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Structural Dynamics and Vibrations of Damped, Aircraft-Type Structures

Maurice I. Young

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Maurice I. Young
ViGYAN, Inc.
Hampton, Virginia

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TABLE OF CONTENTS

Page

TABLE OF CONTENTS

(continued)

TABLE OF CONTENTS

(concluded)

1.0 SUMMARY

Engineering preliminary design methods for approximating and predicting the effects of viscous or equivalent viscous-type damping treatments on the free and forced vibration of lightly damped aircraft-type structures are developed. Similar developments are presented for dynamic hysteresis-viscoelastic-type damping treatments. It is shown by both engineering analysis and numerical illustrations that the intermodal coupling of the undamped modes arising from the introduction of damping may be neglected in applying these preliminary design methods, except when dissimilar modes of these lightly damped, complex aircraft-type structures have identical or nearly identical natural frequencies. In such cases it is shown that a relatively simple, additional interaction calculation between pairs of modes exhibiting this "modal resonance" phenomenon suffices in the prediction of interacting modal damping fractions. The accuracy of the methods is shown to be very good to excellent, depending on the normal natural frequency separation of the system modes, thereby permitting a relatively simple preliminary design approach. This approach is shown to be a natural precursor to elaborate finite element, digital computer design computations in evaluating the type, quantity and location of damping treatments. It is expected that in many instances these simplified computations will supplant the more elaborate ones.

2.0 NOMENCLATURE

2.1 Notation

A	cross-sectional area of rod, in^2; an arbitrary constant
a	coefficient in interaction quartic equation; membrane dimension, inch
B	coefficient in resolvent cubic equation; an arbitrary constant
b	coefficient in interaction quartic equation; membrane dimension, inch
C	coefficient in resolvent cubic equation; damping coefficient, lb-sec/in
c	coefficient in interaction quartic equation; damping coefficient, lb-sec/in
D	plate flexural rigidity, lb-in^2; coefficient in resolvent cubic equation
d	coefficient in interaction quartic equation; damping coefficient, lb-sec/in
E	Young's modulus of elasticity, lb/in^2
F	function symbol; force
f	force, lb
G	shear modulus of elasticity, lb/in^2
I	area moment of inertia, in^4; mass moment of inertia, lb-sec^2-in
i	ordinal number; a subscript
J	torsional section constant, in^4
j	ordinal number; complex operator, $\sqrt{-1}$
K	stiffness, spring rate, lb/in
k	stiffness, spring rate, lb/in

M	mass, lb-sec^2/in
m	mass, lb-sec^2/in; an ordinal number; a subscript
N	an ordinal number; aspect ratio
n	an ordinal number; a subscript
o	naught, a subscript
p	generalized coordinate
Q	a response quantity; a generalized force
q	a generalized coordinate
R	a response quantity; a parameter
r	an ordinal number; a subscript; a ratio
s	an ordinal number; a subscript
T	membrane tension, lb/in
t	time, sec
u	a displacement, inch
v	a displacement, inch
w	a displacement, inch
x	a Cartesian coordinate, inch
y	a Cartesian coordinate, inch
z	a Cartesian coordinate, inch
α	an ordinal number; a subscript; a constant
β	an ordinal number; a subscript; a constant
γ	a parameter

Δ	dilatation
δ	a damping coefficient, lb-sec/in; perturbation symbol
ζ	fraction of critical damping
η	loss factor
θ	torsional displacement, radian
Λ, λ	characteristic number
μ	mass per unit length, lb-sec^2/in^2; mass per unit area, lb-sec^2/in^3
ν	frequency ratio
ξ	axial coordinate position
ρ	mass density, lb-sec^2/in^4; frequency ratio
σ	damping per unit length, lb-sec/in^2
Ω	forcing frequency, rad/sec
ω	frequency, rad/sec

2.2 Symbols

$\dot{}$	dot, differentiation with respect to time
$\bar{}$	bar, amplitude of
$*$	asterisk, complex conjugate
∇^2	del squared, the Laplacian operator
$\vec{}$	arrow, vector quantity
$\tilde{}$	tilde, a modified quantity
$\int(\ )$	integral, the integral of ( )
$\|\ \|$	the magnitude of, the determinant of

[ ]	square matrix
⌊⌋	row vector
{ }	column vector
d ( )	differential of ( )
∂ ( )	partial differential of ( )
div ( )	the divergence of ( )
$\overrightarrow{grad}$ ( )	the gradient of ( )

3.0 INTRODUCTION

Aircraft, spacecraft, and especially rotorcraft airframes are subject to steady forced vibrations due to a variety of rotating or oscillatory type mechanical and aerodynamic systems. These steady forced vibrations can become severe when resonant or near resonant conditions occur in the airframe. For example, in the case of rotorcraft, excitation frequencies include the rotational frequencies of the main rotor and tail rotor (both different) and harmonic excitations at integer multiples of their blade number times the fundamental frequency. In the case of two-bladed rotors, for example, excitation frequencies at rates of once per revolution, twice per revolution, four times per revolution, etc. are commonplace. In the case of propeller/rotor-type systems, similar families of excitations exist, but the difficulties can be compounded if the propeller/rotor operates at one rate of revolution in hovering flight and another one in forward flight.

Of crucial importance in the case of resonant or near resonant forced vibrations is the fraction of critical damping associated with the particular mode which is responding. At resonance there is a direct, inverse relationship between the magnitude of the response and the magnitude of the fraction of critical damping of the mode. If damping can be increased by an order of magnitude, then the response is reduced by an order of magnitude, etc. In the case of airframes, the inherent damping levels are small and of the order of a few percent of critical damping or less. Accordingly, augmenting the normally small levels of inherent damping can be very beneficial when a severe steady vibratory response is attributable to a resonant or near resonant condition.

Since weight, cost, complexity, etc. are among the primary concerns in airframes, then whenever a "fix" or corrective addition of damping is indicated, a simple but accurate preliminary design type of engineering method of analysis is to be desired. This method should be capable of both rapid and relatively simple, but nevertheless accurate engineering predictions of modal damping. This is especially important in guiding the structural designer-dynamicist in determining the location, type, and quantity of energy dissipation-damping treatments when the airframe is sustaining severe forced vibrations.

The objective of this report is to develop a simple, preliminary design-type of analysis and methodology which can accurately predict the modal damping associated with damping treatment. It is expected that very detailed, lengthy, and complex finite element type computations may also be performed, especially in the development of a new airframe structural design. However, the preliminary design method developed and presented in this report is intended to be a precursor to such finite element type computations. The designer can quickly determine the efficacy of localized damping treatments to within acceptable engineering accuracy prior to undertaking a much more detailed and complex design computation. It is also expected that the accuracy of this preliminary design method will frequently obviate the need for any other computations.

4.0 ANALYSIS

4.1 The Single Degree of Freedom Oscillator With Damping

The special case of a dynamic system whose vibrations can be adequately represented by a single degree of freedom is considered first. This will permit the definition and development of various damping concepts in a simple manner before dealing with the general case of a damped structural dynamic system with many degrees of freedom.

Consider first the case of viscous damping where the damping force is proportional to velocity, but opposite in sense. The governing scalar differential equation of motion is $m\ddot{x}+c\dot{x}+kx = \bar{f}e^{j\Omega t}$, where the complex exponential represents a simple harmonic excitation of amplitude $\bar{f}$ and frequency Ω. The parameters m, c, and k represent the system mass, damping and spring constants, respectively.

In the transient case when $\bar{f} = 0$, the general solution takes the form $x(t) = \bar{x}e^{\lambda t}$. A substitution of this leads to the characteristic polynomial $\lambda^2 + \left(\frac{c}{m}\right)\lambda + \left(\frac{k}{m}\right) = 0$. Defining the natural frequency of the oscillation as ω, $\omega^2 \equiv \left(\frac{k}{m}\right)$, and its fraction of critical damping by $\zeta \equiv \left(\frac{c}{2\sqrt{mk}}\right)$, the characteristic equation becomes $\lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0$. This equation yields the characteristic number λ and its complex conjugate λ^* as follows:

$$\lambda = -\omega\left(\zeta + j\sqrt{1-\zeta^2}\right), \quad \lambda^* = -\omega\left(\zeta - j\sqrt{1-\zeta^2}\right), \quad j^2 = -1.$$

The transient solution then takes the form

$$x(t) = e^{-\zeta\omega t}\left(A\cos\omega\sqrt{1-\zeta^2}t + B\sin\omega\sqrt{1-\zeta^2}t\right),$$

where A and B are constants to be determined from the initial state of the transient oscillation, where

$$x(o) = x_o \text{ and } \dot{x}(o) = v_o.$$

In the case of steady forced vibration the transient oscillation decays rapidly and after several cycles the steady state response is given by $x_{\text{s.s.}}(t) = \bar{x}_{\text{s.s.}}e^{j\Omega t}$. Substitution above results in the equation

$$\bar{x}_{\text{s.s.}} = \frac{\bar{f}}{(k - m\Omega^2) + j(c\Omega)}.$$

Introducing the previous definitions $\bar{f}$, Ω, and ζ gives

$$\bar{x}_{\text{s.s.}} = \frac{\left(\frac{\bar{f}}{k}\right) e^{-j\phi}}{\sqrt{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + \left[2\zeta\left(\frac{\Omega}{\omega}\right)\right]^2}},$$

where

$$\phi \equiv \tan^{-1}\left[\frac{2\zeta\left(\frac{\Omega}{\omega}\right)}{1 - \left(\frac{\Omega}{\omega}\right)^2}\right].$$

Defining the static displacement $\bar{x}_{\text{s.s.}_{\text{Static}}} \equiv (\bar{f}/k)$, the steady state forced response is given by

$$x_{\text{s.s.}}(t) = \frac{\bar{x}_{\text{s.s.}_{\text{Static}}} e^{j(\Omega t - \phi)}}{\sqrt{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + \left[2\zeta\left(\frac{\Omega}{\omega}\right)\right]^2}}.$$

Now consider the case of dynamic hysteresis, where the damping force is proportional to displacement, but opposite in sense to the velocity. Employing the imaginary operator j as a "phase shifter" or "differentiator," the governing equation can be written as follows:

$$m\ddot{x} + k(1 + j\eta)x = \bar{f}e^{j\Omega t},$$

where η, the "loss factor" will be seen later to be a measure of energy dissipation by dynamic hysteresis. In the transient case when $\bar{f} = 0$, the general solution can be written as $x = \bar{x}e^{\lambda t}$, so that the characteristic equation takes the form

$$\lambda^2 + \omega^2(1 + j\eta) = 0.$$

Accordingly

$$\lambda = \pm j\omega(1 + j\eta)^{1/2}.$$

Employing DeMoivre's theorem,

$$(1 + j\eta)^{1/2} = (1 + \eta^2)^{1/4} e^{j(\theta/2)}$$

where

$$\theta \equiv \tan^{-1}\eta,$$

$$\lambda = -\omega(1+\eta^2)^{1/4}\left(\sin\frac{\theta}{2} \pm j\cos\frac{\theta}{2}\right).$$

In the typical engineering case of $\eta < 1$ and, generally, $\eta \ll 1$,

$$\lambda \cong -\omega\left(\frac{\eta}{2} \pm j\sqrt{1-\left(\frac{\eta}{2}\right)^2}\right).$$

Thus the loss factor divided by two may be interpreted as equivalent viscous damping with $\zeta_{\text{Equivalent}} = \eta/2$. As an alternative to the foregoing transient vibration analysis employing the operator j, the governing equation with dynamic hysteresis may also be written as

$$m\ddot{x} + \left(\frac{k\eta}{\omega}\right)\dot{x} + kx = 0.$$

Taking the solution once again in the form $x(t) = \bar{x}e^{\lambda t}$, the characteristic equation follows as

$$\lambda^2 + \omega\eta\lambda + \omega^2 = 0.$$

It is seen at once that $\zeta_{\text{Equivalent}} = \eta/2$.

In the case of steady-state forced vibration with dynamic hysteresis employing either the formulation

$$m\ddot{x} + k(1+j\eta)x = \bar{f}e^{j\Omega t},$$

or

$$m\ddot{x} + \left(\frac{k\eta}{\omega}\right)\dot{x} + kx = \bar{f}e^{j\Omega t}$$

leads to a steady-state solution with comparable results. First with the operator j:

$$x_{\text{s.s.}}(t) = \frac{\bar{x}_{\text{s.s.}_{\text{Static}}}e^{j(\Omega t-\phi)}}{\sqrt{\left[1-\left(\frac{\Omega}{\omega}\right)\right]^2+\eta^2}},$$

where

$$\phi = \tan^{-1}\left[\frac{\eta}{1-\left(\frac{\Omega}{\omega}\right)^2}\right].$$

The equivalent viscous damping then follows as

$$\zeta_{\text{Equivalent}} \equiv \left[\frac{\eta/2}{(\Omega/\omega)}\right],$$

so that strictly speaking, only at resonance $(\Omega/\omega) = 1$ is $\zeta_{\text{Equivalent}} = \eta/2$ (however this is the case of greatest practical interest); in the case of the other formulation

$$x_{\text{s.s.}}(t) = \frac{\bar{x}_{\text{s.s.Static}} e^{j(\Omega t - \phi)}}{\sqrt{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2 + \left[\eta\left(\frac{\Omega}{\omega}\right)\right]^2}}, \qquad \phi \equiv \tan^{-1}\left[\frac{\eta\left(\frac{\Omega}{\omega}\right)}{1 - \left(\frac{\Omega}{\omega}\right)^2}\right].$$

The equivalent viscous damping then follows as $\zeta_{\text{Equivalent}} \equiv \eta/2$.

That η is a measure of energy dissipated per cycle can be seen as follows: the dynamic hysteresis damping force is given by

$$f_{\text{damping}} = -k\eta x\left(\frac{\dot{x}}{|\dot{x}|}\right).$$

Computing the work done against f_{damping} per cycle of simple harmonic response at amplitude x_o and frequency Ω,

$$w/\text{cycle} = -\int_0^{(2\pi/\Omega)} f_d dx = -\int_0^{(2\pi/\Omega)} f_d \dot{x} dt = k\eta x_o^2 \Omega \int_0^{(2\pi/\Omega)} \cos^2 \Omega t dt = \pi k \eta x_o^2.$$

By contrast when the damping force is modeled as viscous with $f_{\text{damping}} = -c_{\text{Equivalent}}\dot{x}$, the energy dissipated per cycle follows in a similar manner as

$$w/\text{cycle} = \pi c \Omega x_o^2.$$

Equating the two expressions for work per cycle

$$c_{\text{Equivalent}} = \frac{k\eta}{\Omega}.$$

Dividing through by m

$$\frac{c_{\text{Equivalent}}}{m} = 2\zeta_{\text{Equivalent}}\omega = \left(\frac{k}{m}\right)\left(\frac{\eta}{\Omega}\right) = \left(\frac{\omega^2}{\Omega}\right)\eta.$$

Then

$$\zeta_{\text{Equivalent}} = \frac{\frac{1}{2}\eta}{\left(\frac{\Omega}{\omega}\right)}.$$

This is seen to be the same as the result obtained above when forced vibration with the operator j is taken as the basis of defining the relationship between the loss factor η and $\zeta_{\text{Equivalent}}$. It is to be noted that it is primarily at or near resonance that $\zeta_{\text{Equivalent}}$ takes on special significance. Accordingly, from an engineering perspective the relationship is taken as a universal one that

$$\zeta_{\text{Equivalent}} = \frac{\eta}{2}.$$

Various other forms of energy dissipation are also of interest during forced vibration. Once again special interest centers on the response at or near resonance. Also, the cases of relatively small amounts of damping are assumed, so that despite the nonlinear aspects and character of the dissipative force, the steady state response exists, is dynamically stable and for practical purposes is simple harmonic in time. Employing the concept once again of energy dissipated per cycle as a basis for deducing and defining the equivalent viscous damping coefficient, the following summary can be made for the most frequently encountered cases of damping:

(1) Dynamic Hysteresis

$$c_{\text{Equivalent}} = (k\eta/\Omega)\,, \qquad \zeta_{\text{Equivalent}} = \left[\frac{\left(\frac{\eta}{2}\right)}{\left(\frac{\Omega}{\omega}\right)}\right];$$

(2) Coulomb Friction

$$c_{\text{Equivalent}} = \frac{\left(\frac{4}{\pi}\right)\left(\frac{f_{\text{friction}}}{x_o}\right)}{\Omega}, \qquad \zeta_{\text{Equivalent}} = \left[\frac{\left(\frac{2}{\pi}\right)\left(\frac{f_{\text{friction}}}{x_o}\right)}{k\left(\frac{\Omega}{\omega}\right)}\right];$$

(3) Velocity Squared Damping $\left(f_{\text{Damping}} = -\alpha\dot{x}^2\left(\frac{\dot{x}}{|\dot{x}|}\right)\right)$

$$c_{\text{Equivalent}} = \left(\frac{8\alpha}{3\pi}\right)(\Omega x_o), \qquad \zeta_{\text{Equivalent}} = \left[\left(\frac{4\alpha}{3\pi}\right)\left(\frac{\Omega}{\omega}\right)\left(\frac{x_o}{m}\right)\right];$$

(4) Velocity Cubed Damping $\left(f_{\text{damping}} = -\alpha\dot{x}^3\right)$

$$c_{\text{Equivalent}} = \frac{3}{4}\alpha\Omega^2 x_o^2, \qquad \zeta_{\text{Equivalent}} = \left[\left(\frac{3}{8}\alpha\right)\left(\frac{\Omega^2 x_o^2}{\sqrt{mk}}\right)\right].$$

4.2 Free Vibration of Systems With Viscous or Equivalent Viscous Damping

The governing matrix differential equation of free motion of systems with viscous or equivalent viscous damping is

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{0\},$$

where $\{x(t)\}$ is the vector displacement, $[m]$, $[k]$, and $[c]$ are, respectively, the system mass, stiffness, and damping matrices. The $[m]$ and $[k]$ matrices are symmetric and positive definite; the $[c]$ matrix is symmetric and either positive definite or positive semi-definite.

Consider a solution in the form

$$\{x(t)\} = \{\bar{x}\}_i e^{\lambda_i t},$$

where $\{\bar{x}\}_i$ and λ_i are, respectively, the i^{th} complex modal vector and the associated complex i^{th} characteristic number, where $i = 1, 2, \ldots, N$, and

$$\{\bar{x}\}_i \equiv \{\bar{x}\}_{i_{\text{Real}}} + j\{\bar{x}\}_{i_{\text{Imaginary}}}, \qquad j^2 = -1,$$

$$\lambda_i \equiv \lambda_{i_{\text{Real}}} + j\lambda_{i_{\text{Imaginary}}}.$$

There are also N complex conjugate vectors $\{\bar{x}\}_i^*$ and their associated characteristic numbers λ_i^* which satisfy the differential equation. The assumed form of solution leads to a system of algebraic equations which follows in matrix form as

$$\left[\lambda_i^2[m] + \lambda_i[c] + [k]\right]\{\bar{x}\}_i = \{0\}, \qquad i = 1, 2, \ldots, N.$$

Pre-multiplying by the transposed complex conjugate modal vector $\lfloor \bar{x} \rfloor_i^*$ yields N scalar equations

$$\lambda_i^2 \left(\lfloor \bar{x} \rfloor_i^* [m] \{\bar{x}\}_i \right) + \lambda_i \left(\lfloor \bar{x} \rfloor_i^* [c] \{\bar{x}\}_i \right) + \left(\lfloor \bar{x} \rfloor_i^* [k] \{\bar{x}\}_i \right) = 0, \qquad i = 1, 2, \ldots, N.$$

This is re-written as

$$\lambda_i^2 + 2\zeta_i \omega_i \lambda_i + \omega_i^2 = 0,$$

where

$$\omega_i^2 \equiv \left(\frac{\lfloor \bar{x} \rfloor_i^* [k] \{\bar{x}\}_i}{\lfloor \bar{x} \rfloor_i^* [m] \{\bar{x}\}_i} \right) \qquad \text{and} \qquad \zeta_i \equiv \left(\frac{1}{2\omega_i} \right) \left(\frac{\lfloor \bar{x} \rfloor_i^* [c] \{\bar{x}\}_i}{\lfloor \bar{x} \rfloor_i^* [m] \{\bar{x}\}_i} \right).$$

These N scalar characteristic equations yield the N characteristic numbers and their N complex conjugates which are as follows:

$$\lambda_i = -\omega_i \left(\zeta_i \pm j\sqrt{1 - \zeta_i^2} \right), \qquad i = 1, 2, \ldots, N.$$

Orthogonality relationships which are useful in analyzing forced vibrations are now developed. Writing the r^{th} and s^{th} matrix algebraic relations:

$$\left[\lambda_r^2 [m] + \lambda_r [c] + [k] \right] \{\bar{x}\}_r = \{0\},$$

$$\left[\lambda_s^2 [m] + \lambda_s [c] + [k] \right] \{\bar{x}\}_s = \{0\}.$$

Converting both of these to scalar relationships by pre-multiplying by $\lfloor \bar{x} \rfloor_s$ and $\lfloor \bar{x} \rfloor_r$, respectively, yields:

$$\lambda_r^2 \left(\lfloor \bar{x} \rfloor_s [m] \{\bar{x}\}_r \right) + \lambda_r \left(\lfloor \bar{x} \rfloor_s [c] \{\bar{x}\}_r \right) + \left(\lfloor \bar{x} \rfloor_s [k] \{\bar{x}\}_r \right) = 0,$$

$$\lambda_s^2 \left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s \right) + \lambda_s \left(\lfloor \bar{x} \rfloor_r [c] \{\bar{x}\}_s \right) + \left(\lfloor \bar{x} \rfloor_r [k] \{\bar{x}\}_s \right) = 0.$$

Substracting the s^{th} equation from the r^{th} and noting the symmetry of $[m]$, $[c]$, and $[k]$ yields

$$\left(\lambda_r^2 - \lambda_s^2 \right) \left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s \right) + (\lambda_r - \lambda_s) \left(\lfloor \bar{x} \rfloor_r [c] \{\bar{x}\}_s \right) = 0.$$

Factoring out $(\lambda_r - \lambda_s)$ when $r \neq s$, the orthogonality relationship follows as

$$\left(\lfloor \bar{x} \rfloor_r [c] \{\bar{x}\}_s\right) + (\lambda_r + \lambda_s)\left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s\right) = 0.$$

It is to be noted in passing that if $\lambda_s = \lambda_r^*$, then the previously derived result for ζ_r follows from this orthogonality relationship.

An alternative orthogonality relationship can be deduced by adding the r^{th} and s^{th} scalar equations above. Noting the symmetry of the $[m]$, $[c]$, and $[k]$ matrices once again, there results the scalar equation

$$\left(\lambda_r^2 + \lambda_s^2\right)\left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s\right) + (\lambda_r + \lambda_s)\left(\lfloor \bar{x} \rfloor_r [c] \{\bar{x}\}_s\right) + 2\left(\lfloor \bar{x} \rfloor_r [k] \{\bar{x}\}_s\right) = 0.$$

Noting from above that

$$(\lambda_r + \lambda_s)\left(\lfloor \bar{x} \rfloor_r [c] \{\bar{x}\}_s\right) = -(\lambda_r + \lambda_s)^2\left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s\right)$$

and substituting yields

$$\left[\left(\lambda_r^2 + \lambda_s^2\right) - (\lambda_r + \lambda_s)^2\right]\left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s\right) + 2\left(\lfloor \bar{x} \rfloor_r [k] \{\bar{x}\}_s\right) = 0.$$

Simplification results in the alternate orthogonality relationship

$$\left(\lfloor \bar{x} \rfloor_r [k] \{\bar{x}\}_s\right) - \lambda_r \lambda_s \left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s\right) = 0, \qquad r \neq s.$$

It is to be noted in passing that if $\lambda_s = \lambda_r^*$, then the previously derived result for ω_r^2 follows from this form of the orthogonality relation.

