - dp/\rho \qquad (D.1.1)$$

where s is entropy and h is enthalpy. For isentropic flow of an ideal gas,

$$ds = 0, \quad \text{and} \quad dh = d(c_p T) = d\left(c_p \frac{p}{\rho R}\right) = d\left(\frac{\gamma}{\gamma - 1}\frac{p}{\rho}\right). \qquad (D.1.2)$$

Upon substituting relations from (D.1.2) into (D.1.1) we obtain (5.3).

D.2 Upon dividing (5.7) by $\rho^2 D^2$ we obtain, respectively, for the first, second, and third term on the L.H.S. and for the term on the R.H.S.,

$$\frac{g_{xi},x}{\rho D}, \qquad (D.2.1)$$

$$-\frac{g_{xi}}{\rho^2 D^2}(\rho D),_x = g_{xi}\left(\frac{1}{\rho D}\right)_{,x}, \qquad (D.2.2)$$

$$-\frac{1}{2}\frac{g_{xx},i}{\rho D}, \qquad)D.2.3)$$

$$\frac{\rho D}{2}(g_{xx}/\rho^2 D^2),_i = \frac{1}{2\rho D}g_{xx},i + g_{xx}\left(\frac{1}{\rho D}\right)_{,i}. \qquad (D.2.4)$$

Upon combining (D.2.1) with (D.2.2) we obtain the first term on L.H.S. of (5.8), and upon combining (D.2.3) and (D.2.4) we obtain the second term on the L.H.S. of (5.8).

ACKNOWLEDGMENTS

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