In summary when $r \neq s$,

$$\left(\lfloor \bar{x} \rfloor_r [c] \{\bar{x}\}_s\right) + (\lambda_r + \lambda_s)\left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s\right) = 0$$

and

$$\left(\lfloor \bar{x} \rfloor_r [k] \{\bar{x}\}_s\right) - \lambda_r \lambda_s \left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s\right) = 0.$$

When the s^{th} mode corresponds to the complex conjugate of the r^{th} mode, then

$$2\zeta_r\omega_r = \left(\frac{\lfloor\bar{x}\rfloor_r^*[c]\{\bar{x}\}_r}{\lfloor\bar{x}\rfloor_r^*[m]\{\bar{x}\}_r}\right)$$

and

$$\omega_r^2 = \left(\frac{\lfloor\bar{x}\rfloor_r^*[k]\{\bar{x}\}_r}{\lfloor\bar{x}\rfloor_r^*[m]\{\bar{x}\}_r}\right)$$

It is to be noted that, except in the case of differing modes with closely matched natural frequencies, the undamped modes may be employed in calculating ω_r and, in turn, ζ_r. This is discussed at length below and in Reference 1.

4.3 Free Vibration With Two Different Modes Having the Same Natural Frequency

To illustrate consider the case when modes "r" and "s" have the same natural frequency, although the modal patterns differ. A simple example of this is provided by a rectangular membrane of length a and width b with interchanged nodal lines such as the modes $\bar{w}_{mn}(x, y)$ and $\bar{w}_{nm}(x, y)$, where

$$\bar{W}_r(x, y) \equiv \bar{W}_{mn}(x, y) = \sin\left(\frac{m\pi}{a}\right) x \sin\left(\frac{n\pi}{b}\right) y,$$

$$\bar{W}_s(x, y) \equiv \bar{W}_{nm}(x, y) = \sin\left(\frac{n\pi}{a}\right) x \sin\left(\frac{m\pi}{b}\right) y,$$

and

$$\omega_{mn}^2 = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\left(\frac{T}{\mu}\right)$$

$$\omega_{nm}^2 = \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]\left(\frac{T}{\mu}\right),$$

where T is the membrane tension force per unit length and μ is the membrane mass per unit area. Equating the foregoing frequency expressions, a frequency match occurs whenever $a = b$, which corresponds to a square membrane. A more general case occurs when $\omega_{mn} = \omega_{rs}$ and the membrane aspect ratio $N = (a/b)$ satisfies the constraint $N = \left|\frac{m^2 - r^2}{s^2 - n^2}\right|^{1/2}$, and m, n, r, and s are integers. For example when $m = 1$, $n = 2$, $r = 3$, and $s = 4$ a frequency match occurs for the aspect ratio $N = (a/b) = .8165$. Other detailed examples will be provided later in the section presenting numerical results.

Now consider an N degree of freedom system in free damped vibration as above where the r^{th} modal vector $\{\bar{x}\}_r$ differs from the s^{th} modal vector $\{\bar{x}\}_s$, but the r^{th} natural frequency ω_r equals the s^{th} natural frequency ω_s. Now expand the modal vectors into an undamped mode series. That is with undamped modal vector participation factors or series coefficients $\bar{p}_n$, where

$$\{\bar{x}\} = \sum_{n=1}^{N} \bar{p}_n \{\bar{x}\}_{n_{\text{Undamped}}}.$$

The system algebraic equations when $\{x(t)\} = \{\bar{x}\}e^{\lambda t}$ then take the form

$$\left[\lambda^2[m] + \lambda[c] + [k]\right]\{\bar{x}\} = \sum_{n=1}^{N} \bar{p}_n \left[\lambda^2[m] + \lambda[c] + [k]\right]\{\bar{x}\}_{n_{\text{Undamped}}} = \{0\}.$$

Noting that in the case of the undamped modes

$$\left[-\omega_n^2[m] + [k]\right]\{\bar{x}\}_n = \{0\}, \qquad n = 1, 2, \ldots, N,$$

$$\sum_{n=1}^{N} \bar{p}_n \left[\left(\lambda^2 + \omega_n^2\right)[m] + \lambda[c]\right]\{\bar{x}\}_{n_{\text{Undamped}}} = \{0\}.$$

Also noting the orthogonality of the undamped modes given by

$$M_{ij} \equiv \lfloor \bar{x} \rfloor_{i_{\text{Undamped}}} [m] \{\bar{x}\}_{j_{\text{Undamped}}} = 0, \qquad i \neq j,$$

the matrix algebraic equation simplifies to

$$\left[\lambda^2[I] + \lambda[d] + \begin{bmatrix} \omega_1^2 & \ddots & 0 \\ \ddots & \omega_2^2 & \ddots \\ 0 & \ddots & \omega_N^2 \end{bmatrix}\right] \begin{Bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \vdots \\ \bar{p}_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix},$$

where $[I]$ is the identity or unit matrix and

$$d_{ij} \equiv \left(\frac{\lfloor \bar{x} \rfloor_{i_{\text{Undamped}}} [c] \{\bar{x}\}_{j_{\text{Undamped}}}}{M_{ii}} \right), \qquad i, j = 1, 2, \ldots, N.$$

Note also the interactive damping fractions ζ_{ij} where

$$2\omega_i\zeta_{ij} \equiv d_{ij}.$$

Also let $\nu \equiv \left(\frac{\lambda}{\omega_1}\right)$ and $r_i \equiv \left(\frac{\omega_i}{\omega_1}\right)$. Then

$$\left[\nu^2[I] + \nu \begin{bmatrix} d_{11}d_{12} & \cdots & d_{1N} \\ d_{21}d_{22} & \cdots & d_{2N} \\ \vdots & \ddots & \vdots \\ d_{N1}d_{N2} & \cdots & d_{NN} \end{bmatrix} + \begin{bmatrix} r_1^2 & \ddots & 0 \\ \ddots & r_2^2 & \ddots \\ 0 & \ddots & r_N^2 \end{bmatrix}\right] \begin{Bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \vdots \\ \bar{p}_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}.$$

The characteristic determinant can, of course, now be expanded to obtain the characteristic roots of the system. However suppose that $r_\alpha = r_\beta$, the case of different undamped modes with the same natural frequency. In this case of the degenerate modes "α" and "β", the damped modal vectors are linear combinations of "α" and "β". That is

$$\bar{p}'_\alpha = \gamma\bar{p}_\alpha + \delta\bar{p}_\beta,$$

$$\bar{p}'_\beta = \epsilon\bar{p}_\alpha + \rho\bar{p}_\beta,$$

or

$$\begin{Bmatrix} \bar{p}'_\alpha \\ \bar{p}'_\beta \end{Bmatrix} = \begin{bmatrix} \gamma & \delta \\ \epsilon & \rho \end{bmatrix} \begin{Bmatrix} \bar{p}_\alpha \\ \bar{p}_\beta \end{Bmatrix}.$$

Since only "α" and "β" interact in this special case, then the two algebraic equations for the modes "α" and "β" decouple from the rest of the system due to this degeneracy. This results in a relatively simple and informative quartic characteristic equation for these modes. That is

$$\left[\nu^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \nu \begin{bmatrix} d_{\alpha\alpha} & d_{\alpha\beta} \\ d_{\beta\alpha} & d_{\beta\beta} \end{bmatrix} + r_2^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{Bmatrix} \bar{p}'_\alpha \\ \bar{p}'_\beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

and

$$\nu^4 + (d_{\alpha\alpha} + d_{\beta\beta})\nu^3 + \left(2r_\alpha^2 + d_{\alpha\alpha}d_{\beta\beta} - d_{\alpha\beta}d_{\beta\alpha}\right)\nu^2 + r_\alpha^2(d_{\alpha\alpha} + d_{\beta\beta})\nu + r_\alpha^4 = 0.$$

When

$$(d_{\alpha\alpha}d_{\beta\beta} - d_{\alpha\beta}d_{\beta\alpha}) \equiv \begin{vmatrix} d_{\alpha\alpha} & d_{\alpha\beta} \\ d_{\beta\alpha} & d_{\beta\beta} \end{vmatrix} = 0,$$

$$\nu^4 + (d_{\alpha\alpha} + d_{\beta\beta})\nu^3 + 2r_\alpha^2\nu^2 + r_\alpha^2(d_{\alpha\alpha} + d_{\beta\beta})\nu + r_\alpha^4 = 0.$$

This quartic is now seen to be factorable into the product of the quadratic factors

$$\left(\nu^2 + r_\alpha^2\right)\left[\nu^2 + (d_{\alpha\alpha} + d_{\beta\beta})\nu + r_\alpha^2\right] = 0.$$

The implication of this is that when matching natural frequencies of different modes occur, one of these modes is undamped, while the other has the damping of both. More about this will be presented in the section on numerical results where it will be seen that many cases of engineering interest satisfy the foregoing conditions, or are numerically comparable to this when either $\begin{vmatrix} d_{\alpha\alpha} & d_{\alpha\beta} \\ d_{\beta\alpha} & d_{\beta\beta} \end{vmatrix}$ is exactly zero or negligibly small compared to $2r_\alpha^2$ above.

4.4 Employing the Undamped Modes to Determine the Damped Modes

It has been shown above that the exact undamped natural frequencies and modal damping fractions for the various modes of the dynamic system are given by

$$\omega_i^2 = \left(\frac{\lfloor\bar{x}\rfloor_i^*[k]\{\bar{x}\}_i}{\lfloor\bar{x}\rfloor_i^*[m]\{\bar{x}\}_i}\right) \quad \text{and} \quad \zeta_i = \left(\frac{1}{2\omega_i}\right)\left(\frac{\lfloor\bar{x}\rfloor_i^*[c]\{\bar{x}\}_i}{\lfloor\bar{x}\rfloor_i^*[m]\{\bar{x}\}_i}\right).$$

Clearly the efficacy of an engineering approximation depends on accurately approximating the i^{th} damped modal vector $\lfloor\bar{x}\rfloor_i$ and its complex conjugate $\lfloor\bar{x}\rfloor_i^*$. The undamped modal vector $\lfloor\bar{x}\rfloor_{i\mathrm{U}}$ provides the necessary approximation, provided that the system is lightly damped and that two dissimilar modes do not have the same or nearly the same natural frequencies, the "modal resonance" case. Express the i^{th} damped modal vector as the i^{th} undamped modal vector and a perturbation effect due to damping, $\{\bar{x}\}_{u_i}$ and $\{\delta\bar{x}\}_i$, respectively, where

$$\{\bar{x}\}_i = \{\bar{x}\}_{u_i} + \zeta_{ii}\{\delta\bar{x}\}_i,$$

$$\zeta_{ii} = \left(\frac{1}{2\omega_{u_i}}\right)\left(\frac{\lfloor\bar{x}\rfloor_{u_i}[c]\{\bar{x}\}_{u_i}}{\lfloor\bar{x}\rfloor_{u_i}[m]\{\bar{x}\}_{u_i}}\right), \qquad \omega_{u_i}^2 = \left(\frac{\lfloor\bar{x}\rfloor_{u_i}[k]\{\bar{x}\}_{u_i}}{\lfloor\bar{x}\rfloor_{u_i}[m]\{\bar{x}\}_{u_i}}\right).$$

In approximating the i^{th} mode fraction of critical damping for use, for example, in forced vibration calculations, it is clear that if $\{\delta\bar{x}\}_i$ is the same order of magnitude as $\{\bar{x}\}_{u_i}$ and

ζ_{ii} is very small compared to unity, then $\{\bar{x}\}_i$ and $\{\bar{x}\}_i^*$ will approach $\{\bar{x}\}_{u_i}$ and $\zeta_{ii} \simeq \zeta_i$. Rayleigh's theorem for damped linear systems states that the i^{th} eigenvalue λ_i is stationary with respect to perturbations in the i^{th} eigenvector: that is,

$$d\lambda_i/d\{\bar{x}\}_i = 0.$$

Employing the perturbation form for $\{\bar{x}\}_i$ yields

$$\frac{d\lambda_i}{d\{\bar{x}\}_{u_i} + \zeta_{ii}d\{\delta\bar{x}\}_i} = \left[\frac{d\lambda_i/d\{x\}_{u_i}}{1 + \zeta_{ii}d\{\delta\bar{x}\}_i/d\{\bar{x}\}_{u_i}}\right] = 0.$$

For very small values of $\{\zeta_{ii}\}$ and absence of a "modal resonance" which can result in a very large $\{\delta\bar{x}\}_i$, the i^{th} eigenvalue is seen to also be stationary with respect to the i^{th} undamped modal vector $\{\bar{x}\}_{u_i}$, so that

$$d\lambda_i/d\{\bar{x}\}_i = 0.$$

In this case

$$\lambda_i = -\omega_{u_i}\left(\zeta_{ii} + j\sqrt{1-\zeta_{ii}^2}\right), \qquad j^2 = -1.$$

The perturbation vector $\{\delta\bar{x}\}_i$ is now examined by expanding the modal vector $\{\bar{x}\}$ into an undamped modal vector series,

$$\{x(t)\} = \sum_{i-1}^{N} p_i(t)\{\bar{x}\}_{u_i},$$

where the $p_i(t)$ are generalized coordinates or undamped modal participation factors. In view of the orthogonality of the undamped modes with respect to $[m]$ and $[k]$,

$$\ddot{p}_i + 2\zeta_{ii}\omega_{u_i}\dot{p}_i + \omega_{u_i}^2 p_i = -2\zeta_{ii}\omega_{u_i}\sum_{j=1}^{N}(\zeta_{ij}/\zeta_{ii})(1-\delta_{ij})\dot{p}_j,$$

where δ_{ij} is the Kronecker delta with magnitude zero or unity as $i \neq j$ or $i = j$, respectively. Also, ζ_{ij} is given by

$$\zeta_{ij} = (1/2\omega_{u_i})\left(\lfloor\bar{x}\rfloor_{u_i}[c]\{\bar{x}\}_{uj}/\lfloor\bar{x}\rfloor_{u_i}[m]\{\bar{x}\}_{u_i}\right).$$

Clearly the perturbation in p_i (and in turn the i^{th} damped modal vector) is of the order of ζ_{ii}, unless a "modal resonance" occurs.

4.5 Forced Vibration of Systems With Viscous Damping

The system is assumed to be in steady forced vibration under a periodic excitation. This periodic excitation has a typical simple harmonic component $\{x(t)\} = \{\bar{f}\}e^{i\Omega t}$, where $\{\bar{f}\}$ is the vector amplitude and Ω is the excitation frequency; the complex exponential represents the simple harmonic variation in time. As in the case of free vibration $\{x(t)\}$ is the system displacement vector, $[k]$ and $[m]$ are symmetric, positive definite matrices representing system stiffness and system mass, respectively. $[c]$ is a symmetric matrix which is either positive definite or semi-definite; it represents either actual viscous damping or an equivalent viscous damping representation for other forms of energy dissipation. Accordingly the governing matrix differential equation is

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{f(t)\} = \{\bar{f}\}e^{j\Omega t}$$

The system is now represented in a canonical matrix form by augmenting the system displacement vector with the system velocity vector. The augmenting identity equation $[m]\{\dot{x}\} - [m]\{\dot{x}\} = \{0\}$ yields the following partitioned matrix format for the system forced vibration:

$$\left[\begin{array}{c|c} [c] & [m] \\ \hline [m] & [0] \end{array}\right] \left\{\begin{array}{c} \{\dot{x}\} \\ \hline \{\ddot{x}\} \end{array}\right\} + \left[\begin{array}{c|c} [k] & [0] \\ \hline [0] & -[m] \end{array}\right] \left\{\begin{array}{c} \{x\} \\ \hline \{\dot{x}\} \end{array}\right\} = \left\{\begin{array}{c} \{\bar{f}\} \\ \hline \{0\} \end{array}\right\} e^{j\Omega t}$$

A still more condensed form results by employing $\{z(t)\} \equiv \left\{\begin{array}{c} \{x\} \\ \{\dot{x}\} \end{array}\right\}$; the governing equation then becomes

$$[A]\{\dot{z}\} + [B]\{z\} = \{\bar{F}\}e^{j\Omega t},$$

where

$$\{\bar{F}\} \equiv \left\{ \begin{array}{c} \{\bar{f}\} \\ -- \\ \{0\} \end{array} \right\}, \qquad [A] \equiv \left[\begin{array}{c|c} [c] & [m] \\ \hline [m] & [0] \end{array} \right], \qquad [B] \equiv \left[\begin{array}{c|c} [k] & [0] \\ \hline [0] & -[m] \end{array} \right].$$

In the case of homogeneous free vibration there are $2N$ damped modes and their associated damped characteristic values $\{\bar{z}\}_r$ and λ_r, respectively (i.e., there are N complex modes and their associated complex characteristic numbers and their N complex conjugates). These satisfy the damped modal orthogonality relationships

$$\lfloor \bar{z} \rfloor_r [A] \{\bar{z}\}_s = \lfloor \bar{z} \rfloor_r [B] \{\bar{z}\}_s = 0, \qquad r \neq s.$$

A modal expansion solution is now developed for the steady state forced vibration of the system. In the steady state $\{z(t)\}$ is represented as

$$\{z(t)\} = \sum_{r=1}^{2N} q_r(t) \{\bar{z}\}_r,$$

where the $q_r(t)$ are generalized coordinates or "damped modal participation factors." Substituting above and employing the orthogonality relationships, $2N$ uncoupled, scalar response equations result. These are

$$\mu_{rr}\dot{q}(t) + \nu_{rr} q_r(t) = \bar{Q}_r e^{j\Omega t}, \qquad r = 1, 2, \ldots, 2N,$$

where

$$\begin{aligned} \mu_{rr} &\equiv \left(\lfloor \bar{z} \rfloor_r [A] \{\bar{z}\}_r \right) \equiv \left(\lfloor \bar{x} \rfloor_r [c] \{\bar{x}\}_r \right) + 2\lambda_r \left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_r \right) \\ \nu_{rr} &\equiv \left(\lfloor \bar{z} \rfloor_r [B] \{\bar{z}\}_r \right) \equiv \left(\lfloor \bar{x} \rfloor_r [k] \{\bar{x}\}_r \right) - \lambda_r^2 \left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_r \right) \\ \bar{Q}_r &\equiv \left(\lfloor \bar{z} \rfloor_r \{\bar{F}\} \right) \equiv \left(\lfloor \bar{x} \rfloor_r \{\bar{f}\} \right) \end{aligned}$$

In the steady-state the modal participation factors $q_r(t)$ have the solution $q_r(t) = \bar{q}_r e^{j\Omega t}$. Substitution above yields the complex amplitude $\bar{q}_r$ given by the equation

$$\bar{q}_r = \bar{Q}_r (\nu_{rr} + j\Omega \mu_{rr})^{-1}.$$

The complex closed form solution for steady-state forced vibration then follows as

$$\{x(t)\}_{\text{Steady-state}} = e^{j\Omega t} \sum_{r=1}^{2N} \bar{Q}_r(\nu_{rr} + j\Omega\mu_{rr})^{-1}\{\bar{x}\}_r.$$

Substituting from above and employing the previously defined modal scalar quantities, the steady-state response of the damped system is

$$\{x(t)\}_{\text{Steady-state}} = e^{j\Omega t} \sum_{r=1}^{2N} \left(\lfloor\bar{x}\rfloor_r\{\bar{f}\}\right) \left\{ \begin{array}{c} \left(\lfloor\bar{x}\rfloor_r[k]\{\bar{x}\}_r\right) - \lambda_r^2\left(\lfloor\bar{x}\rfloor_r[m]\{\bar{x}\}_r\right) \\ +(j\Omega)\left[\left(\lfloor\bar{x}\rfloor_r[c]\{\bar{x}\}_r\right) + 2\lambda_r\left(\lfloor\bar{x}\rfloor_r[m]\{\bar{x}\}_r\right)\right] \end{array} \right\}^{-1} \{\bar{x}\}_r,$$

where

$$\lambda_r \equiv -\omega_r\left(\zeta_r + j\sqrt{1-\zeta_r^2}\right).$$

Also

$$\omega_r^2 \equiv \left(\frac{\lfloor\bar{x}\rfloor_r^*[k]\{\bar{x}\}_r}{\lfloor\bar{x}\rfloor_r^*[m]\{\bar{x}\}_r}\right)$$

and

$$\zeta_r \equiv \left(\frac{1}{2\omega_r}\right)\left(\frac{\lfloor\bar{x}\rfloor_r^*[c]\{\bar{x}\}_r}{\lfloor\bar{x}\rfloor_r^*[m]\{\bar{x}\}_r}\right)$$

where the asterisk (*) denotes the complex conjugate modal vector.

The amplitude of the r^{th} generalized coordinate is

$$\bar{q}_r = \bar{Q}_r(\nu_{rr} + j\Omega\mu_{rr})^{-1}.$$

This can be rewritten as

$$\bar{q}_r = \bar{q}_{r\,\text{Static}}\left[1 + j\Omega\left(\frac{\mu_{rr}}{\nu_{rr}}\right)\right]^{-1},$$

where the r^{th} mode static displacement is defined as

$$\bar{q}_r \equiv \left(\frac{\bar{Q}_r}{\nu_{rr}}\right).$$

In the case of free damped vibration, the r^{th} mode has the Rayleigh-type quotient for the r^{th} characteristic exponent

$$\lambda_r \equiv -\left(\frac{\lfloor\bar{z}\rfloor_r[B]\{\bar{z}\}_r}{\lfloor\bar{z}\rfloor_r[A]\{\bar{z}\}_r}\right) \equiv -\left(\frac{\nu_{rr}}{\mu_{rr}}\right).$$

Accordingly the ratio $\left(\frac{\bar{q}_r}{\bar{q}_{r\text{Static}}}\right)$ follows as

$$\left(\frac{\bar{q}_r}{\bar{q}_{r\text{Static}}}\right) = \left[1 - j\left(\frac{\Omega}{\lambda_r}\right)\right]^{-1}.$$

Similarly for the $(r+1)$ mode

$$\left(\frac{\bar{q}_r}{\bar{q}_{r+1\text{Static}}}\right) = \left[1 - j\left(\frac{\Omega}{\lambda_{r+1}}\right)\right]^{-1}.$$

Since the damped modes will occur in complex conjugate pairs for all cases of practical engineering interest, let the $(r+1)$ damped mode be the complex conjugate of the r^{th} damped mode and denote this by an asterisk superscript.† Then

$$\left(\frac{\bar{q}_r^*}{\bar{q}_{r\text{Static}}^*}\right) = \left[1 - j\left(\frac{\Omega}{\lambda_r^*}\right)\right]^{-1}.$$

Consider the portion of the steady-state response due to the r^{th} damped mode and its conjugate:

$$\bar{q}_r\{\bar{x}\}_r + \bar{q}_r^*\{\bar{x}\}_r^* = \bar{q}_{r\text{Static}}\{\bar{x}\}_r\left[1 - j\left(\frac{\Omega}{\lambda_r}\right)\right]^{-1} + \bar{q}_{r\text{Static}}^*\{\bar{x}\}_r^*\left[1 - j\left(\frac{\Omega}{\lambda_r^*}\right)\right]^{-1},$$

or

$$\bar{q}_r = \frac{\bar{q}_{r\text{Static}}\left[1 - j\left(\frac{\Omega}{\lambda_r^*}\right)\right]\{\bar{x}\}_r + \bar{q}_{r\text{Static}}^*\left[1 - j\left(\frac{\Omega}{\lambda_r}\right)\right]\{\bar{x}\}_r^*}{\left[1 - j\left(\frac{\Omega}{\lambda_r}\right)\right]\left[1 - j\left(\frac{\Omega}{\lambda_r^*}\right)\right]}.$$

Continuing to simplify

$$\bar{q}_r = \frac{\bar{q}_{r\text{Static}}\left[1 - j\left(\frac{\Omega}{\lambda_r^*}\right)\right]\{\bar{x}\}_r + \bar{q}_{r\text{Static}}^*\left[1 - j\left(\frac{\Omega}{\lambda_r}\right)\right]\{\bar{x}\}_r^*}{\left\{\left[1 - \left(\frac{\Omega^2}{\lambda_r\lambda_r^*}\right)\right] - j\Omega\left(\frac{1}{\lambda_r} + \frac{1}{\lambda_r^*}\right)\right\}}.$$

Noting that $\lambda_r\lambda_r^* = \omega_r^2$ and $(\lambda_r + \lambda_r^*) = -2\zeta_r\omega_r$,

$$\bar{q}_r\{\bar{x}\}_r + \bar{q}_r^*\{\bar{x}\}_r^* = \frac{\bar{q}_{r\text{Static}}\left[1 - j\left(\frac{\Omega}{\lambda_r^*}\right)\right]\{\bar{x}\}_r + \bar{q}_{r\text{Static}}^*\left[1 - j\left(\frac{\Omega}{\lambda_r}\right)\right]\{\bar{x}\}_r^*}{\left\{\left[1 - \left(\frac{\Omega}{\omega_r}\right)^2\right] + j\left[2\zeta_r\left(\frac{\Omega}{\omega_r}\right)\right]\right\}}.$$

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† Subsidence or critically damped modes will not occur in the aerospace type structures of interest.

Then the complete closed-form solution for the steady-state response of the system to a simple harmonic excitation follows as

$$\{x(t)\}_{\text{Steady-state}} = e^{j\Omega t}\sum_{r=1}^{N}\left\{\left[1-\left(\frac{\Omega}{\omega_r}\right)^2\right] + j\left[2\zeta_r\left(\frac{\Omega}{\omega_r}\right)\right]\right\}^{-1}\left\{\bar{q}_{r\text{Static}}\left[1-j\left(\frac{\Omega}{\lambda_r^*}\right)\right]\{\bar{x}\}_r + \bar{q}^*_{r\text{Static}}\left[1-j\left(\frac{\Omega}{\lambda_r}\right)\right]\{\bar{x}\}^*_r\right\},$$

where

$$\bar{q}_{r\text{Static}} \equiv \frac{\bar{Q}_r}{\nu_{rr}}, \qquad \bar{q}^*_{r\text{Static}} \equiv \frac{\bar{Q}^*_r}{\nu^*_{rr}},$$

$$\bar{Q}_r \equiv \lfloor\bar{x}\rfloor_r\{\bar{f}\}, \qquad \bar{Q}^*_r \equiv \lfloor\bar{x}\rfloor^*_r\{\bar{f}\},$$

$$\nu_{rr} \equiv K_{rr} - \lambda_r^2 M_{rr}, \qquad \nu^*_{rr} \equiv K^*_{rr} - \lambda_r^{*2} M^*_{rr},$$

$$K_{rr} \equiv \lfloor\bar{x}\rfloor_r[k]\{\bar{x}\}_r, \qquad K^*_{rr} \equiv \lfloor\bar{x}\rfloor^*_r[k]\{\bar{x}\}^*_r$$

$$M_{rr} \equiv \lfloor\bar{x}\rfloor_r[m]\{\bar{x}\}_r, \qquad M^*_{rr} \equiv \lfloor\bar{x}\rfloor^*_r[m]\{\bar{x}\}^*_r,$$

$$\lambda_r \equiv -\omega_r\left(\zeta_r + j\sqrt{1-\zeta_r^2}\right), \qquad \lambda^*_r \equiv -\omega_r\left(\zeta_r - j\sqrt{1-\zeta_r^2}\right),$$

$$\omega_r \equiv \left(\frac{\lfloor\bar{x}\rfloor^*_r[k]\{\bar{x}\}_r}{\lfloor\bar{x}\rfloor^*_r[m]\{\bar{x}\}_r}\right)^{1/2}, \qquad \zeta_r \equiv \left(\frac{1}{2\omega_r}\right)\left(\frac{\lfloor\bar{x}\rfloor^*_r[c]\{\bar{x}\}_r}{\lfloor\bar{x}\rfloor^*_r[m]\{\bar{x}\}_r}\right).$$

The foregoing steady-state form of the solution is now rewritten as follows:

$$\{x(t)\}_{\text{Steady-state}} = \sum_{r=1}^{N} e^{j(\Omega t-\phi_r)}\left[\frac{\bar{q}_{r\text{Static}}\left[1-j\left(\frac{\Omega}{\lambda^*_r}\right)\right]\{\bar{x}\}_r + \bar{q}^*_{r\text{Static}}\left[1-j\left(\frac{\Omega}{\lambda_r}\right)\right]\{\bar{x}\}^*_r}{\sqrt{\left[1-\left(\frac{\Omega}{\omega_r}\right)^2\right]^2 + \left[2\zeta_r\left(\frac{\Omega}{\omega_r}\right)\right]^2}}\right],$$

where

$$\phi_r \equiv \tan^{-1}\left[\frac{2\zeta_r\left(\frac{\Omega}{\omega_r}\right)}{1-\left(\frac{\Omega}{\omega_r}\right)^2}\right].$$

It is to be especially noted that each damped mode (and its conjugate) respond with dynamic magnification and phase shift characteristics similar to single degree of freedom damped oscillators. The magnification depends on the ratio of the forcing frequency to the natural frequency (Ω/ω_r) and the modal fraction of critical damping ζ_r. It is seen that a resonant or near resonant modal response magnitude is crucially dependent on ζ_r, the fraction of critical damping.

4.5.1 Approximating the Response of Lightly Damped Systems

Consider the numerators of the response solution. Defining this as A_r:

$$A_r \equiv \left(\frac{\lfloor \bar{x} \rfloor_r \{\bar{f}\}}{K_{rr} - \lambda_r^2 M_{rr}}\right) \left[1 - j\left(\frac{\Omega}{\lambda_r^*}\right)\right] \{\bar{x}\}_r$$
$$+ \left(\frac{\lfloor \bar{x} \rfloor_r^* \{\bar{f}\}}{K_{rr}^* - \lambda_r^{*2} M_{rr}^*}\right) \left[1 - j\left(\frac{\Omega}{\lambda_r}\right)\right] \{\bar{x}\}_r^*, \qquad r = 1, 2, \ldots, 2N.$$

Neglecting the imaginary parts of $\{\bar{x}\}_r$ and $\{\bar{x}\}_r^*$ for small damping yields $\{\bar{x}\}_r \cong \{\bar{x}\}_r^* \cong \{\bar{x}\}_{r_{\mathbf{Undamped}}}$. Then

$$A_r \cong \left(\frac{\lfloor \bar{x} \rfloor_{r_{\mathrm{U.}}} \{\bar{f}\}}{K_{rr}}\right) \{\bar{x}\}_{r_{\mathrm{U.}}} \left\{ \left[\frac{1 - j\left(\frac{\Omega}{\lambda_r^*}\right)}{1 - \left(\frac{\lambda_r}{\omega_r}\right)^2}\right] + \left[\frac{1 - j\left(\frac{\Omega}{\lambda_r}\right)}{1 - \left(\frac{\lambda_r^*}{\omega_r}\right)^2}\right] \right\}$$

Noting that

$$\left(\frac{\Omega}{\lambda_r}\right) = \frac{\Omega}{-\omega_r\left(\zeta_r + j\sqrt{1-\zeta_r^2}\right)} = -\left(\frac{\Omega}{\omega_r}\right)\left(\zeta_r - j\sqrt{1-\zeta_r^2}\right)$$

$$\left(\frac{\Omega}{\lambda_r^*}\right) = \frac{\Omega}{-\omega_r\left(\zeta_r - +j\sqrt{1-\zeta_r^2}\right)} = -\left(\frac{\Omega}{\omega_r}\right)\left(\zeta_r + j\sqrt{1-\zeta_r^2}\right)$$

and that

$$\left(\frac{\lambda_r}{\omega_r}\right)^2 = +\left(\zeta_r + j\sqrt{1-\zeta_r^2}\right)^2 = \left(-1 + 2j\zeta_r\sqrt{1-\zeta_r^2}\right) + 2\zeta_r^2$$

and

$$\left(\frac{\lambda_r^*}{\omega_r}\right)^2 = \left(-1 - 2j\zeta_r\sqrt{1-\zeta_r^2}\right) + 2\zeta_r^2$$

$$A_r \cong \gamma_r \left(\frac{\lfloor \bar{x} \rfloor_{r_{\mathrm{U.}}} \{\bar{f}\}}{\lfloor \bar{x} \rfloor_{r_{\mathrm{U.}}} [k] \{\bar{x}\}_{r_{\mathrm{U.}}}}\right),$$

where

$$\gamma_r \equiv \left(\frac{1}{2}\right)\left\{\left[\frac{\left[1-\left(\frac{\Omega}{\omega_r}\right)\sqrt{1-\zeta_r^2}\right]+j\left[\zeta_r\left(\frac{\Omega}{\omega_r}\right)\right]}{1-j\left(\zeta_r\sqrt{1-\zeta_r^2}\right)}\right]\right.$$

$$\left.+\left[\frac{\left[1+\left(\frac{\Omega}{\omega_r}\right)\sqrt{1+\zeta_r^2}\right]+j\left[\zeta_r\left(\frac{\Omega}{\omega_r}\right)\right]}{1+j\left(\zeta_r\sqrt{1-\zeta_r^2}\right)}\right]\right\}.$$

For practical purposes, if $\zeta_r \ll 1$, the complex parameter γ_r in the r^{th} damped mode response is unity. Accordingly the lightly damped system response can be computed, mode-by-mode, with a quasi-single degree of freedom response function R_r. That is

$$\{x(t)\} \cong \sum_{\gamma=1}^{N} e^{j(\Omega t-\phi_r)} R_r \{\bar{x}\}_{r\mathrm{U.}},$$

where

$$R_r \equiv \left(\frac{\lfloor\bar{x}\rfloor_{r\mathrm{U.}}\{\bar{f}\}}{\lfloor\bar{x}\rfloor_{r\mathrm{U.}}[k]\{\bar{x}\}_{r\mathrm{U.}}}\right)\left\{\left[1-\left(\frac{\Omega}{\omega_r}\right)^2\right]^2+\left[2\zeta_r\left(\frac{\Omega}{\omega_r}\right)\right]^2\right\}^{-1/2},$$

$\lfloor\bar{x}\rfloor_{r\mathrm{U.}}\{\bar{f}\}$ is the scalar energy or work function of the load distribution $\{\bar{f}\}$ acting in the r^{th} (undamped) mode and $\lfloor\bar{x}\rfloor_{r\mathrm{U.}}[k]\{\bar{x}\}_{r\mathrm{U.}}$ is the r^{th} generalized or modal stiffness scalar. The ratio of these two scalars is analogous to the static deflection of a single degree of freedom system. That is, $\left(\frac{\lfloor\bar{x}\rfloor_{r\mathrm{U.}}\{\bar{f}\}}{\lfloor\bar{x}\rfloor_{r\mathrm{U.}}[k]\{\bar{x}\}_{r\mathrm{U.}}}\right)$ is a measure of the r^{th} mode response when the load $\{f\}$ is a static or very slowly varying one so that

$$\{f(t)\} = \{\bar{f}\}e^{j\Omega t} \cong \{\bar{f}\},$$

or in effect

$$\left(\frac{\Omega}{\omega_r}\right) <<< 1.$$

4.6 Free Vibration of Linear Systems With Dynamic Hysteresis-Viscoelastic Damping

The governing system differential equation in matrix form is

$$[m]\{\ddot{x}\} + [k]\{x\} + j[\delta k]\{x\} = \{0\},$$

where $\{x\}$ is global displacement vector for the N-degree of freedom system. $[m]$ and $[k]$ are symmetric, positive-definite matrices. $[\delta k]$ is a symmetric matrix which is positive-semi-definite. The complex operator j, where $j^2 = -1$, represents and effectuates the dynamic hysteresis damping force character where these forces have the sense of being opposite to the system velocity components, but are at the same time proportional to the system displacement components.

The general solution for free vibration displacements is taken in the form $\{x(t)\} = \{\bar{x}\}e^{\lambda t}$, where $\{\bar{x}\}$ is a complex modal vector to be determined, and λ is a complex scalar characteristic value associated with the modal vector. This reduces the matrix differential equation to the following matrix algebraic equation for the i^{th} mode:

$$\left[\lambda_i^2[m] + [k] + j[\delta k]\right]\{\bar{x}\}_i = \{0\}.$$

Pre-multiplying this equation by the transposed complex conjugate i^{th} modal vector $\lfloor\bar{x}\rfloor_i^*$, a scalar equation for λ_i follows below.

$$\lambda_i^2\left(\lfloor\bar{x}\rfloor_i^*[m]\{\bar{x}\}_i\right) + \left(\lfloor\bar{x}\rfloor_i^*[k]\{\bar{x}\}_i\right) + j\left(\lfloor\bar{x}\rfloor_i^*[\delta k]\{\bar{x}\}_i\right) = 0.$$

Defining the real scalar quantities

$$M_{ii} \equiv \left(\lfloor\bar{x}\rfloor_i^*[m]\{\bar{x}\}_i\right), \qquad K_{ii} \equiv \left(\lfloor\bar{x}\rfloor_i^*[k]\{\bar{x}\}_i\right), \qquad \omega_i^2 \equiv \left(\frac{K_{ii}}{M_{ii}}\right),$$

$$\delta K_{ii} \equiv \left(\lfloor\bar{x}\rfloor_i^*[\delta k]\{\bar{x}\}_i\right), \qquad \eta_i \equiv \left(\frac{\delta K_{ii}}{K_{ii}}\right),$$

the scalar equation for λ_i is rewritten as follows:

$$\lambda_i^2 + \omega_i^2(1 + j\eta_i) = 0.$$

Solving for λ_i,

$$\lambda_i^2 = j^2\omega_i^2(1 + j\eta_i),$$

and

$$\lambda_i = j\omega_i(1 + j\eta_i)^{1/2}.$$

Employing Euler's theorem

$$1 + j\eta_i = \left(1 + \eta_i^2\right)^{1/2} e^{j(\tan^{-1}\eta_i)},$$

and

$$(1 + j\eta_i)^{1/2} = \left(1 + \eta_i^2\right)^{1/4} e^{j[\frac{1}{2}(\tan^{-1}\eta)]}.$$

Accordingly

$$\lambda_i = -\omega_i \left(1 + \eta_i^2\right)^{1/4} \left[+\sin\left(\frac{1}{2}\tan^{-1}\eta_i\right) - j\cos\left(\frac{1}{2}\tan^{-1}\eta_i\right)\right].$$

Its complex conjugate λ_i^* is

$$\lambda_i^* = -\omega_i \left(1 + \eta_i^2\right)^{1/4} \left[+\sin\left(\frac{1}{2}\tan^{-1}\eta_i\right) + j\cos\left(\frac{1}{2}\tan^{-1}\eta_i\right)\right]$$

The i^{th} mode decay factor is the real part of the i^{th} characteristic value given by

$$\lambda_{i_{\text{Real}}} = -\omega_i \left(1 + \eta_i^2\right)^{1/4} \sin\left(\frac{1}{2}\tan^{-1}\eta_i\right).$$

By analogy with viscous damping where the decay factor is $\lambda_{i_{\substack{\text{Real}\\\text{Viscous}}}} = -\zeta_{i_{\text{Viscous}}}\omega_i$, it follows that

$$\zeta_{i_{\substack{\text{Real}\\\text{Equivalent}}}} = \left(1 + \eta_i^2\right)^{1/4} \sin\left(\frac{1}{2}\tan^{-1}\eta_i\right),$$

where η_i is the modal analogy of the material loss factor η for a simple structural element with complex moduli

$$\widetilde{E} \equiv E(1 + j\eta) \quad \text{or} \quad \widetilde{G} \equiv G(1 + j\eta).$$

Generally η is much less than unity, so that for $\eta \ll 1$

$$\zeta_{i_{\substack{\text{Viscous}\\\text{Equivalent}}}} \cong \eta_i/2.$$

In the extreme cases of synthetic rubber materials (specially compounded and impregnated with carbon-black particles to enhance damping), the loss factor can approach unity in magnitude. Then for $\eta_i = 1$

$$\zeta_{i_{\substack{\text{Equivalent}\\\text{Viscous}}}} = (2)^{1/4}\sin\left(\frac{1}{2}\tan^{-1}1\right) = (2)^{1/4}\sin\left(\frac{\pi}{8}\right) = 0.4550.$$

It is seen that in this extreme case, the linearized loss factor prediction of the equivalent viscous damping fraction of critical damping of 0.5000 overestimates the exact value by about nine percent (9%). A graph of the modal equivalent viscous damping fraction over the range of modal loss factors from zero to unity is shown in Figure 1. It is seen that negligible error occurs over the practical range of interest from $\eta_i = 0$ to $\eta_i = 0.5$; where at $\eta_i = 0.5$, $\zeta_{i_{\substack{\text{Equivalent}\\\text{Viscous}}}} \equiv .243$ and the error of the linearized approximation is only about three percent (3%).

Returning to the definition of η_i,

$$\eta_i \equiv \left(\frac{\lfloor \bar{x} \rfloor_i^* [\delta k] \{\bar{x}\}_i}{\lfloor \bar{x} \rfloor_i^* [k] \{\bar{x}\}_i} \right)$$

and expanding the i^{th} complex modal vector into the representation

$$\{\bar{x}\}_i = \{\bar{x}\}_{i_{\text{Undamped}}} + j\eta_i \{\delta\bar{x}\}_i,$$

where the real, undamped modal vector $\{\bar{x}\}_{i_{\text{Undamped}}}$ is perturbed by the complex vector $\{\delta\bar{x}\}_i$ as η_i increases from zero. Substituting above, the equation for η_i becomes

$$\eta_i = \left[\frac{\left(\lfloor \bar{x} \rfloor_{i_{\text{U.}}} [\delta k] \{\bar{x}\}_{i_{\text{U.}}} \right) + \eta_i^2 \left(\lfloor \delta\bar{x} \rfloor_i [\delta k] \{\delta\bar{x}\}_i \right)}{\left(\lfloor \bar{x} \rfloor_{i_{\text{U.}}} [k] \{\bar{x}\}_{i_{\text{U.}}} \right) + \eta_i^2 \left(\lfloor \delta\bar{x} \rfloor_i [k] \{\delta\bar{x}\}_i \right)} \right].$$

Except in the case of a modal resonance, the perturbation vector $\{\delta\bar{x}\}_i$ is small, generally very small and of the order of magnitude of η_i itself or less as in the case of lightly damped viscous systems (discussed above). Accordingly the approximation for η_i employing the undamped modes is seen to be a valid engineering approximation for η_i. That is

$$\eta_i \cong \left(\frac{\lfloor \bar{x} \rfloor_{i_{\text{U.}}} [\delta k] \{\bar{x}\}_{i_{\text{U.}}}}{\lfloor \bar{x} \rfloor_{i_{\text{U.}}} [k] \{\bar{x}\}_{i_{\text{U.}}}} \right).$$

4.7 Pervasive Dynamic Hysteresis-Viscoelastic Damping

In the special case when the matrix $[\delta k]$ is given by $[\delta k] \equiv \eta[k]$ an exact decoupling of the damped modes is possible employing the undamped modes. In this case the matrix differential equation for free vibration is

$$[m]\{\ddot{x}\} + (1 + j\eta)[k]\{x\} = \{0\}.$$

Taking the i^{th} damped mode solution in the form $\{x(t)\} = \{\bar{x}\}_{i_{\text{Undamped}}} e^{\lambda_i t}$, it follows that

$$\left[\lambda_i^2[m] + (1 + j\eta)[k]\right] \{\bar{x}\}_{i_{\text{Undamped}}} = \{0\}.$$

Employing the undamped mode orthogonality condition

$$\lfloor \bar{x} \rfloor_{r_{\text{U.}}} [m]\{\bar{x}\}_{s_{\text{U.}}} = \lfloor \bar{x} \rfloor_{r_{\text{U.}}} [k]\{\bar{x}\}_{s_{\text{U.}}} = 0, \qquad r \neq s,$$

then $\lambda_i^2 + \omega_i^2(1 + j\eta) = 0$, where ω_i^2 is given exactly by Rayleigh's quotient

$$\omega_i^2 = \left(\frac{\lfloor \bar{x} \rfloor_{i_{\text{U.}}} [k]\{\bar{x}\}_{i_{\text{U.}}}}{\lfloor \bar{x} \rfloor_{i_{\text{U.}}} [m]\{\bar{x}\}_{i_{\text{U.}}}} \right).$$

It follows as in the general case that

$$\lambda_{i_{\text{Real}}} = -\omega_i (1 + \eta^2)^{1/4} \sin\left(\frac{1}{2}\tan^{-1}\eta\right) \cong -\omega_i \frac{\eta}{2}$$

$$\lambda_{i_{\text{Imaginary}}} = \omega_i (1 + \eta^2)^{1/4} \cos\left(\frac{1}{2}\tan^{-1}\eta\right) \cong \omega_i$$

for $\eta \ll 1$. In this special case where the loss factor η is more a universal structural property of the system, rather than a modal one, the equivalent viscous damping ratio of every mode is the same, namely

$$\zeta_{i_{\substack{\text{Equivalent} \\ \text{Viscous}}}} = \frac{\eta}{2}.$$

4.8 Forced Vibration of Systems With Dynamic Hysteresis

Consider a simple harmonic system excitation at frequency Ω (radians/second) represented by the complex exponential vector $\{f(t)\} = \{\bar{f}\}e^{j\Omega t}$. The system matrix differential equation is

$$[m]\{\ddot{x}\} + [k]\{x\} + j[\delta k]\{x\} = \{\bar{f}\}e^{j\Omega t}.$$

The steady-state forced vibratory response can then be represented as $\{x(t)\}_{\text{S.S.}} = \{\bar{x}\}_{\text{S.S.}}e^{j\Omega t}$. Substitution above yields the matrix algebraic equation

$$\left[[k] - \Omega^2[m] + j[\delta k]\right]\{\bar{x}\}_{\text{S.S.}} = \{\bar{f}\}.$$

Now consider the damped modal expansion

$$\{\bar{x}\}_{\text{S.S.}} = \sum_{r=1}^{N} \bar{p}_r\{\bar{x}\}_r$$

and rewrite the foregoing equations

$$\sum_{r=1}^{N} \bar{p}_r \left[[k] - \Omega^2[m] + j[\delta k]\right]\{\bar{x}\}_r = \{\bar{f}\}.$$

Continuing, this equation is equivalent to

$$\sum_{r=1}^{N} \bar{p}_r \left[[k] + j[\delta k] + \lambda_r^2[m] - \left(\Omega^2 + \lambda_r^2\right)[m]\right]\{\bar{x}\}_r = \{\bar{f}\}.$$

Since

$$\left[[k] + j[\delta k] + \lambda_r^2[m]\right]\{\bar{x}\}_r = \{0\},$$

then it follows that

$$-\sum_{r=1}^{N} \bar{p}_r \left(\Omega^2 + \lambda_r^2\right)[m]\{\bar{x}\}_r = \{\bar{f}\}.$$

Pre-multiplying by the transposed complex modal vector $\lfloor\bar{x}\rfloor_s$ and employing the dynamic hysteresis-viscoelastic orthogonality condition

$$\lfloor\bar{x}\rfloor_r[m]\{\bar{x}\}_s = 0 \qquad r \neq s$$

$$-\left(\Omega^2+\lambda_r^2\right)\bar{p}_r M'_{rr} = \lfloor \bar{x} \rfloor_r \{\bar{f}\}, \qquad M'_{rr} \equiv \left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_r\right),$$

and

$$\bar{p}_r = -\left(\frac{1}{\Omega^2+\lambda_r^2}\right)\left(\frac{\bar{Q}_r}{M'_{rr}}\right), \qquad \bar{Q}_r \equiv \lfloor \bar{x} \rfloor_r \{\bar{f}\}.$$

The r^{th} and s^{th} damped modes can be interrelated as follows:

$$\lfloor \bar{x} \rfloor_s \left[[k]+j[\delta k]+\lambda_r^2[m]\right] \{\bar{x}\}_r = 0$$

and

$$\lfloor \bar{x} \rfloor_r \left[[k]+j[\delta k]+\lambda_s^2[m]\right] \{\bar{x}\}_s = 0.$$

Since $[k]$, $[\delta k]$, and $[m]$ are symmetric matrices, subtraction of the second equation from the first yields

$$\left(\lambda_r^2-\lambda_s^2\right)\left(\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s\right) = 0.$$

Accordingly, since $\lambda_r \neq \lambda_s$, the orthogonality condition follows as

$$\lfloor \bar{x} \rfloor_r [m] \{\bar{x}\}_s = 0, \qquad r \neq s.$$

Defining

$$\bar{p}_{r\,\text{Static}} \equiv \frac{\bar{Q}_r}{M'_{rr}\omega_r^2}$$

and noting from above that

$$\lambda_r^2 = -\omega_r^2(1+j\eta_r),$$

$$\left(\frac{\bar{p}_r}{\bar{p}_{r\,\text{Static}}}\right) = \left[\frac{\omega_r^2}{-\Omega^2+\omega_r^2(1+j\eta_r)}\right] = \left\{\frac{1}{\left[1-\left(\frac{\Omega}{\omega_r}\right)^2\right]+j\eta_r}\right\},$$

and the magnitude of this response ratio, "the damped magnification factor with modal hysteresis" follows as the magnitude

$$\left|\frac{\bar{p}_r}{\bar{p}_{r\,\text{Static}}}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\Omega}{\omega_r}\right)^2\right]^2+\eta_r^2}}$$

4.9 Derivative Operator Formulation For Systems With Dynamic Hysteresis

Consider the free vibration of a structural dynamic system with dynamic hysteresis type energy dissipation. We seek the equivalent viscous damping fraction of critical damping for the i^{th} damped mode of vibration. The governing matrix differential equation for the free vibration is taken as follows:

$$[m]\{\ddot{x}\} + [k]\{x\} + \left(\frac{1}{\omega_i}\right)[\delta k]\{\dot{x}\} = \{0\}.$$

The incremental viscoelastic type forces have the phase of viscous damping forces, but do not increase in magnitude with increase of natural frequency of vibration (hence the factor $(\omega)^{-1}$).

The general solution to the governing equation for the i^{th} mode of free vibration is taken in the form

$$\{x(t)\} = \{\bar{x}\}_i e^{\lambda_i t}.$$

Then

$$\lambda_i^2 + \left(\frac{\delta K_{ii}}{M_{ii}\omega_i}\right)\lambda_i + \omega_i^2 = 0,$$

where

$$\omega_i^2 \equiv \frac{K_{ii}}{M_{ii}}, \qquad K_{ii} \equiv \left(\lfloor\bar{x}\rfloor_i^*[k]|\{\bar{x}\}_i\right), \qquad M_{ii} \equiv \lfloor\bar{x}\rfloor_i^*[m]\{\bar{x}\}_i.$$

$$\delta K_{ii} \equiv \lfloor\bar{x}\rfloor_i^*[\delta k]\{\bar{x}\}_i, \qquad \{\bar{x}\}_i \equiv \{\bar{x}\}_{i_{\text{Real}}} + j\{\bar{x}\}_{i_{\text{Imaginary}}}, \qquad \{\bar{x}\}_i^* \equiv \{\bar{x}\}_{i_{\text{Real}}} - j\{\bar{x}\}_{i_{\text{Imaginary}}}$$

Defining

$$\frac{\delta K_{ii}}{M_{ii}\omega_i} \equiv 2\omega_i\,\zeta_{i_{\text{Equivalent}}}.$$

Then

$$\xi_{i_{\text{Equivalent}}} = \frac{1}{2}\left(\frac{1}{M_{ii}\omega_i^2}\right)(\delta K_{ii}).$$

Substituting for ω_i^2,

$$\xi_{i_{\text{Equivalent}}} = \frac{1}{2}\left(\frac{\delta K_{ii}}{K_{ii}}\right).$$

Defining the i^{th} the modal loss factor as

$$\eta_i \equiv \left(\frac{\delta K_{ii}}{K_{ii}}\right) \equiv \left(\frac{\lfloor \bar{x} \rfloor_i^* [\delta k] \{\bar{x}\}_i}{\lfloor \bar{x} \rfloor_i [k] \{\bar{x}\}_i}\right),$$

then

$$\zeta_{i_{\text{Equivalent}}} = \frac{1}{2}\eta_i.$$

It is to be noted that for practical purposes of modal loss factors (and internal loss factors also) of the order of 0.50 or less, there is negligible difference in this result and the one employing a complex stiffness approach for modeling the viscoelastic-dynamic hysteresis effects in the damping of structural vibrations.

4.10 The Global Equations With Proportional Damping

Consider the case of so-called proportional damping, when the damping matrix $[C]$ is expressible as a linear combination of the system global mass and stiffness matrices. That is

$$[C] \equiv \alpha[m] + \beta[k].$$

Accordingly the governing matrix differential equation for free vibration of the structural dynamic system takes the form

$$[m]\{\ddot{x}\} + [\alpha[m] + \beta[k]]\{\dot{x}\} + [k]\{x\} = \{0\}.$$

In the undamped case the i^{th} characteristic vector $\{\bar{x}\}_{ui}$ and the associated undamped natural frequency ω_{ui} satisfy the matrix algebraic equation

$$\left[[k] - \omega_{ui}^2[m]\right]\{\bar{x}\}_{ui} = \{0\},$$

and the orthogonality relations

$$\lfloor \bar{x} \rfloor_{ui}[m]\{\bar{x}\}_{uj} = \lfloor \bar{x} \rfloor_{ui}[k]\{\bar{x}\}_{uj} = 0, \qquad i \neq j$$

In the damped case, consider a solution for the i^{th} damped characteristic solution in the form

$$\{x\}_i = \{\bar{x}\}_{ui}\, e^{\lambda_i t}$$

If this is valid, then

$$\left[\lambda_i^2[m] + \lambda_i[\alpha[m] + \beta[k]] + [k]\right] \{\bar{x}\}_{ui} = \{0\}.$$

Pre-multiplying by the transposed i^{th} undamped characteristic vector $\lfloor \bar{x} \rfloor_{ui}$ results in this matrix algebraic equation being reduced to the scalar algebraic equation

$$\begin{aligned} \lambda_i^2 \left(\lfloor \bar{x} \rfloor_{ui}[m]\{\bar{x}\}_{ui}\right) + \lambda_i \left(\lfloor \bar{x} \rfloor_{ui}[\alpha[m] + \beta[k]]\{\bar{x}\}_{ui}\right) \\ + \left(\lfloor \bar{x} \rfloor_{ui}[k]\{\bar{x}\}_{ui}\right) = 0 \end{aligned}$$

It can also be inferred that $\{\bar{x}\}_i \equiv \{\bar{x}\}_{ui}$, since other potential terms in $\{\bar{x}\}_i$ involving the modes $\{\bar{x}\}_j \neq \{\bar{x}\}_i$ will vanish due to the orthogonality of the $\{\bar{x}\}_{ui}$ with respect to $[m]$ and $[k]$. Accordingly the foregoing scalar equation is an exact relationship for determining the i^{th} damped mode characteristic value when proportional type damping is present. It follows then that in this case

$$\lambda_i^2 + 2\zeta_i\omega_i\lambda_i + \omega_i^2 = 0, \qquad i = 1, 2, \ldots, N,$$

where

$$\omega_i^2 \equiv \left(\frac{\lfloor \bar{x} \rfloor_{ui}[k]\{\bar{x}\}_{ui}}{\lfloor \bar{x} \rfloor_{ui}[m]\{\bar{x}\}_{ui}}\right)$$

$$\zeta_i \equiv \left(\frac{1}{2\omega_i}\right)\left(\alpha + \beta\omega_i^2\right)$$

For example when proportional viscoelastic or proportional dynamic hysteretic type damping is assumed where

$$\beta \equiv \frac{\beta_o}{\omega_i},$$

then

$$\zeta_{i_{\text{Equivalent}}} \equiv \frac{1}{2}\beta_o.$$

5.0 APPROXIMATING THE DAMPING FRACTION FOR CONTINUOUS LINEAR SYSTEMS

5.1 The Damped Rod in Axial Vibration

Figure 2 illustrates a uniform rod which is built-in at one end and free at the other. It is embedded in viscoelastic material capable of dissipating energy during vibration. Neglecting the stiffness of the damping material compared to that of the rod itself, a one-dimensional wave type equation follows which describes the damped, free axial vibrations of the rod

$$AEu'' = \mu\ddot{u} + \sigma\dot{u},$$

with the boundary conditions

$$u(o,t) = \frac{\partial u}{\partial x}(\ell,t) = 0.$$

Noting that the natural frequencies of the undamped free vibrations are given by

$$\omega_n = \left(\frac{2n-1}{2}\right)\left(\frac{\pi}{\ell}\right)\sqrt{\frac{AE}{\mu}}, \qquad n = 1,2,\ldots,$$

and that

$$\left(\frac{\sigma}{\mu}\right) \equiv \frac{(c/\ell)}{(m/\ell)} \equiv \left(\frac{c}{m}\right) \equiv 2\zeta_n\omega_n,$$

the governing partial differential equation is separated with the solution

$$u(x,t) = \bar{u}(x)e^{\lambda t}.$$

This leads to the characteristic equation

$$\lambda_n^2 + 2\zeta_n\omega_n\lambda_n + \omega_n^2 = 0$$

from which

$$\lambda_n = -\omega_n\left(\zeta_n \pm j\sqrt{1-\zeta_n^2}\right),$$

where

$$\zeta_n = \frac{c}{2m\omega_n},$$

and where

$$c \equiv \sigma\ell \quad \text{and} \quad m \equiv \mu\ell.$$

Collecting the various definitions and substituting above yields

$$\zeta_n = \left[\frac{1}{(2n-1)\pi}\right]\left[\frac{c}{\sqrt{AE\mu}}\right], \qquad n = 1, 2, \ldots\ .$$

Figure 3 illustrates the case of the rod once again, but with a damper of rate c placed at the previously free end. The governing equation is now the wave equation

$$AEu'' = \mu\ddot{u},$$

with the boundary conditions

$$u(o,t) = 0$$

$$AEu'(\ell,t) = -c\dot{u}(\ell,t)$$

In view of the foregoing boundary conditions and the solution in the form

$$u(x,t) = \bar{u}(x)e^{\lambda t},$$

there results the characteristic equation

$$\tanh\left[(\lambda\ell)\sqrt{\frac{\mu}{AE}}\right] + \frac{\sqrt{\mu AE}}{c} = 0.$$

Since the damping fraction as a function of the system parameters is desired, the characteristic exponent is transformed as follows to reveal the real part explicitly. Let

$$\sqrt{\frac{\mu}{AE}}(\lambda\ell) \equiv \Lambda_R + j\Lambda_I, \qquad j^2 = -1.$$

Defining

$$\gamma \equiv \left(\frac{c}{\sqrt{\mu AE}}\right) \geqq 0,$$

employing the trigonometric-hyperbolic function identities

$$\sinh\ jz \equiv j \sin\ z$$

$$\cosh\ jz \equiv \cos\ z,$$

the transcendental, complex characteristic equation simplifies to two separate equations, each of which must vanish. These are

$$\cos\ \Lambda_I = 0,$$

and

$$\tanh\ \Lambda_R = -\gamma.$$

Employing the foregoing definitions

$$\Lambda_I \equiv \left(\sqrt{\frac{\mu}{AE}}\right)(\omega_n \ell)$$

and since the admissible values of Λ_I are

$$\Lambda_I = \left(\frac{2n-1}{2}\right)\pi, \qquad n = 1, 2, \ldots,$$

the natural frequency of the damped oscillation is

$$\omega_n = \left(\frac{2n-1}{2}\right)\left(\frac{\pi}{\ell}\right)\sqrt{\frac{AE}{\mu}}, \qquad n = 1, 2, \ldots .$$

This is seen to be identical to the results for the undamped case. Continuing,

$$\Lambda_R \equiv \left(\sqrt{\frac{\mu}{AE}}\right)(\lambda_R \ell)$$

and the transcendental equation for the decay factor and damping fractions for the various modes follows as

$$\tanh\ \sqrt{\frac{\mu}{AE}}(\lambda_R \ell) = -\frac{c}{\sqrt{\mu AE}}.$$

Inverting the hyperbolic tangent function

$$\lambda_{n_{\text{Real}}} \equiv -\omega_n \zeta_n = -\left(\frac{1}{\ell}\right)\sqrt{\frac{AE}{\mu}}\ \tanh^{-1}\left(\frac{c}{\sqrt{\mu AE}}\right).$$

Since lightly damped systems are the ones of primary interest, an engineering approximation follows at once by employing the power series expansion of the hyperbolic tangent function for small argument z:

$$\tanh\ z = z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots (z^2 < 1);$$

taking

$$z = \frac{c}{\sqrt{\mu AE}},$$

$$\lambda_{n_{\text{Real}}} \cong -\left(\frac{1}{\ell}\right)\left(\sqrt{\frac{AE}{\mu}}\right)\left(\frac{c}{\sqrt{\mu AE}}\right) = -\left(\frac{c}{\mu\ell}\right) \equiv -\left(\frac{c}{m}\right),$$

and

$$\zeta_n\omega_n \cong \left(\frac{c}{m}\right), \quad \text{and}$$

$$\zeta_n \cong \left[\frac{2}{(2n-1)\pi}\right]\left(\frac{c}{\sqrt{\mu AE}}\right).$$

It is noteworthy that a damper of rate c at the otherwise free end of the rod yields twice the damping fraction of the pervasive, uniformly distributed damping $c \equiv \sigma\ell$ previously determined.

The engineering solution for the damping fraction for the case being considered is now shown to be identical to that obtainable by neglecting the intermodal coupling due to damping, which is seen to be valid for lightly damped systems. The n^{th} undamped mode of free axial vibration of the rod is

$$\bar{u}_n(x) = \sin\left(\frac{2n-1}{2}\right)\left(\frac{\pi}{\ell}\right)x, \qquad n = 1, 2, \ldots .$$

The governing, uncoupled differential equation is by analogy with a single degree of freedom system

$$M_{n_{\text{Effective}}}\ddot{q}_n + C_{n_{\text{Effective}}}\dot{q}_n + M_n\omega_n^2 q_n = 0,$$

and

$$\left(\frac{C_{n_{\text{Effective}}}}{M_{n_{\text{Effective}}}}\right) \equiv 2\zeta_n\omega_n,$$

where

$$M_{n_{\text{Effective}}} \equiv \int_0^\ell \mu(x)\bar{u}_n^2(x)dx = \mu \int_0^\ell \sin^2\left(\frac{2n-1}{2}\right)\left(\frac{\pi}{\ell}\right) x\ dx = \frac{\mu\ell}{2} = \frac{m}{2},$$

$$C_{n_{\text{Effective}}} \equiv \int_0^\ell \sigma(x)\bar{u}_n^2(x)dx = (\sigma\ell)\ \sin^2\left(\frac{2n-1}{2}\right)\pi = C.$$

Accordingly

$$2\zeta_n\omega_n = \frac{2c}{m}, \qquad \zeta_n\omega_n = \left(\frac{c}{m}\right),$$

and

$$\zeta_n = \left[\frac{2}{(2n-1)\pi}\right]\left(\frac{c}{\sqrt{\mu AE}}\right).$$

This, of course, is identical to the formal solution of the result above employing an exact closed form solution for a lightly damped system.

To illustrate this point further by a direct numerical comparison, consider the rod with a free-end damper designed to yield a fundamental mode damping fraction $\zeta_{1_{\text{Nominal}}} = 0.10$. This is a relatively large fraction of critical damping for an engineering structure. Employing the exact, closed form transcendental equation solution,

$$\zeta_{1_{\text{Exact}}} = 0.1014.$$

This implies that the exact solution is 1.4 percent greater than that calculated from the engineering approximation. This small error decreases to zero as the nominal fraction of critical damping desired is decreased. For example for a nominal damping fraction

$$\zeta_{1_{\text{Nominal}}} = 0.02 \quad \text{and} \quad \zeta_{1_{\text{Exact}}} = 0.02006,$$

the approximation error is approximately 0.3 percent.

5.2 Damped Structural Members Other Than Rods

Since the governing equations for simple, St. Venant type torsion are also one dimensional wave type equations, it follows that by analogy the damping fraction results are identical to those of stretching, axial type rod vibrations. For engineering cases which are lightly damped and the damping treatment ranges from $x = x_1$ to $x = x_2$, the damping fraction follows in general as

$$\zeta_n \cong \left(\frac{1}{2\omega_N}\right)\left(\frac{C_{n_{\text{Effective}}}}{M_{n_{\text{Effective}}}}\right),$$

where

$$M_{\text{Effective}} \equiv \int_0^\ell \mu(x)\bar{\theta}_n^2(x)\; dx$$

$$C_{\text{Effective}} \equiv \int_0^\ell \sigma(x)\bar{\theta}_n^2(x)\; dx \equiv \sigma \int_{x_1}^{x_2} \bar{\theta}_n^2(x)\; dx.$$

Similar formulas for biharmonic or flexural systems such as beams, plates, and shells can be written by inspection. They are expected to be even more accurate than those for wave equation type systems. This follows from the greater separation of natural frequencies and, in turn, still weaker intermodal coupling due to damping. For example, a comparison of natural frequency spacing for a simply supported, uniform beam shows a mode number squared separation. That is, the second mode natural frequency is four times the fundamental, the third mode is nine times the fundamental, etc. The same structural element in stretching or twisting tends to have an integer spacing such as 1, 3, 5, ... for the rod fixed at one end and free at the other. As an example of the engineering approximation for a damped beam in bending, consider the simply supported uniform beam with a damper of rate c at $x = \xi$. For such a beam the frequencies and mode shapes are given by

$$\omega_n = \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\mu}}, \qquad n = 1, 2, \ldots,$$

$$\bar{w}_n(x) = \sin\left(\frac{n\pi}{\ell}\right) x$$

and, as is always the case,

$$2\zeta_n\omega_n = \frac{C_{n_{\text{Effective}}}}{M_{n_{\text{Effective}}}}$$

where

$$M_{n_{\text{Effective}}} \equiv \int_0^\ell \mu(x)\bar{w}_n^2(x)\; dx = \frac{\mu\ell}{2} = \frac{m}{2}$$

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Introducing the displacement vector $\vec{q}$ and the unit vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$,

$$\vec{q} \equiv u(x,y,z)\vec{\imath} + v(x,y,z)\vec{\jmath} + w(x,y,z)\vec{k},$$

and noting that

$$\Delta \equiv \mathrm{div}\vec{q}$$

and

$$\overrightarrow{\mathrm{grad}}\Delta \equiv \frac{\partial \Delta}{\partial x}\vec{\imath} + \frac{\partial \Delta}{\partial y}\vec{\jmath} + \frac{\partial \Delta}{\partial z}\vec{k},$$

$$\lambda \equiv \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad G \equiv \frac{E}{2(1+\nu)},$$

the foregoing scalar Navier equations are combined into the single vector equation

$$(\lambda + G)\overrightarrow{\mathrm{grad}}\Delta + G\nabla^2\vec{q} = \rho\ddot{\vec{q}} + \sigma\dot{\vec{q}}.$$

In free, undamped oscillations when $\sigma \equiv 0$, $\vec{q} = \bar{\vec{q}}e^{j\omega_n t}$, and

$$(\lambda + G)\overrightarrow{\mathrm{grad}}\bar{\Delta}_n + G\nabla^2\bar{\vec{q}}_n = -\rho\omega_n^2\bar{\vec{q}}_n.$$

Now taking the damped oscillation solution in the form

$$\vec{q} = \bar{\vec{q}}_n e^{\lambda_n t},$$

there results the characteristic equation

$$\lambda_n^2 + \left(\frac{\sigma}{G}\right)\lambda_n + \omega_n^2 = 0.$$

It follows then that the damping fraction for the n^{th} mode of oscillation is given by

$$\zeta_n = \left(\frac{1}{2\omega_n}\right)\left(\frac{\sigma}{G}\right).$$

In the cases of spot damping or concentrated and localized damping, it is intuitively obvious that the effective n^{th} mode damping fraction will exceed the result above, provided that the total distributed pervasive damping equals the concentrated damping in magnitude, and that an anti-nodal region is employed for the damping treatment. Needless to say, dual modal degeneracy is excluded in this observation, where it has been seen that one of the two degenerate modes can be undamped in such cases.

6.3 Examples of Damped Beam Vibration

The governing differential equation for a viscously damped beam in transverse bending vibrations is

$$EI\frac{\partial^4 w}{\partial x^4} = -\left(\mu\frac{\partial^2 w}{\partial t^2} + \sigma\frac{\partial w}{\partial t}\right).$$

The beam is assumed to be uniform in its flexural rigidity EI and mass per unit length μ. The damping constant per unit length σ will take several forms in the examples which follow, ranging from a constant to a single or multiple discrete dampers. In every case an energy relationship which is now deduced provides the exact solution for the damping fraction when the exact, damped eigenvectors are known. It is also the basis for an acceptably accurate engineering approximation when the damped eigenvectors are approximated by the undamped ones.

Taking the exact solution in the form

$$w(x,t) = \bar{w}_n(x)e^{\lambda_n t},$$

the partial differential equation is reduced to an ordinary differential equation which follows below as

$$EI\frac{d^4\bar{w}_n}{dx^4} + \mu\lambda^2\bar{w}_n + \sigma\lambda_n\bar{w}_n = 0, \qquad n = 1, 2, \ldots .$$

Since these eigenvectors $\bar{w}_n$ are, in general, complex vectors, multiply the foregoing equation by the complex conjugate eigenvector $\bar{w}_n^*$ and take the definite integrals over the beam span from $x = 0$ to $x = \ell$. This results in the equation

$$EI\int_0^\ell \frac{d^4\bar{w}_n}{dx^4}\bar{w}_n^*\,dx + \mu\lambda_n^2\int_0^\ell \bar{w}_n\bar{w}_n^* + \lambda_n\int_0^\ell \sigma\bar{w}_n\bar{w}_n^*\,dx = 0.$$

Integrating by parts and invocating the beam's boundary conditions yields the algebraic equation

$$\lambda_n^2 + \left(\frac{\int_0^\ell \sigma\bar{w}_n\bar{w}_n^*\,dx}{\mu\int_0^\ell \bar{w}_n\bar{w}_n^*\,dx}\right)\lambda_n + \left(\frac{EI\int_0^\ell \frac{d^2\bar{w}_n}{dx^2}\frac{d^2\bar{w}_n^*}{dx^2}dx}{\mu\int_0^\ell \bar{w}_n\bar{w}_n^*\,dx}\right) = 0.$$

Defining

$$\omega_n^2 \equiv \left(\frac{EI}{\mu}\right)\left(\frac{\int_0^\ell \frac{d^2\bar{w}_n}{dx^2}\frac{d^2\bar{w}_n^*}{dx^2}dx}{\mu\int_0^\ell \bar{w}_n\bar{w}_n^*\,dx}\right),$$

then

$$\zeta_n \equiv \left(\frac{1}{2\omega_n}\right)\left(\frac{\int_0^\ell \sigma\,\bar{w}_n\bar{w}_n^*\,dx}{\mu\int_0^\ell \bar{w}_n\bar{w}_n^*\,dx}\right).$$

In the case of pervasive damping when σ is a constant,

$$\zeta_{n_{\text{Pervasive}}} \equiv \left(\frac{1}{2\omega_n}\right)\left(\frac{\sigma}{\mu}\right), \qquad n = 1, 2, \dots\ .$$

More generally, when σ is not constant over the span and the modes have the usual numerical spacing, $\bar{w}_n$ is approximated by $\bar{w}_{nu}$, the undamped modal vector. Accordingly,

$$\omega_n^2 \cong \left(\frac{EI}{\mu}\right)\left(\frac{\int_0^\ell \left(\frac{d^2\bar{w}_{nu}}{dx^2}\right)^2 dx}{\int_0^\ell \bar{w}_{nu}^2\,dx}\right),$$

and

$$\zeta_n \cong \left(\frac{1}{2\omega_n}\right)\left(\frac{\int_0^\ell \sigma\bar{w}_{nu}^2\,dx}{\mu\int_0^\ell \bar{w}_{nu}^2\,dx}\right).$$

For example, consider the case of the simply supported beam with pervasive damping. In this case the exact solution follows as

$$\zeta_n = \left(\frac{1}{2\omega_n}\right)\left(\frac{\sigma}{\mu}\right), \qquad \omega_n = \left(\frac{n\pi}{\ell}\right)^2\sqrt{\frac{EI}{\mu}}, \qquad n = 1, 2, \dots\ ,$$

so that

$$\zeta_n = \frac{\ell C}{2\pi n^2\sqrt{\mu EI}}, \qquad n = 1, 2, \dots\ .$$

Consider the case now of three discrete dampers of rate $(C/3)$ located at $x = \ell/4$, $\ell/2$, and $3\ell/4$. Approximating $\bar{w}_1$ by the undamped mode

$$\bar{w}_{1_u} = \sin\frac{\pi x}{\ell}, \quad \zeta_1 \cong \frac{C\ell}{3\pi^2\sqrt{\mu EI}}\left[\sin^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{3\pi}{4}\right)\right],$$

and

$$\zeta_1 \cong \left(\frac{2}{3\pi^2}\right)\frac{C\ell}{\sqrt{\mu EI}}.$$

This approximation for the damping fraction is

$$\zeta_1 \cong \left(\frac{1}{2\omega_1}\right)\left[\frac{C\bar{w}_{1u}^2(\xi)}{\mu\int_0^\ell \bar{w}_{1u}^2\,dx}\right].$$

The fundamental mode of the cantilever is now approximated by a polynomial approximation which satisfies the undamped cantilever beam boundary conditions. This is

$$\bar{w}_1(\xi) = \bar{w}_1(1)\left(2\xi^2 - \frac{4}{3}\xi^3 + \frac{1}{3}\xi^4\right), \qquad \xi \equiv \left(\frac{x}{\ell}\right).$$

Computing the various terms in the approximation and comparing the results to the beam tip mounted damper yields the tabular comparison below.

$\bar{\xi} \equiv \left(\frac{\bar{x}}{\ell}\right)$	$\left(\zeta_{1_{\text{Effective}}}/\zeta_{1_{\xi=1}}\right)$
0.00	0
0.50	0.1254
0.75	0.4462
1.00	1.0000

6.4 Bending Vibration of a Discretely Modelled Damped Beam

Consider a cantilever beam of length ℓ and constant flexural rigidity EI. Concentrated masses m_1, m_2, and m_3 are attached at $x = \ell/3$, $x = 2\ell/3$, and $x = \ell$ as illustrated in Figure 9. A damper of rate C is placed at $x = \ell$. Taking $m_1 = m_2 = m_3 = m$ and neglecting the distributed mass of the beam compared to the effects of the concentrated masses, the three coupled equations of motion in matrix format are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \end{Bmatrix} + \left(\frac{C}{m}\right)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{Bmatrix}$$

$$+\left(\frac{81EI}{13m\ell^3}\right)\begin{bmatrix} 80 & -46 & 12 \\ -46 & 44 & -16 \\ 12 & -16 & 7 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

A detailed numerical comparison is now carried out between the fundamental mode damping fraction approximation and an exact solution employing a digital computer code for the damped eigenvectors and associated complex eigenvalues. These digital computer results are presented in Appendix C.

The data employed is as follows:

$$m_1 = m_2 = m_3 = m = 1.0,$$

$$\left(\frac{3EI}{m\ell^3}\right) = 1.0,$$

and

$$[k_{\text{Effective}}] \equiv \begin{bmatrix} 166.154 & -95.538 & 24.923 \\ -95.538 & 91.385 & -33.231 \\ 24.923 & -33.231 & 14.538 \end{bmatrix};$$

the damper constant C is varied to determine the influence of the magnitude of C on the accuracy of the approximation with

$$C = 0.00,\ 0.10,\ 0.20,\ 0.30,\ 0.40,\ 0.50,\ 1.00,\ 1.50,\ 2.00.$$

The engineering approximation employs a Dunkerley type approximation followed by a matrix interaction to approximate the fundamental mode natural frequency and undamped modal pattern employing a pocket type calculator. This approximation yields the following data:

$$\omega_1 \cong 0.8775 \text{ (radians/second)},$$

$$\begin{Bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \end{Bmatrix} \cong \begin{Bmatrix} 0.1565 \\ 0.5318 \\ 1.000 \end{Bmatrix},$$

and the formula

$$\zeta_1 \cong 0.436C.$$

This permits a direct tabular comparison which follows below and which illustrates the excellent accuracy afforded by the method of approximation.

C	$\zeta_{1\text{“Approx.”}}$	$\zeta_{1\text{“Exact”}}$	% Error*
0.00	0	0	0
0.10	0.0436	0.043608	0.018
0.20	0.0872	0.087227	0.031
0.30	0.1308	0.130851	0.0390
0.40	0.1744	0.174499	0.0568
0.50	0.2180	0.2181739	0.0798
1.00	0.4360	0.437176	0.2697
1.50	0.6540	0.657797	0.5806
2.00	0.8720	0.8807479	1.003

*Percent error is defined as “E” $\equiv \left[\left(\frac{\zeta_{1\text{“Exact”}}}{\zeta_{1\text{“Approx.”}}}\right) - 1\right] \times 100.$

It is seen that for practical engineering levels of damping, the percent error is a small fraction of one percent. Even at levels approaching critical damping the error is still only of the order of one percent.

6.5 Coupled Bending and Torsion Vibrations of a Discretely Modelled Damped Beam

Now consider the system illustrated in Figure 10. This six degree of freedom system extends the previous illustration by introducing coupled bending and torsional oscillations. The rationale is that torsional modes are, generally speaking, not as widely separated numerically as bending modes, so that the method of approximation might yield larger errors as the damper constant is increased. Once again

$$m_1 = m_2 = m_3 = m = 1,$$

but mass moment of inertia effects are introduced with

$$I_1 = I_2 = I_3 = 1.$$

The offset of m_3 is taken to be $d_3 = 1$. Also

$$(GJ/L)_1 = (GJ/L)_2 = (GJ/L)_3 = 1$$

and

$$(EI/L^3)_1 = (EI/L^3)_2 = (EI/L^3)_3 = 9.$$

To test the hypothesis that the engineering approximation will still yield results of acceptable engineering accuracy, the third of the six modes of damped coupled bending and torsion is examined. The natural frequency and associated undamped modal vector are computed and are as follows:

$$\omega_3 \cong 1.341(\text{radians/second})$$

$$\begin{Bmatrix} \bar{w}_1 \\ \bar{\theta}_1 \\ \bar{w}_2 \\ \bar{\theta}_2 \\ \bar{w}_3 \\ \bar{\theta}_3 \end{Bmatrix}_3 \cong \begin{Bmatrix} 0.1683 \\ 0.7863 \\ 0.5504 \\ 0.1592 \\ 1.0000 \\ -.7540 \end{Bmatrix}.$$

The approximation for ζ_3 is found to be $\zeta_{3\,\text{“Approx,”}} \cong .2249C$. The exact digital computer results are presented in Appendix D and yield the following tabular comparison:

C	$\zeta_{3\,\text{“Approx.”}}$	$\zeta_{3\,\text{“Exact”}}$	% Error
0.10	0.0225	0.0233	3.56%
0.20	0.0449	0.0469	4.45%
0.30	0.0675	0.0709	5.03%
0.50	0.1124	0.1213	7.92%

Here the percentage errors are significantly larger than for bending vibrations only, but are still of acceptable engineering accuracy for preliminary design purposes when the type, location, and quality of damping are of principal interest in the system design.

6.6 Vibration of a Plate With Spot Damping and Different Modes With Matching or Nearly Matching Natural Frequencies

Figure 11 shows a simply supported uniform rectangular plate with a concentrated damper force at the interior point $(\tilde{x}, \tilde{y})$. The governing differential equation of free motion is

$$D\nabla^4 w(x,y,t) + \mu\ddot{w}(x,y,t) + d\dot{w}(\tilde{x},\tilde{y},t) = 0.$$

The undamped modal patterns are readily seen to be

$$\bar{w}_{mn}(x,y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

The associated natural frequencies are

$$\omega_{mn}^2 = \left(\frac{\pi}{a}\right)^4 (m^2 + N^2 n^2)^2 \left(\frac{D}{\mu}\right),$$

where $N \equiv \left(\frac{a}{b}\right)$. Now consider the case where differing modal patterns $\bar{w}_{rs}$ have the same natural frequency. The plate aspect ratio N is then related to the modal integers $m,\ n,\ r$, and s by

$$N = \left|(r^2 - m^2)/(n^2 - s^2)\right|^{1/2}.$$

It is shown analytically in References 2 and 3 that this "spot" damping treatment fails due to the degeneracy of modes mn and rs when their frequencies match. In effect a nodal point of the damped mode occurs at the damper location $(\tilde{x},\tilde{y})$.

A numerical approach also provides a demonstration of this anomaly. A modal expansion of undamped plate modes is employed followed by a numerical solution of the characteristic determinant and polynomial by computing the damped characteristic values as the plate aspect ratio is systematically varied to produce two modes whose natural frequencies are close to one another and ultimately match. The first nine modes are coupled by the damper as the aspect ratio is varied for several nominal damping levels ranging from 5 percent to 50 percent of critical in the fundamental plate mode. The frequency match occurs in the fourth and fifth modes. The table which follows below is for a plate of constant area of one square meter as the aspect ratio is varied. The damper is placed at $(\tilde{x}/a) = (\tilde{y}/b) = (1/\pi)$. It is seen that the loss of damping is almost complete over a significant range of frequencies above and below the match at aspect ratio $N \cong .7744$ and frequency ratio $\rho \cong 1$ ($a \cong .880,\ b \cong 1.1364$).

Table: $A = ab = 1.0$ Meters Squared; $(\widetilde{x}/a) = (\widetilde{y}/b) = (1/\pi)$

a	ρ	$N \equiv a/b \equiv a^2$	$\delta_{\text{ref}} = .05$	$\delta_{\text{ref}} = .10$	$\delta_{\text{ref}} = .25$	$\delta_{\text{ref}} = .50$
.65	0.5529	.4225	1.7005	3.3506	8.7111	12.2702
.70	0.6372	.4900	1.7019	3.3618	8.7960	10.2591
.75	0.7307	.5625	1.7029	3.3696	8.9389	0.2759
.80	0.8316	.6400	0.0423	0.0834	0.1856	0.2574
.83	0.8937	.6889	0.0387	0.0743	0.1392	0.1346
.85	0.9345	.7225	0.0357	0.0616	0.0726	0.0379
.86	0.9572	.7396	0.0325	0.0469	0.0328	0.0071
.87	0.9785	.7569	0.0224	0.0193	0.0045	0.0052
.88	0.9998	.7744	0.0000	0.0000	0.0001	0.0002
.89	1.0211	.7921	0.0205	0.0196	0.0160	0.0170
.90	1.0423	.8100	0.0287	0.0401	0.0404	0.0406
.91	1.0635	.8281	0.0306	0.0500	0.0632	0.0658
.93	1.1058	.8649	0.0309	0.0564	0.0942	0.1170
.95	1.1478	.9025	0.0302	0.0572	0.1097	0.1437

7.0 A SMART DYNAMIC VIBRATION ABSORBER: A COLLATERAL DAMPING APPLICATIONS TECHNOLOGY

The dynamic vibration absorber is a subsidiary dynamic system to be attached to the primary system. Typically the primary system is exhibiting an undesirable dynamic response to a simple harmonic forced vibratory excitation, usually inherent in the system and not subject to significant frequency or magnitude changes to reduce the undesirable response. Accordingly the absorber is tuned to the offending forcing frequency and generates an opposing force, thereby reducing the response to zero (i.e., enforcing a node) or to an acceptable level. However in the absence of damping in the absorber, two new responses result at neighboring frequencies near the original offending one. When damping is introduced into the absorber these neighboring responses can be significantly attenuated. However this results in a significant reduction in the efficiency of the absorber compared to an undamped one as illustrated in Figures 12 and 13 (cf. References 4, 5, and 6).

A smart dynamic absorber would be one which benefits from damping at the "side" frequencies by attenuating these new responses to a negligible level, while ignoring the presence of damping at its primary frequency, thereby having the potential to have the "best of both worlds": full vibration suppression at the primary frequency with negligible, damped response at the "side" frequencies. This can be accomplished via the principle of the "notch filter." In effect the damper valve is sharply frequency sensitive so that it produces its normal, large damping magnitude, except over a very narrow range of frequencies centered at the primary excitation frequency. A classical dynamic system where this is the case is the "Bridge-T Network" familiar to electrical engineers. Figures 14 and 15 illustrate such a network and its frequency response characteristics. Figures 16, 17, and 18 illustrate a purely mechanical dynamic system with similar characteristics. It should be recognized that the frequency sensitive damper valve results in a closed loop dynamic system. As in all such systems an engineering "trade-off" must be examined. The "loop gain" and optimum efficiency of the "smart" dynamic absorber as a closed loop system must be weighed against the transient response characteristics of the new system. Generally the transients can be expected to be more "skittish" with the "smart" absorber. In fact, dynamic instability can ensue in the closed loop system if insufficient care is taken in adjusting the loop gain of the valve-absorber-primary system.

8.0 DYNAMIC STABILITY BOUNDARIES FOR BINARY SYSTEMS

The notion of a quartic damping interaction polynomial equation developed above can be extended to include the dynamic stability boundaries for binary systems (Ref. 7). That is systems which can be characterized by two degrees of freedom and in which both energy input and energy dissipation are factors. In such systems, the various parameters, especially the frequency proximity of the pair of oscillators comprising the system, determine the boundaries of dynamic stability and the "trade-offs" useful in preliminary design analysis.

Dynamic stability boundaries are now developed for linear, two-degree-of-freedom systems with damping and elastic couplings. Special emphasis is placed on the influence of natural frequency proximity and those instabilities which stem from skew symmetric stiffness properties. These arise in numerous engineering and physical systems, but especially in aeroelasticity and flight dynamics, as in the case of wing flutter and aircraft stability and control characteristics, respectively. New insight is provided into the destabilizing effects of the dreaded modal "resonance."

The generalized co-ordinates of the binary system are denoted by q_1 and q_2. These represent modes of undamped free vibration, the natural frequencies of which are ω_1 and ω_2, and the generalized masses of which are M_{11} and M_{22}, respectively. The damping matrix is symmetric and positive definite with elements $C_{11},\ C_{12}$, and C_{22}. The stiffness matrix is skew-symmetric and positive definite with elements $K_{11},\ K_{12}$, and K_{22}, and $K_{11} = M_{11}\omega_1^2$, $K_{22} = M_{22}\omega_2^2$. The equation of the system is

$$\begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ -K_{12} & K_{22} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = 0.$$

The system characteristic values λ can be developed as follows. Let

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \bar{q}_1 \\ \bar{q}_2 \end{Bmatrix} e^{\lambda t}$$

and refer these characteristic values to the natural frequency ω_1, where $\eta \equiv (\lambda/\omega_1)$. Upon introducing the dimensionless parameters $\rho \equiv (C_{22}/C_{11})/(M_{22}/M_{11}) > 0$, $\zeta \equiv (C_{11}/2M_{11}\omega_1) > 0$, $\delta^2 \equiv (C_{11}C_{12} - C_{12}^2)/M_{11}M_{22}\omega_1^2 > 0$, $\Gamma^2 \equiv K_{12}^2/M_{11}M_{12}\omega_1^4 > 0$, and $r \equiv \omega_2/\omega_1$, the system characteristic equation becomes

$$\eta^4 + 2\zeta(1+\rho)\eta^3 + (1+r^2+\delta^2)\eta^2 + 2\zeta(r^2+\rho)\eta + (r^2+\Gamma^2) = 0.$$

At neutral stability a sustained oscillation will occur at dimensionless frequency ν, so that $\eta = \pm j\nu$. It follows at once that

$$\nu^2 = (r^2+\rho)/(1+\rho) \quad \text{and} \quad \nu^4 - (1+r^2+\delta^2)\nu^2 + (r^2+\Gamma^2) = 0.$$

Solving for Γ^2 as a function of δ^2, r^2, and ρ gives

$$\Gamma^2 = \left(\frac{1+r^2+\delta^2}{1+\rho}\right)(r^2+\rho^2) - \left(\frac{r^2+\rho^2}{1+\rho}\right) - r^2.$$

Upon introducing scale changes through the new variable E, where

$$E \equiv (r^2-1)/(1+\rho) \quad \text{and} \quad r^2 \equiv 1 + (1+\rho)E,$$

and noting that

$$\nu^2 = 1 + E,$$

the stability boundaries take the simple parabolic form

$$(\Gamma^2/\rho) = E^2 + (\delta^2/\rho)(1+E).$$

In this form the destabilizing and stabilizing parameters Γ^2 and δ^2 are related via the new frequency proximity variable E. The dreaded "modal resonance" phenomenon is now especially transparent. When the natural frequencies ω_1 and ω_2 are equal, $E = 0$ and $\Gamma = \delta$: that is, the damping characterized by δ is needed to provide neutral stability in the presence of the destabilizing effects of Γ. Anything less results in a divergent oscillation. It is also seen that when $\Gamma = \delta$ and $E = 0$, the frequency of the sustained oscillations is $\omega_1 = \omega_2$.

More generally, the frequency of the sustained oscillations at neutral stability differs from ω_1 when $r \neq 1$. In Figures 19 and 20 are shown the stability boundaries and instability or "flutter" frequency, respectively, as functions of the frequency proximity variable E.

The physical significance of the results illustrated in Figures 19 and 20 can be considered by an example calculation. Suppose the modal damping parameters C_{11} and C_{22} and the associated generalized mass parameters M_{11} and M_{22} are such that $\rho \equiv (C_{22}/C_{11})/(M_{22}/M_{11})$ $= 0{\cdot}50$. Also, the consolidated damping-energy dissipation parameter $\delta^2 = 0{\cdot}20$ and the modal frequency ratio parameter $\omega_2/\omega_1 = 1{\cdot}10$. The frequency proximity variable E then is $0{\cdot}1400$. It follows at once that $\Gamma^2/\rho = 0{\cdot}4756$ and, for $\rho = 0{\cdot}50$, $\Gamma^2 = 0{\cdot}2378$. Accordingly, neutral dynamic stability results when $\Gamma^2 \equiv K_{12}^2/M_{11}M_{22}\omega_1^4 = 0{\cdot}2378$. When $K_{12} >$ $0{\cdot}4876(M_{11}M_{22})^{1/2}\omega_1^2$, a divergent oscillation occurs. The frequency of this instability, the "flutter" frequency, follows as illustrated in Figure 20 as $\omega_{\text{flutter}} = 1{\cdot}068\omega_1$. Since also

$$\Gamma^2 \equiv K_{12}^2/M_{11}M_{22}\omega_1^4 \equiv \left(K_{12}^2/K_{11}K_{22}\right)(\omega_2/\omega_1)^2 = 0{\cdot}2378$$

and $(\omega_2/\omega_1)^2$ $1{\cdot}21$, it follows that dynamic instability occurs for $K_{12}/(K_{11}K_{22})^{1/2} > 0.4433$.

In conclusion, it is to be noted that the binary system stability boundaries are universal ones. They yield the levels of the destabilizing parameters in terms of generalized damping and frequency proximity parameters. It is also clear that the modal natural frequency ratio is of crucial importance in binary system dynamic stability. Severe and perhaps unstainable damping requirements result when the two natural frequencies match or nearly match. Hence the concept of the "dreaded modal resonance."

9.0 CONCLUSIONS

Methods for approximating the effects of viscous or equivalent-viscous type damping treatments on the free and forced vibration of lightly damped aircraft-type structures have been developed. Similar methods have been developed for dynamic hysteresis-viscoelastic-type damping treatments. In all cases it is clear that these relatively simple, energy-based methods yield acceptably accurate engineering approximations for preliminary design purposes and in most instances can supplant much more complex, finite element-type digital computer computations.

Selected illustrative computational examples for a variety of structural elements have been carried out. This type of computation should be continued and extended to illustrate the procedures and methodology for an entire resonating airframe. This should be accompanied by suitable experimental validation as well as comparison with the digital computer, finite element approach.

It is noteworthy that in the case of lightly damped structures, the apparent complication of intermodal coupling due to damping can be neglected with one exception. In the case of differing modes having matching or nearly matching natural frequencies it is necessary to carry out a relatively simple interaction calculation which determines the distribution of damping between such modes.

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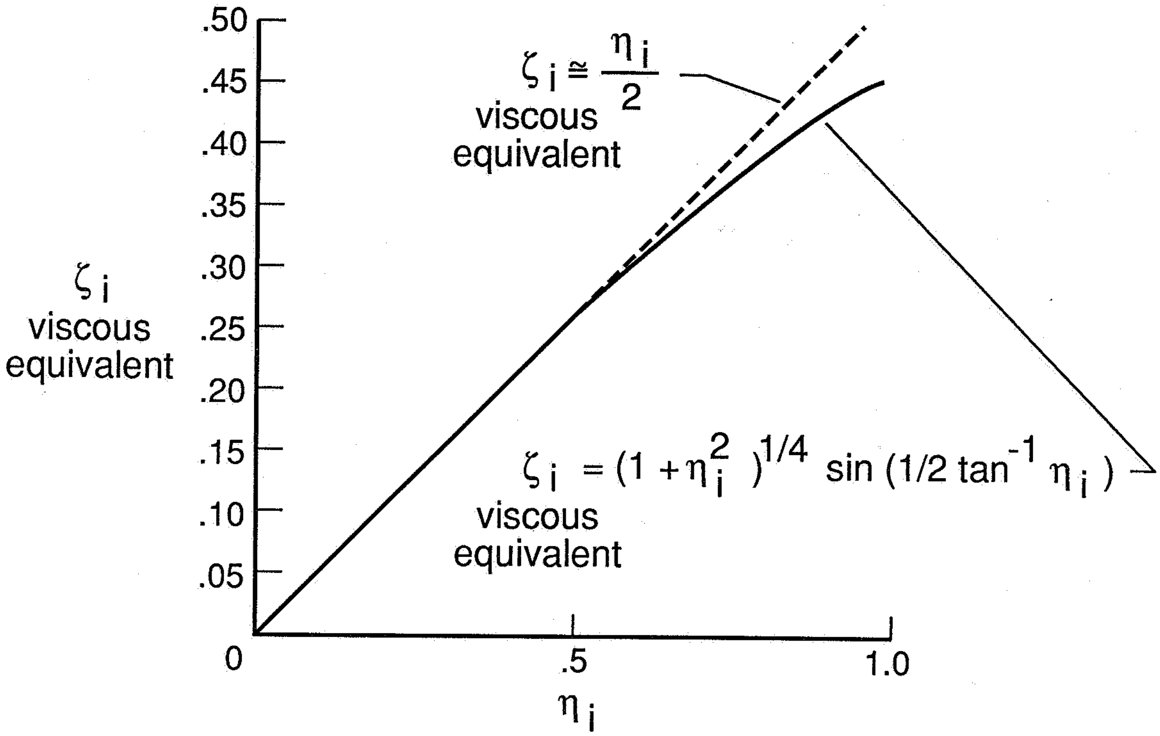

Figure 1.- Equivalent viscous damping as a function of loss factor.

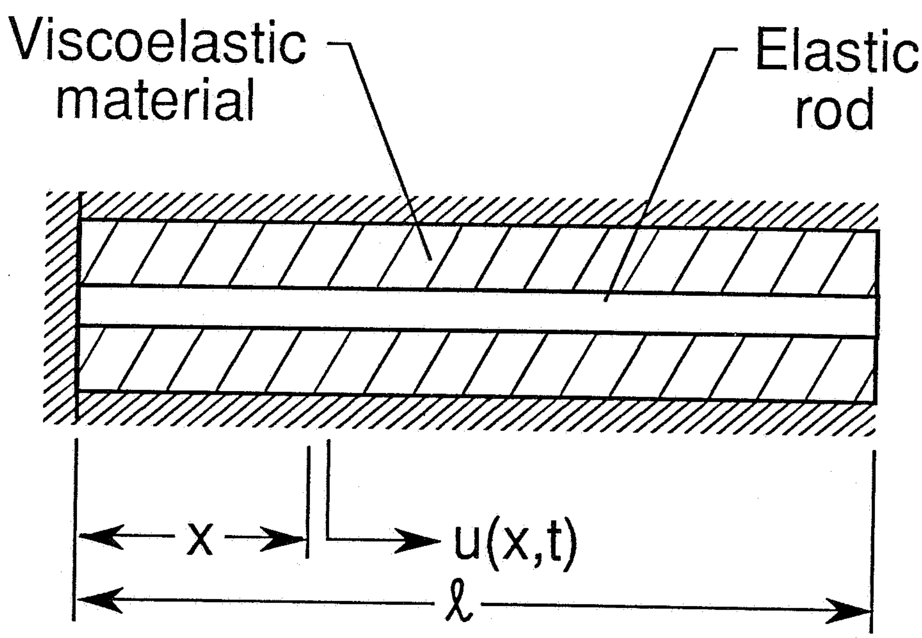

Figure 2.- Elastic rod embedded in viscoelastic material.

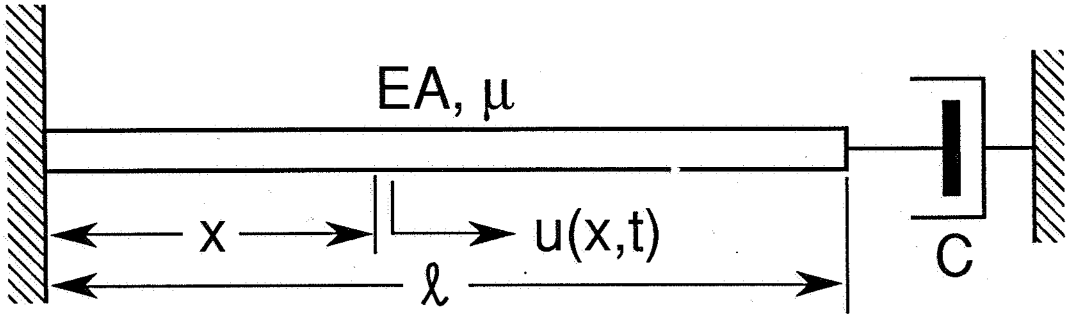

Figure 3.- Elastic rod with viscous damper at free end.

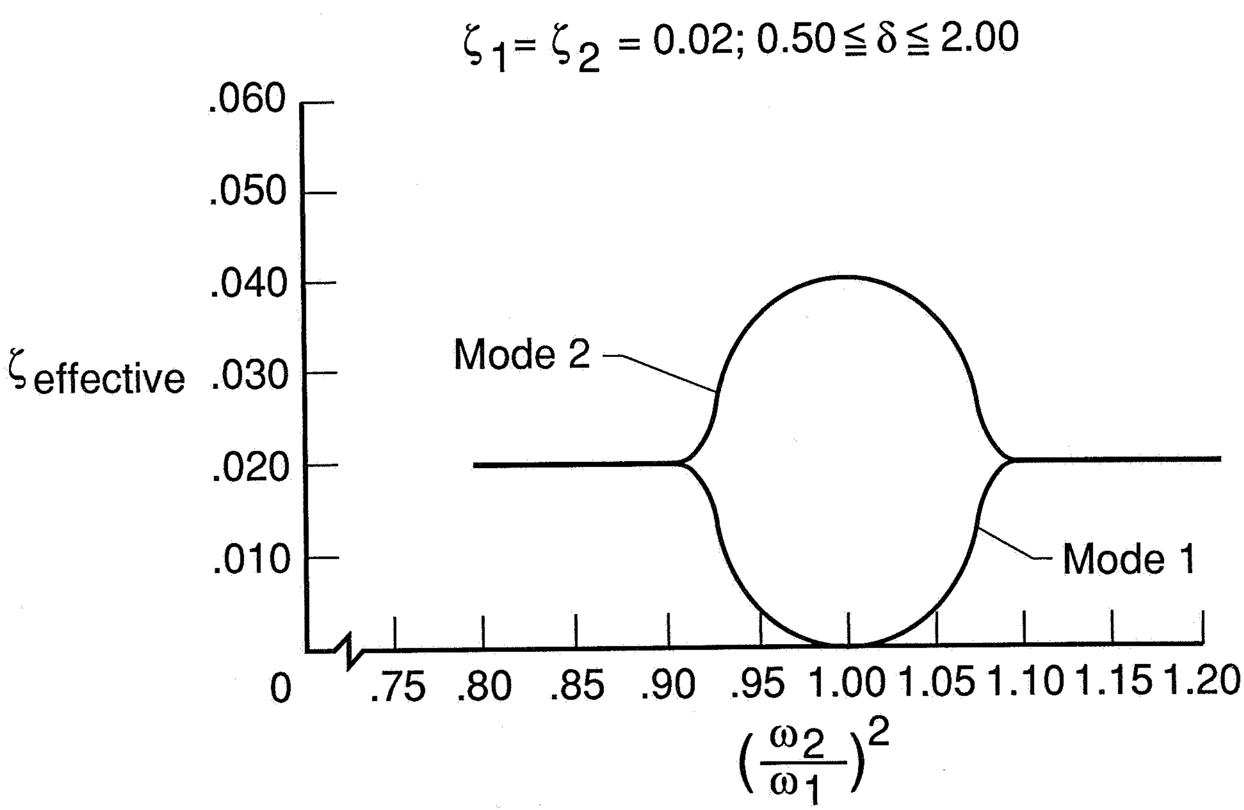

Figure 4.- Effective viscous damping ratio as a function of neighboring mode frequency separation ($\zeta_1 = \zeta_2 = .02$).

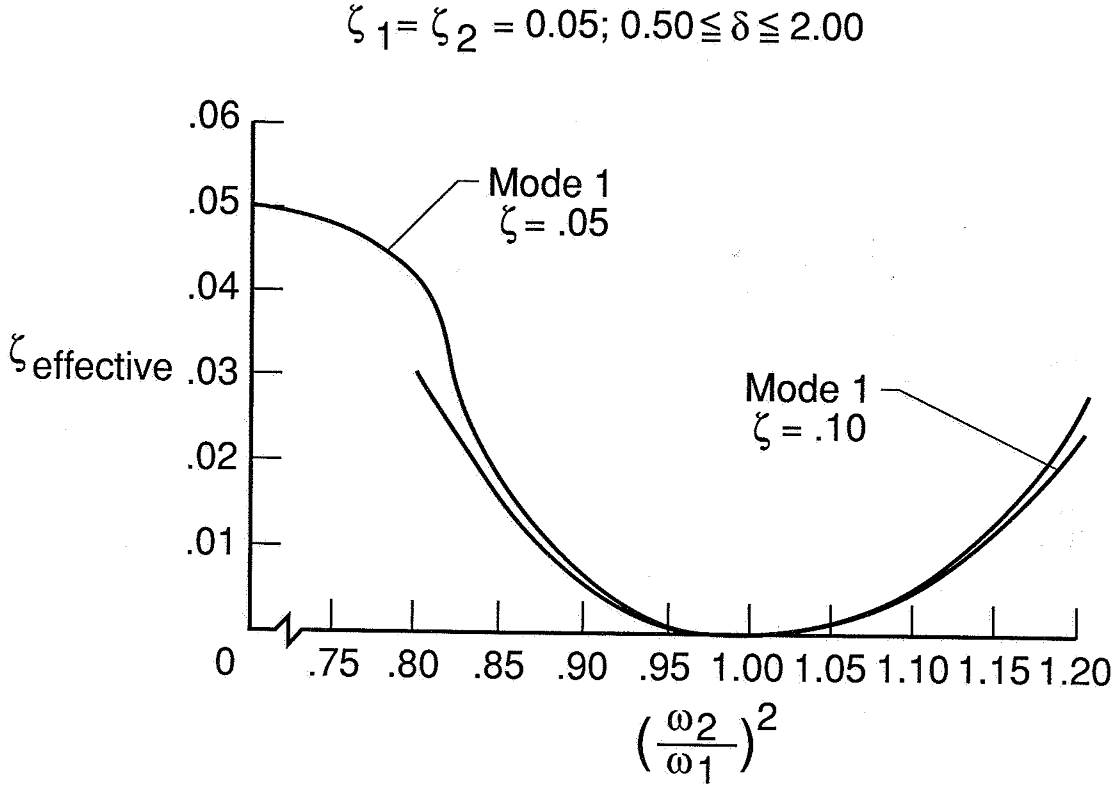

Figure 5.- Effective viscous damping ratio as a function of neighboring mode frequency separation ($\zeta_1 = \zeta_2 = .05$).

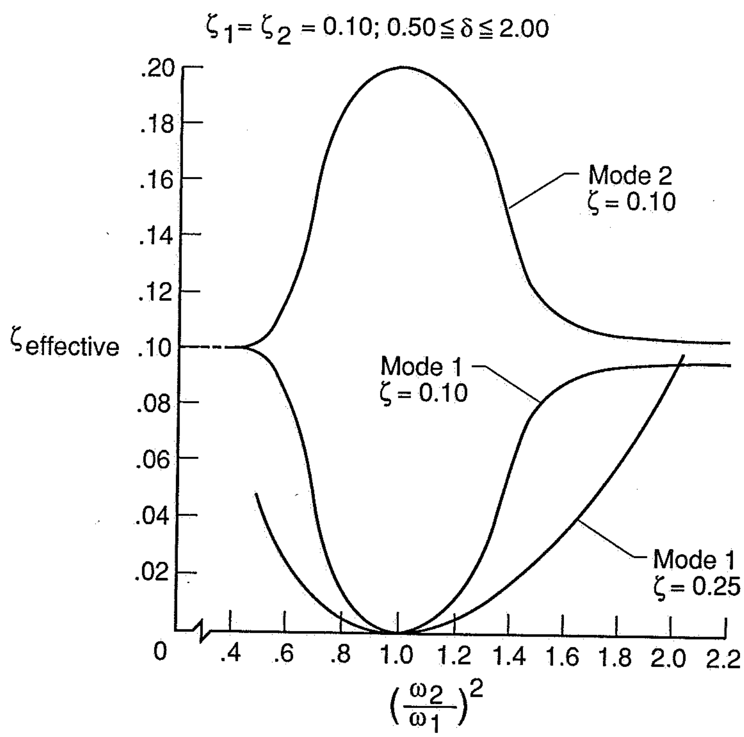

Figure 6.- Effective viscous damping ratio as a function of neighboring mode frequency separation ($\zeta_1 = \zeta_2 = .10$).

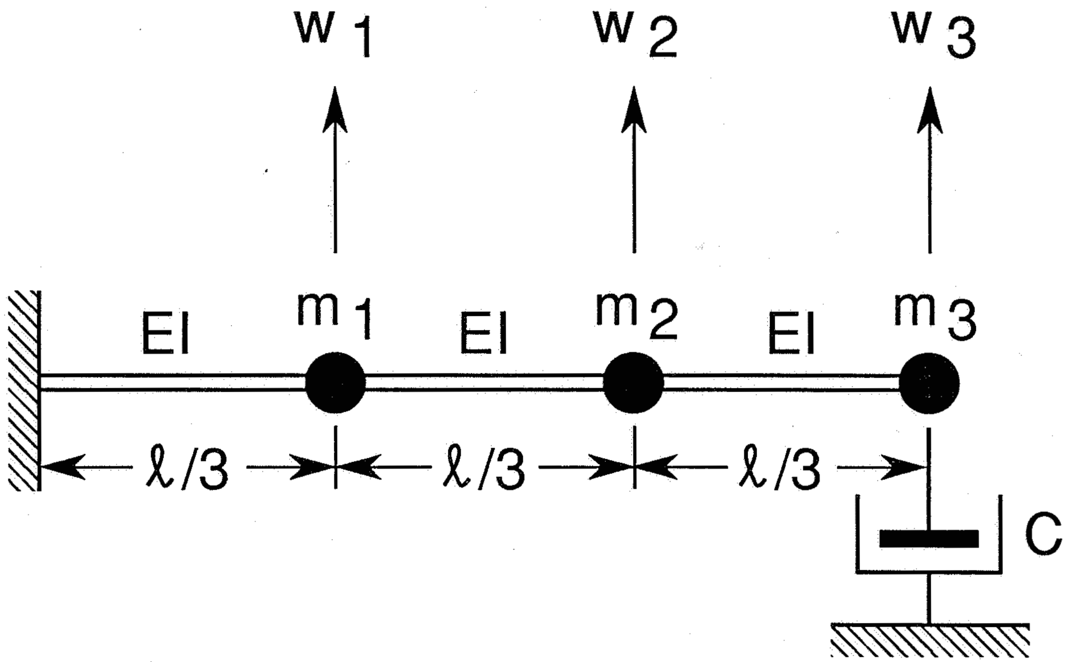

Figure 9.- Discrete model of a cantilever beam with a viscous damper at the free end.

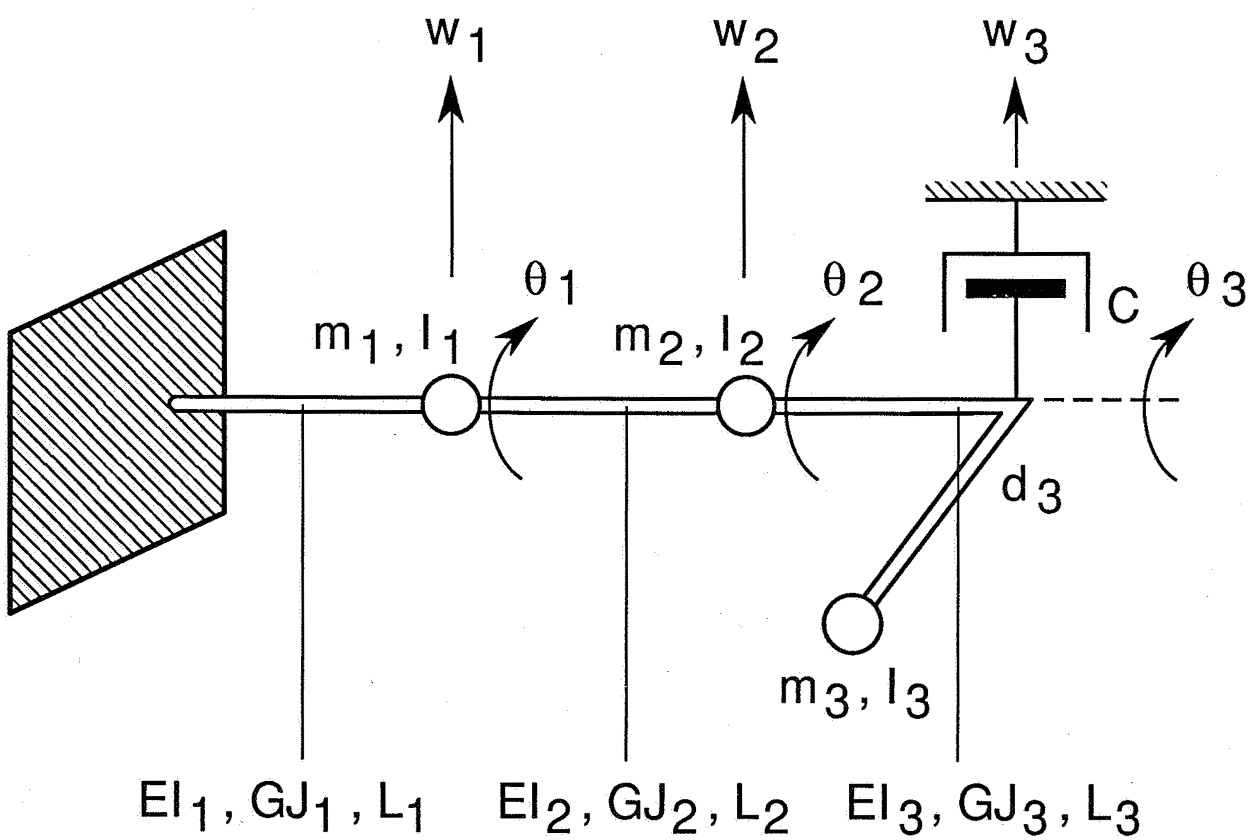

Figure 10.- A discrete model of coupled bending and torsion with a viscous damper for bending displacement.

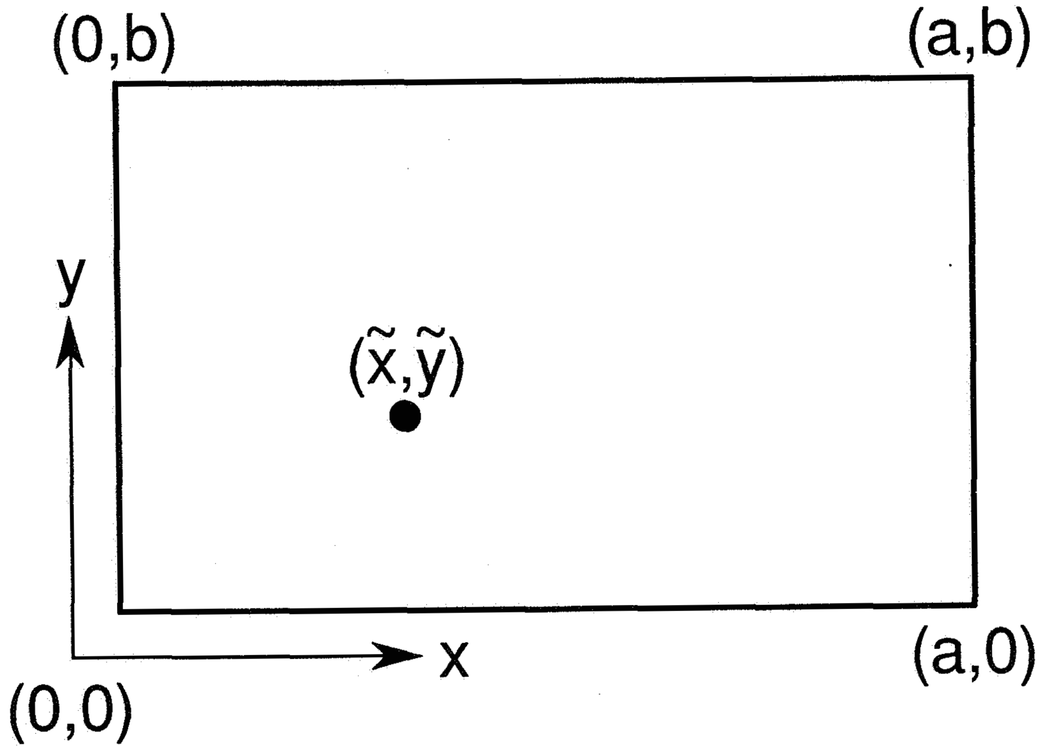

Figure 11.- Rectangular plate with spot damping.

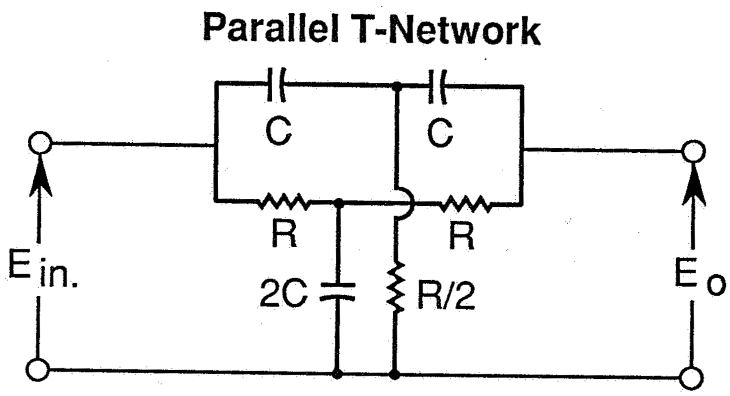

Figure 14.- Parallel T-network notch filter.

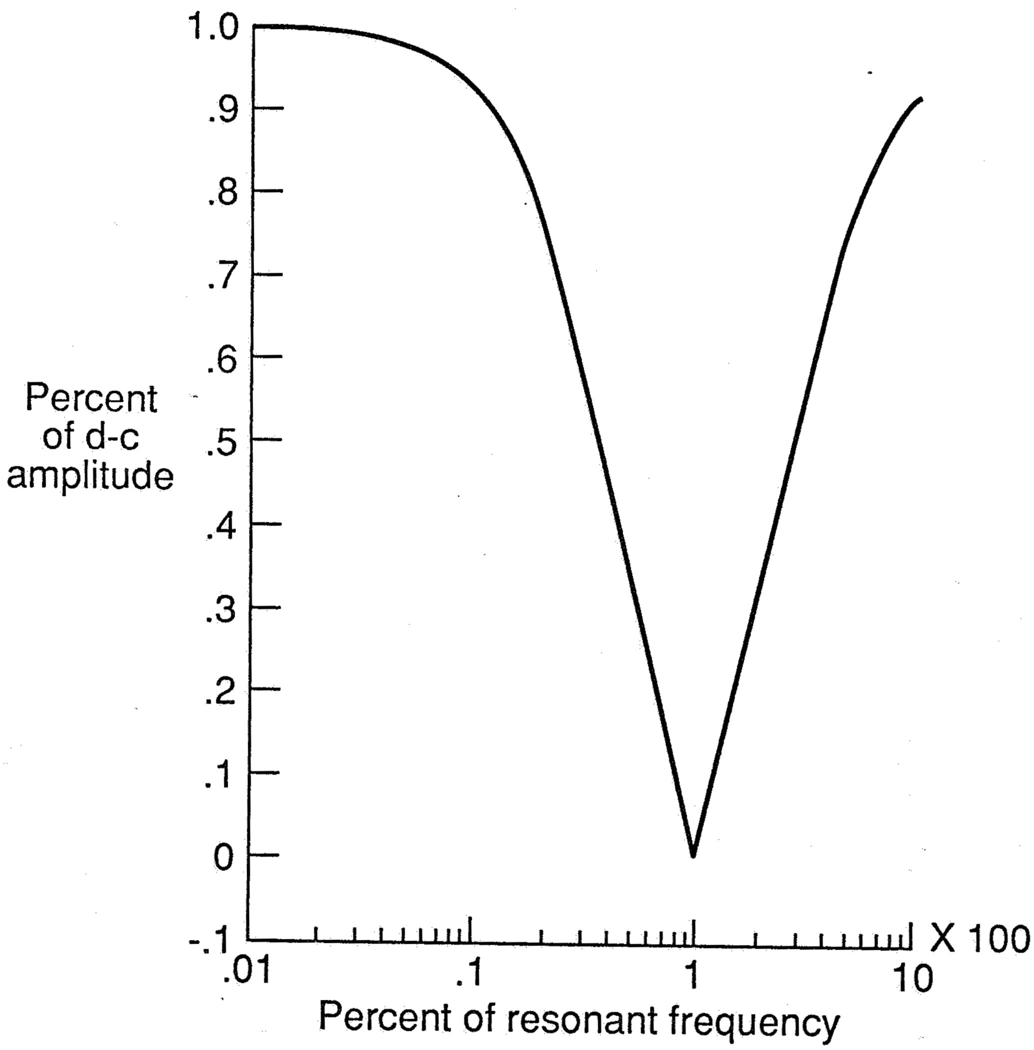

Figure 15.- Frequency response characteristic of notch filter.

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70

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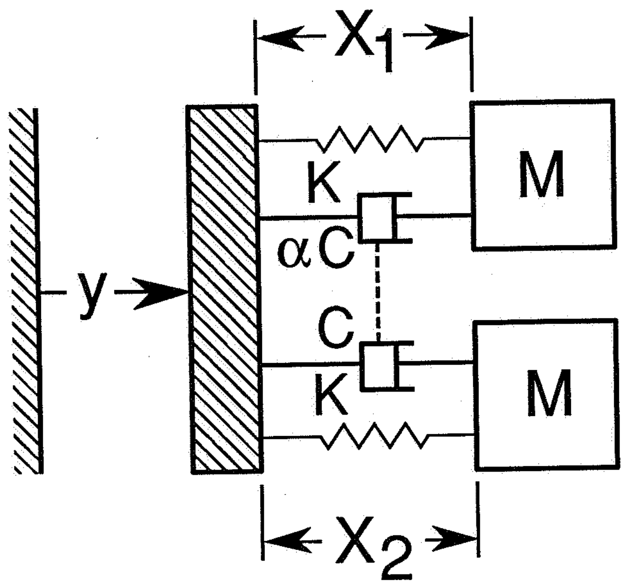


Figure 16.- Mechanical analog of notch filter - smart dynamic absorber.

$$\left| \frac{\bar{X}_1(j\beta)}{\bar{X}_2(j\beta)} \right| = \sqrt{\frac{(1-\beta^2)^2 + \alpha^2(4\zeta^2\beta^2)}{(1-\beta^2)^2 + (4\zeta^2\beta^2)}} \;;$$

$$\zeta \equiv \frac{C}{2\sqrt{MK}} \;;\; \omega = \sqrt{K/M}$$

Figure 17.- Frequency response of mechanical notch filter as a function of frequency ratio, viscous damping ratio, and feedback gain.

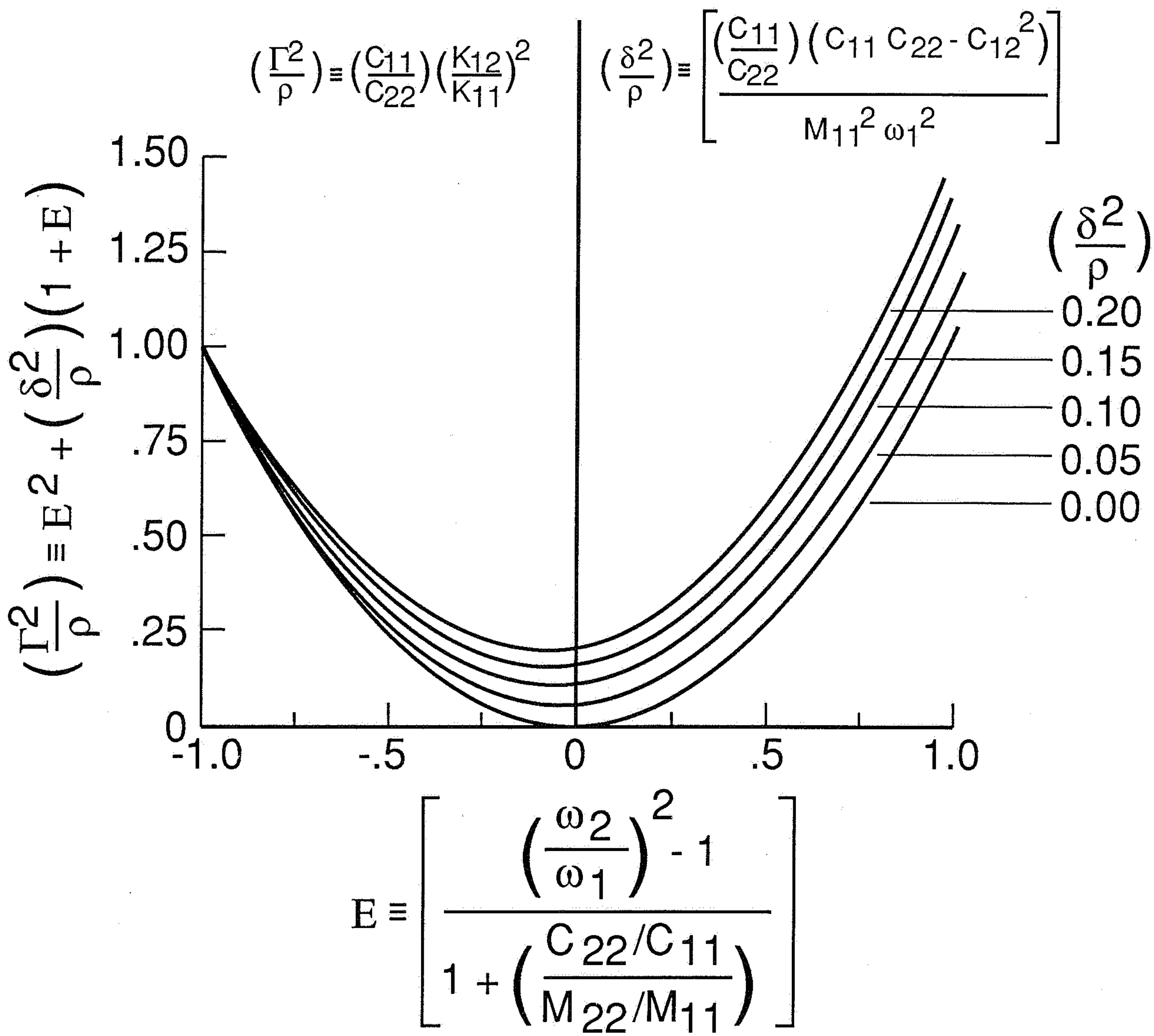
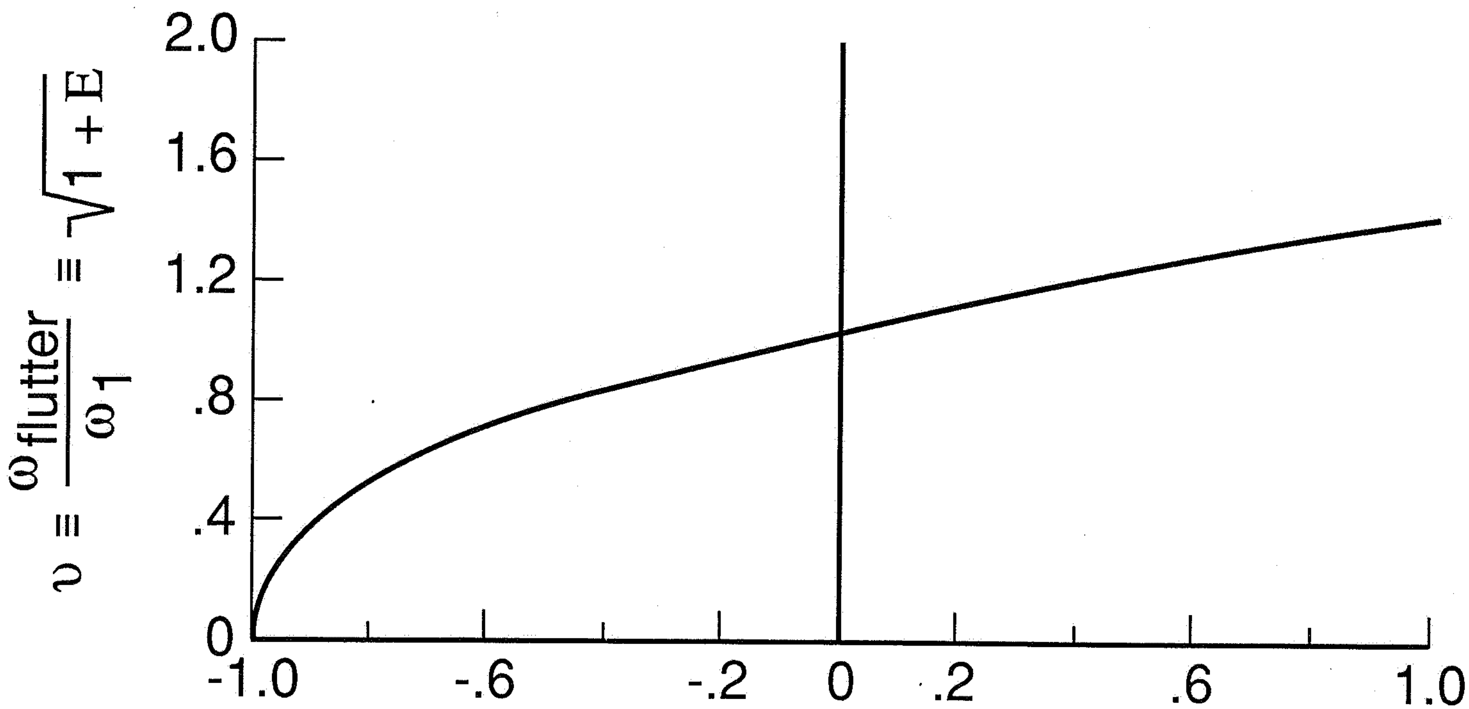


Figure 19.- Dynamic stability boundaries for binary systems.



$$E \equiv \left[\frac{\left(\frac{\omega_2}{\omega_1}\right)^2 - 1}{1 + \rho} \right] \equiv \left[\frac{\left(\frac{\omega_2}{\omega_1}\right)^2 - 1}{1 + \left(\frac{C_{22}/C_{11}}{M_{22}/M_{11}}\right)} \right]$$

Figure 20.- Binary flutter frequency as a function of modal frequency ratio.

APPENDIX A
A CLOSED FORM SOLUTION TO THE DAMPING INTERACTION QUARTIC EQUATION

The damping interaction quartic equation can be written in the form

$$f(\eta) \equiv \eta^4 + a\eta^3 + b\eta^2 + c\eta + d = 0,$$

where

$$\eta \equiv \left(\frac{\lambda}{\omega_\alpha}\right), \qquad a \equiv 2(\zeta_{\alpha\alpha} + \zeta_{\beta\beta}), \qquad b \equiv \left[1 + \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 + 4(\zeta_{\alpha\alpha}\zeta_{\beta\beta} - \zeta_{\alpha\beta}\zeta_{\beta\alpha})\right],$$

$$c \equiv 2\left[\zeta_{\alpha\alpha}\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 + \zeta_{\beta\beta},\right] \qquad d \equiv \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2$$

and

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix} e^{\lambda t}.$$

The quartic polynomial may be factored into the product of two quadratic polynomials $f(\eta) = f_1(\eta) f_2(\eta)$, where

$$f_1(\eta) \equiv \eta^2 + \left[\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 + 2y - b}\right]\eta + \left(y + \sqrt{y^2 - d}\right) = 0$$

and

$$f_2(\eta) \equiv \eta^2 + \left[\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + 2y - b}\right]\eta + \left(y - \sqrt{y^2 - d}\right) = 0.$$

The parameter y is known as the resolvent and is a real root of the cubic equation

$$F(y) \equiv 8y^3 - 4by^2 + 2(ac - 4d)y - \left[c^2 + d(a^2 - 4b)\right] = 0.$$

This approach is due to Ferrari. The cubic itself also has a closed form solution due to Tartaglia and Cardan, thereby completing the closed form solution to the quartic equation. Rewrite the cubic equation for the resolvent in the form

$$F(y) \equiv y^3 + By^2 + Cy + D = 0,$$

where

$$B \equiv -\frac{1}{2}b, \qquad C \equiv \frac{1}{4}(ac - 4d), \qquad D \equiv -\frac{1}{8}\left[c^2 + d(a^2 - 4b)\right].$$

The cubic equation can be reduced to one with the second degree term absent by the change of variable $x = y - \frac{B}{3}$. This results in the reduced cubic

$$F(x) \equiv x^3 + px + q = 0,$$

where

$$p \equiv C - \frac{B^2}{3} \quad \text{and} \quad q \equiv D - \frac{BC}{3} + \frac{2B^3}{27}.$$

Changing variables once again, let $x = z - \frac{p}{3z}$. This yields a quadratic equation in the variable z^3:

$$(z^3)^2 + q(z^3) - \frac{p^3}{27} = 0.$$

Solution of this quadratic equation yields

$$z^3 = -\frac{q}{2} \pm \sqrt{R},$$

where

$$R \equiv \frac{p^3}{27} + \frac{q^2}{4}.$$

The three cube roots of z^3 yield x_1, x_2, and x_3:

$$x_1 = z_1 - \frac{p}{3z_1}, \qquad x_2 = z_2 - \frac{p}{3z_2}, \qquad x_3 = z_3 - \frac{p}{3z_3}.$$

In turn the three roots of the resolvent cubic follow:

$$y_1 = z_1 - \frac{p}{3z_1} - \frac{b}{3}, \qquad y_2 = z_2 - \frac{p}{3z_2} - \frac{b}{3}, \qquad y_3 = z_3 - \frac{p}{3z_3} - \frac{b}{3}.$$

In cases of practical engineering interest y_1 will be the real root of interest in determining the effective damping in each of the interacting modes whose natural frequencies are close to one another. Then from $f_1(\eta)$ and $f_2(\eta)$ above and neglecting the very small term

$4(\zeta_{\alpha\alpha}\zeta_{\beta\beta} - \zeta_{\alpha\beta}\zeta_{\beta\alpha})$ compared to unity*

$$\zeta_{\text{Effective}} \equiv \frac{1}{2}\left\{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta}) \pm \sqrt{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2 + 2y_1 - \left[1 + \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2\right]}\right\}.$$

In the case of a frequency match when

$$\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 = 1, \qquad y_1 = 1,$$

and

$$\zeta_{\text{Effective}} = 0 \quad \text{or} \quad (\zeta_{\alpha\alpha} + \zeta_{\beta\beta}).$$

In the case of widely separated natural frequencies where $\left(\frac{\omega_\beta}{\omega_\alpha}\right) \ll 1$ and the system is either spot damped or lightly damped

$$\zeta_{\text{Effective}} \cong \frac{1}{2}\left[(\zeta_{\alpha\alpha} + \zeta_{\beta\beta}) \pm \sqrt{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2 + (2y_1 - 1)}\right].$$

Since neighboring modes will then be effectively uncoupled by damping

$$\zeta_{\text{Effective}} \cong \zeta_{\alpha\alpha} \quad \text{or} \quad \zeta_{\beta\beta}.$$

This implies that

$$(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2 + (2y - 1) \cong (\zeta_{\alpha\alpha} - \zeta_{\beta\beta})^2.$$

Solving for y_1 when

$$\left(\frac{\omega_\beta}{\omega_\alpha}\right) \ll 1, \qquad y \cong \frac{1}{2}(1 - 4\zeta_{\alpha\alpha}\zeta_{\beta\beta}).$$

Thus it is seen that the resolvent varies over the range

$$\frac{1}{2}(1 - 4\zeta_{\alpha\alpha}\zeta_{\beta\beta}) \leqq y_1 \leqq 1,$$

as $\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2$ varies over the range

$$0 < \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 \leqq 1.$$

*This is exactly zero in the case of spot damping.

APPENDIX B
AN APPROXIMATIVE SOLUTION TO THE DAMPING INTERACTION QUARTIC EQUATION

It has been shown in Appendix A that the effective damping fraction is given by

$$\zeta_{\text{Effective}} = \frac{1}{2}\left\{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta}) \pm \sqrt{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2 + 2y_1 - \left[1 + \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2\right]}\right\},$$

where y_1 is the real root of the resolvent cubic equation. It has also been shown that as $\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2$ varies from zero to unity, the extremes of widely separated and matching natural frequencies of neighboring modes, the resolvent varies in the range

$$\frac{1}{2}(1 - 4\zeta_{\alpha\alpha}\zeta_{\beta\beta}) < y_1 \leq 1.$$

Now consider the mean damping fraction $\bar{\zeta}$ given by

$$\bar{\zeta} \equiv \frac{1}{2}(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})$$

and the ratio r given by

$$r \equiv \left(\frac{\zeta_{\text{Effective}}}{\bar{\zeta}}\right).$$

The foregoing equation for the effective damping fraction can be rewritten as follows:

$$r = \left\{1 \pm \sqrt{1 - \left[\frac{1 + \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 - 2y_1}{4\bar{\zeta}^2}\right]}\right\}.$$

It is seen that as $\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 \to 1$, $y_1 \to 1$, and $r \to 0$ or 2. It is also seen that as $\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 \to 0$, $y_1 \to \frac{1}{2}(1 - 4\zeta_{\alpha\alpha}\zeta_{\beta\beta})$ and $r \to \left(\frac{\zeta_{\alpha\alpha}}{\bar{\zeta}}\right)$ or $\left(\frac{\zeta_{\beta\beta}}{\bar{\zeta}}\right)$. $\zeta_{\text{Effective}}$ approaches $\zeta_{\alpha\alpha}$ or $\zeta_{\beta\beta}$ of practical engineering interest in the range of modal frequency ratios near a frequency match. In this case

$$\left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 = 1 - \delta^2,$$

where

$$0 < \delta^2 \ll 1.$$

$$r = \left\{ 1 \pm \sqrt{1 - 2\left[\frac{(1-y_1) - \left(\frac{\delta^2}{2}\right)}{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2}\right]} \right\}.$$

For $\delta^2 \ll 1$, the binomial expansion gives the approximation for r:

$$r \cong 1 \pm \left\{ 1 - \left[\frac{(1-y_1) - \left(\frac{\delta^2}{2}\right)}{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2}\right] \right\} = \left[\frac{(1-y_1) - \left(\frac{\delta^2}{2}\right)}{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2}\right], \quad \left\{ 2 - \left[\frac{(1-y_1) - \left(\frac{\omega^2}{2}\right)}{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2}\right] \right\}.$$

Approximating the resolvent y_1 as

$$y_1 \cong 1 - \left(\frac{\delta^2}{2}\right) - \gamma^2\delta^4$$

yields the approximation for r as

$$r \cong \left(\frac{\gamma\delta^2}{\zeta_{\alpha\alpha} + \zeta_{\beta\beta}}\right)^2 \quad \text{and} \quad 2 - \left(\frac{\gamma\delta^2}{\zeta_{\alpha\alpha} + \zeta_{\beta\beta}}\right)^2.$$

γ may be approximated by reference to several numerical solutions to the interaction quartic. Referring to the data in Figures 4, 5, and 6,

$$\gamma^2 \cong \left(\frac{1}{6}\right).$$

A geometric interpretation of the damping interaction between neighboring modes is also possible as follows: rewrite the Ferrari closed form solution for the effective damping fraction as

$$(r-1)^2 + \left[\frac{1 + \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 - 2y_1}{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})^2}\right] = 1;$$

introducing the new variable

$$q \equiv \pm \left[\frac{\sqrt{1 + \left(\frac{\omega_\beta}{\omega_\alpha}\right)^2 - 2y_1}}{(\zeta_{\alpha\alpha} + \zeta_{\beta\beta})}\right]$$

yields a family of ellipses whose semi-major axis depends on the damping fraction levels of the neighboring modes; the general equation is given by

$$(r-1)^2 + q^2 = 1;$$

this is seen to be a circle of unit radius with center at abscissa $q = 0$ and ordinate $r = 1$. Approximating the resolvent y_1 or solving the resolvent cubic precisely yields the value of q and, in turn, leads to values of r.

APPENDIX C
DIGITAL COMPUTER SOLUTION TABULATION FOR DAMPED BENDING VIBRATION

C = 0

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
0.00000000E+00	0.15436864E+02	0.00000000E+00
0.00000000E+00	-0.15436864E+02	0.00000000E+00
0.00000000E+00	0.57454960E+01	0.00000000E+00
0.00000000E+00	-0.57454960E+01	0.00000000E+00
0.00000000E+00	-0.87721309E+00	0.00000000E+00
0.00000000E+00	0.87721309E+00	0.00000000E+00

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.1064962E-27	0.1000000E+01	-0.4681889E-27
2	-0.6989843E+00	0.2269553E-26	-0.6989843E+00	-0.5413558E-27
3	0.2151913E+00	-0.1010531E-26	0.2151913E+00	0.5053640E-28
4	0.4986863E-25	0.1543686E+02	0.9043501E-26	-0.1543686E+02
5	-0.2343864E-25	-0.1079013E+02	-0.1047923E-25	0.1079013E+02
6	0.1076006E-25	0.3321879E+01	0.4132842E-26	-0.3321879E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.8417192E+00	0.1289842E-21	0.8417192E+00	-0.1291827E-21
2	0.1000000E+01	-0.4634716E-25	0.1000000E+01	0.1565100E-25
3	-0.6632931E+00	-0.7749615E-21	-0.6632931E+00	0.7725248E-21
4	-0.3344544E-21	0.4836094E+01	-0.3350736E-21	-0.4836094E+01
5	-0.5011495E-21	0.5745496E+01	-0.5044752E-21	-0.5745496E+01
6	0.2752724E-20	-0.3810948E+01	0.2746154E-20	0.3810948E+01

SYSTEM EIGENVECTORS

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1564141E+00	-0.1596148E-25	0.1564141E+00	0.3077212E-25
2	0.5316363E+00	-0.1732302E-25	0.5316363E+00	0.3347339E-25
3	0.1000000E+01	-0.3175165E-28	0.1000000E+01	-0.3549874E-29
4	0.4021223E-25	-0.1372085E+00	0.7748807E-25	0.1372085E+00
5	0.2138414E-24	-0.4663584E+00	0.4121986E-24	0.4663584E+00
6	0.4810201E-24	-0.8772131E+00	0.9268212E-24	0.8772131E+00

C = .1

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.15084290E-02	-0.15436854E+02	0.97716086E-02
-0.15084290E-02	0.15436854E+02	0.97716086E-02
-0.10237098E-01	-0.57453482E+01	0.17818036E+00
-0.10237099E-01	0.57453482E+01	0.17818036E+00
-0.38254473E-01	0.87640033E+00	0.43608019E+01
-0.38254473E-01	-0.87640033E+00	0.43608019E+01

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	-0.2283601E-17	0.1000000E+01	-0.2541099E-17
2	-0.6989836E+00	0.1077038E-03	-0.6989836E+00	-0.1077038E-03
3	0.2151816E+00	-0.1455723E-02	0.2151816E+00	0.1455723E-02
4	-0.1508429E-02	-0.1543685E+02	-0.1508429E-02	0.1543685E+02
5	0.2716975E-02	0.1079011E+02	0.2716975E-02	-0.1079011E+02
6	-0.2279638E-01	-0.3321725E+01	-0.2279638E-01	0.3321725E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.8416667E+00	-0.3044757E-02	0.8416667E+00	0.3044757E-02
2	0.1000000E+01	0.1902775E-16	0.1000000E+01	-0.1615461E-16
3	-0.6630879E+00	0.1229338E-01	-0.6630879E+00	-0.1229338E-01
4	-0.2610941E-01	-0.4835637E+01	-0.2610941E-01	0.4835637E+01
5	-0.1023710E-01	-0.5745348E+01	-0.1023710E-01	0.5745348E+01
6	0.7741785E-01	0.3809545E+01	0.7741785E-01	-0.3809545E+01

SYSTEM EIGENVECTORS

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1563800E+00	0.7433139E-03	0.1563800E+00	-0.7433139E-03
2	0.5315826E+00	0.1177009E-02	0.5315826E+00	-0.1177009E-02
3	0.1000000E+01	0.2943609E-16	0.1000000E+01	-0.8267042E-18
4	-0.6633677E-02	0.1370231E+00	-0.6633677E-02	-0.1370231E+00
5	-0.2136694E-01	0.4658341E+00	-0.2136694E-01	-0.4658341E+00
6	-0.3825447E-01	0.8764003E+00	-0.3825447E-01	-0.8764003E+00

C = .2

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.30164747E-02	0.15436825E+02	0.19540772E-01
-0.30164747E-02	-0.15436825E+02	0.19540772E-01
-0.20463016E-01	-0.57449049E+01	0.35619191E+00
-0.20463016E-01	0.57449049E+01	0.35619191E+00
-0.76520510E-01	-0.87395648E+00	0.87222739E+01
-0.76520510E-01	0.87395648E+00	0.87222739E+01

SYSTEM EIGENVECTORS

ROW COL	1 REAL	1 IMAGINARY	2 REAL	2 IMAGINARY
1	0.1000000E+01	0.3388132E-18	0.1000000E+01	-0.1256319E-16
2	-0.6989815E+00	-0.2153812E-03	-0.6989815E+00	0.2153812E-03
3	0.2151526E+00	0.2911066E-02	0.2151526E+00	-0.2911066E-02
4	-0.3016475E-02	0.1543682E+02	-0.3016475E-02	-0.1543682E+02
5	0.5433262E-02	-0.1079005E+02	0.5433262E-02	0.1079005E+02
6	-0.4558662E-01	0.3321263E+01	-0.4558662E-01	-0.3321263E+01

SYSTEM EIGENVECTORS

ROW COL	3 REAL	3 IMAGINARY	4 REAL	4 IMAGINARY
1	0.8415093E+00	-0.6084581E-02	0.8415093E+00	0.6084581E-02
2	0.1000000E+01	0.3000530E-16	0.1000000E+01	0.1818749E-16
3	-0.6624727E+00	0.2456817E-01	-0.6624727E+00	-0.2456817E-01
4	-0.5217516E-01	-0.4834266E+01	-0.5217516E-01	0.4834266E+01
5	-0.2046302E-01	-0.5744905E+01	-0.2046302E-01	0.5744905E+01
6	0.1546980E+00	0.3805340E+01	0.1546980E+00	-0.3805340E+01

SYSTEM EIGENVECTORS

ROW COL	5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.1562779E+00	-0.1481613E-02	0.1562779E+00	0.1481613E-02
2	0.5314214E+00	-0.2346193E-02	0.5314214E+00	0.2346193E-02
3	0.1000000E+01	0.1160096E-16	0.1000000E+01	0.2602085E-17
4	-0.1325333E-01	-0.1364667E+00	-0.1325333E-01	0.1364667E+00
5	-0.4271511E-01	-0.4642597E+00	-0.4271511E-01	0.4642597E+00
6	-0.7652051E-01	-0.8739565E+00	-0.7652051E-01	0.8739565E+00

C = .3

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.45237544E-02	0.15436775E+02	0.29305046E-01
-0.45237544E-02	-0.15436775E+02	0.29305046E-01
-0.30666592E-01	-0.57441670E+01	0.53386601E+00
-0.30666592E-01	0.57441670E+01	0.53386600E+00
-0.11480966E+00	-0.86986468E+00	0.13085085E+02
-0.11480966E+00	0.86986468E+00	0.13085085E+02

SYSTEM EIGENVECTORS

ROW COL	1 REAL	1 IMAGINARY	2 REAL	2 IMAGINARY
1	0.1000000E+01	0.4483176E-16	0.1000000E+01	-0.2626480E-16
2	-0.6989780E+00	-0.3230059E-03	-0.6989780E+00	0.3230059E-03
3	0.2151041E+00	0.4365647E-02	0.2151041E+00	-0.4365647E-02
4	-0.4523754E-02	0.1543678E+02	-0.4523754E-02	-0.1543678E+02
5	0.8148175E-02	-0.1078997E+02	0.8148175E-02	0.1078997E+02
6	-0.6836459E-01	0.3320494E+01	-0.6836459E-01	-0.3320494E+01

SYSTEM EIGENVECTORS

ROW COL	3 REAL	3 IMAGINARY	4 REAL	4 IMAGINARY
1	0.8412475E+00	-0.9114561E-02	0.8412475E+00	0.9114561E-02
2	0.1000000E+01	-0.3377290E-16	0.1000000E+01	-0.2057274E-16
3	-0.6614491E+00	0.3680585E-01	-0.6614491E+00	-0.3680585E-01
4	-0.7815375E-01	-0.4831986E+01	-0.7815375E-01	0.4831986E+01
5	-0.3066659E-01	-0.5744167E+01	-0.3066659E-01	0.5744167E+01
6	0.2317033E+00	0.3798345E+01	0.2317033E+00	-0.3798345E+01

SYSTEM EIGENVECTORS

ROW COL	5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.1561077E+00	-0.2209853E-02	0.1561077E+00	0.2209853E-02
2	0.5311531E+00	-0.3499680E-02	0.5311531E+00	0.3499680E-02
3	0.1000000E+01	0.5149960E-17	0.1000000E+01	0.3815036E-17
4	-0.1984495E-01	-0.1355389E+00	-0.1984495E-01	0.1355389E+00
5	-0.6402575E-01	-0.4616295E+00	-0.6402575E-01	0.4616295E+00
6	-0.1148097E+00	-0.8698647E+00	-0.1148097E+00	0.8698647E+00

C = .4

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.60298852E-02	0.15436706E+02	0.39061991E-01
-0.60298852E-02	-0.15436706E+02	0.39061991E-01
-0.40836695E-01	-0.57431354E+01	0.71103430E+00
-0.40836695E-01	0.57431354E+01	0.71103430E+00
-0.15313342E+00	-0.86409629E+00	0.17449901E+02
-0.15313342E+00	0.86409629E+00	0.17449901E+02

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.4678332E-16	0.1000000E+01	0.4803016E-16
2	-0.6989732E+00	-0.4305517E-03	-0.6989732E+00	0.4305517E-03
3	0.2150363E+00	0.5819086E-02	0.2150363E+00	-0.5819086E-02
4	-0.6029885E-02	0.1543671E+02	-0.6029885E-02	-0.1543671E+02
5	0.1086103E-01	-0.1078984E+02	0.1086103E-01	0.1078984E+02
6	-0.9112416E-01	0.3319418E+01	-0.9112416E-01	-0.3319418E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.8408819E+00	-0.1212982E-01	0.8408819E+00	0.1212982E-01
2	0.1000000E+01	-0.2569559E-16	0.1000000E+01	-0.3174002E-16
3	-0.6600196E+00	0.4898799E-01	-0.6600196E+00	-0.4898799E-01
4	-0.1040020E+00	-0.4828803E+01	-0.1040020E+00	0.4828803E+01
5	-0.4083670E-01	-0.5743135E+01	-0.4083670E-01	0.5743135E+01
6	0.3082977E+00	0.3788581E+01	0.3082977E+00	-0.3788581E+01

SYSTEM EIGENVECTORS

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1558699E+00	-0.2922929E-02	0.1558699E+00	0.2922929E-02
2	0.5307779E+00	-0.4629492E-02	0.5307779E+00	0.4629492E-02
3	0.1000000E+01	0.4878910E-18	0.1000000E+01	0.2981556E-18
4	-0.2639459E-01	-0.1342390E+00	-0.2639459E-01	0.1342390E+00
5	-0.8528017E-01	-0.4579343E+00	-0.8528017E-01	0.4579343E+00
6	-0.1531334E+00	-0.8640963E+00	-0.1531334E+00	0.8640963E+00

C = .5

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.75344860E-02	0.15436617E+02	0.48809172E-01
-0.75344860E-02	-0.15436617E+02	0.48809172E-01
-0.50962256E-01	-0.57418120E+01	0.88752904E+00
-0.50962256E-01	0.57418120E+01	0.88752903E+00
-0.19150326E+00	-0.85661011E+00	0.21817386E+02
-0.19150326E+00	0.85661011E+00	0.21817386E+02

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	-0.2626480E-16	0.1000000E+01	0.5708324E-16
2	-0.6989670E+00	-0.5379923E-03	-0.6989670E+00	0.5379923E-03
3	0.2149493E+00	0.7271004E-02	0.2149493E+00	-0.7271004E-02
4	-0.7534486E-02	0.1543662E+02	-0.7534486E-02	-0.1543662E+02
5	0.1357114E-01	-0.1078968E+02	0.1357114E-01	0.1078968E+02
6	-0.1138592E+00	0.3318035E+01	-0.1138592E+00	-0.3318035E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.8404135E+00	-0.1512553E-01	0.8404135E+00	0.1512553E-01
2	0.1000000E+01	0.5597194E-17	0.1000000E+01	-0.2141299E-17
3	-0.6581876E+00	0.6109639E-01	-0.6581876E+00	-0.6109639E-01
4	-0.1296773E+00	-0.4824725E+01	-0.1296773E+00	0.4824725E+01
5	-0.5096226E-01	-0.5741812E+01	-0.5096226E-01	0.5741812E+01
6	0.3843467E+00	0.3776076E+01	0.3843467E+00	-0.3776076E+01

SYSTEM EIGENVECTORS

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1555649E+00	-0.3615637E-02	0.1555649E+00	0.3615637E-02
2	0.5302966E+00	-0.5727494E-02	0.5302966E+00	0.5727494E-02
3	0.1000000E+01	-0.2954451E-17	0.1000000E+01	-0.4526544E-17
4	-0.3288838E-01	-0.1325661E+00	-0.3288838E-01	0.1325661E+00
5	-0.1064598E+00	-0.4531606E+00	-0.1064598E+00	0.4531606E+00
6	-0.1915033E+00	-0.8566101E+00	-0.1915033E+00	0.8566101E+00

C = 1.0

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.15021291E-01	0.15435879E+02	0.97314081E-01
-0.15021291E-01	-0.15435879E+02	0.97314081E-01
-0.10054050E+00	-0.57309066E+01	0.17540860E+01
-0.10054050E+00	0.57309066E+01	0.17540860E+01
-0.38443821E+00	-0.79088116E+00	0.43717624E+02
-0.38443821E+00	0.79088116E+00	0.43717624E+02

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.2198220E-16	0.1000000E+01	-0.1045577E-16
2	-0.6989152E+00	-0.1072706E-02	-0.6989152E+00	0.1072706E-02
3	0.2142262E+00	0.1449462E-01	0.2142262E+00	-0.1449462E-01
4	-0.1502129E-01	0.1543588E+02	-0.1502129E-01	-0.1543588E+02
5	0.2705677E-01	-0.1078835E+02	0.2705677E-01	0.1078835E+02
6	-0.2269552E+00	0.3306551E+01	-0.2269552E+00	-0.3306551E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.8365826E+00	-0.2964796E-01	0.8365826E+00	0.2964796E-01
2	0.1000000E+01	-0.2481468E-16	0.1000000E+01	-0.7535205E-17
3	-0.6431830E+00	0.1199147E+00	-0.6431830E+00	-0.1199147E+00
4	-0.2540201E+00	-0.4791396E+01	-0.2540201E+00	0.4791396E+01
5	-0.1005405E+00	-0.5730907E+01	-0.1005405E+00	0.5730907E+01
6	0.7518856E+00	0.3673966E+01	0.7518856E+00	-0.3673966E+01

SYSTEM EIGENVECTORS

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1530536E+00	-0.6579089E-02	0.1530536E+00	0.6579089E-02
2	0.5263284E+00	-0.1043489E-01	0.5263284E+00	0.1043489E-01
3	0.1000000E+01	-0.2591243E-16	0.1000000E+01	0.7909255E-16
4	-0.6404294E-01	-0.1185180E+00	-0.6404294E-01	0.1185180E+00
5	-0.2105935E+00	-0.4122516E+00	-0.2105935E+00	0.4122516E+00
6	-0.3844382E+00	-0.7908812E+00	-0.3844382E+00	0.7908812E+00

C = 1.5

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.22413669E-01	0.15434658E+02	0.14521634E+00
-0.22413669E-01	-0.15434658E+02	0.14521634E+00
-0.14740869E+00	-0.57132182E+01	0.25792756E+01
-0.14740868E+00	0.57132182E+01	0.25792756E+01
-0.58017765E+00	0.66431856E+00	0.65779726E+02
-0.58017765E+00	-0.66431856E+00	0.65779726E+02

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.3778445E-16	0.1000000E+01	0.2574980E-18
2	-0.6988296E+00	-0.1600924E-02	-0.6988296E+00	0.1600924E-02
3	0.2130311E+00	0.2162442E-01	0.2130311E+00	-0.2162442E-01
4	-0.2241367E-01	0.1543466E+02	-0.2241367E-01	-0.1543466E+02
5	0.4037305E-01	-0.1078616E+02	0.4037305E-01	0.1078616E+02
6	-0.3385404E+00	0.3287577E+01	-0.3385404E+00	-0.3287577E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.8304759E+00	-0.4301845E-01	0.8304759E+00	0.4301845E-01
2	0.1000000E+01	0.1897354E-16	0.1000000E+01	-0.8605855E-18
3	-0.6191851E+00	0.1743629E+00	-0.6191851E+00	-0.1743629E+00
4	-0.3681931E+00	-0.4738349E+01	-0.3681931E+00	0.4738349E+01
5	-0.1474087E+00	-0.5713218E+01	-0.1474087E+00	0.5713218E+01
6	0.1087446E+01	0.3511837E+01	0.1087446E+01	-0.3511837E+01

SYSTEM EIGENVECTORS

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1489885E+00	0.8088565E-02	0.1489885E+00	-0.8088565E-02
2	0.5198839E+00	0.1285618E-01	0.5198839E+00	-0.1285618E-01
3	0.1000000E+01	-0.1395910E-17	0.1000000E+01	0.1680513E-16
4	-0.9181317E-01	0.9428300E-01	-0.9181317E-01	-0.9428300E-01
5	-0.3101656E+00	0.3379096E+00	-0.3101656E+00	-0.3379096E+00
6	-0.5801776E+00	0.6643186E+00	-0.5801776E+00	-0.6643186E+00

C = 2

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.29666716E-01	-0.15432970E+02	0.19222911E+00
-0.29666716E-01	0.15432970E+02	0.19222911E+00
-0.19036224E+00	-0.56894674E+01	0.33439998E+01
-0.19036224E+00	0.56894674E+01	0.33439998E+01
-0.77997104E+00	0.41939680E+00	0.88074793E+02
-0.77997104E+00	-0.41939680E+00	0.88074793E+02

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.2726768E-16	0.1000000E+01	0.5421011E-19
2	-0.6987111E+00	0.2119552E-02	-0.6987111E+00	-0.2119552E-02
3	0.2113789E+00	-0.2861587E-01	0.2113789E+00	0.2861587E-01
4	-0.2966672E-01	-0.1543297E+02	-0.2966672E-01	0.1543297E+02
5	0.5343944E-01	0.1078312E+02	0.5343944E-01	-0.1078312E+02
6	-0.4478987E+00	-0.3261355E+01	-0.4478987E+00	0.3261355E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.8224784E+00	-0.5479012E-01	0.8224784E+00	0.5479012E-01
2	0.1000000E+01	0.4387631E-18	0.1000000E+01	0.1104531E-16
3	-0.5876064E+00	0.2227034E+00	-0.5876064E+00	-0.2227034E+00
4	-0.4682954E+00	-0.4669034E+01	-0.4682954E+00	0.4669034E+01
5	-0.1903622E+00	-0.5689467E+01	-0.1903622E+00	0.5689467E+01
6	0.1378922E+01	0.3300773E+01	0.1378922E+01	-0.3300773E+01

SYSTEM EIGENVECTORS

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1435474E+00	0.6578427E-02	0.1435474E+00	-0.6578427E-02
2	0.5112140E+00	0.1048759E-01	0.5112140E+00	-0.1048759E-01
3	0.1000000E+01	-0.3930233E-17	0.1000000E+01	0.2347298E-16
4	-0.1147218E+00	0.5507235E-01	-0.1147218E+00	-0.5507235E-01
5	-0.4031306E+00	0.2062215E+00	-0.4031306E+00	-0.2062215E+00
6	-0.7799710E+00	0.4193968E+00	-0.7799710E+00	-0.4193968E+00

APPENDIX D
DIGITAL COMPUTER SOLUTION TABULATION FOR DAMPED, COUPLED BENDING-TORSION VIBRATION

C = 0.00

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
0.00000000E+00	-0.15666482E+02	0.00000000E+00
0.00000000E+00	0.15666482E+02	0.00000000E+00
0.00000000E+00	0.63089979E+01	0.00000000E+00
0.00000000E+00	-0.63089979E+01	0.00000000E+00
0.00000000E+00	0.18001188E+01	0.00000000E+00
0.00000000E+00	-0.18001188E+01	0.00000000E+00
0.00000000E+00	-0.13407144E+01	0.00000000E+00
0.00000000E+00	0.13407144E+01	0.00000000E+00
0.00000000E+00	-0.93963702E+00	0.00000000E+00
0.00000000E+00	0.93963702E+00	0.00000000E+00
0.00000000E+00	0.34696696E+00	0.00000000E+00
0.00000000E+00	-0.34696696E+00	0.00000000E+00

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.2456396E-19	0.1000000E+01	-0.4086934E-19
2	-0.3809764E-05	0.1452009E-23	-0.3809764E-05	-0.5987366E-23
3	-0.7158109E+00	0.1795180E-18	-0.7158109E+00	-0.4518921E-18
4	0.9274438E-03	0.2260271E-21	0.9274438E-03	0.1530077E-20
5	0.4506201E+00	-0.3959879E-19	0.4506201E+00	0.2038173E-18
6	-0.2257719E+00	0.2538452E-19	-0.2257719E+00	-0.1387016E-18
7	-0.1856696E-17	-0.1566648E+02	-0.1467400E-16	0.1566648E+02
8	0.1211044E-22	0.5968560E-04	0.1105449E-21	-0.5968560E-04
9	-0.1506025E-17	0.1121424E+02	0.1684240E-16	-0.1121424E+02
10	0.5558654E-21	-0.1452978E-01	-0.1707963E-19	0.1452978E-01
11	-0.1390828E-17	-0.7059632E+01	-0.8570279E-17	0.7059632E+01
12	-0.2655448E-18	0.3537051E+01	0.4852652E-17	-0.3537051E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	-0.7238354E+00	0.4362551E-20	-0.7238354E+00	0.1379324E-21
2	-0.3546887E-03	-0.1558234E-23	-0.3546887E-03	0.8613099E-25
3	-0.6964225E+00	0.4663644E-20	-0.6964225E+00	0.6896618E-21
4	0.1340846E-01	0.3070908E-22	0.1340846E-01	0.7496324E-24
5	0.1000000E+01	0.5360130E-21	0.1000000E+01	0.5004443E-22
6	-0.5065313E+00	0.6294844E-21	-0.5065313E+00	0.2163077E-21
7	-0.3369603E-19	-0.4566676E+01	0.2767746E-20	0.4566676E+01
8	-0.2681875E-24	-0.2237730E-02	0.3699695E-24	0.2237730E-02
9	-0.3105567E-19	-0.4393728E+01	0.2081186E-20	0.4393728E+01
10	0.5637236E-21	0.8459393E-01	-0.4182432E-22	-0.8459393E-01
11	0.2345222E-19	0.6308998E+01	-0.6882143E-20	-0.6308998E+01
12	-0.7649766E-20	-0.3195705E+01	0.2071260E-20	0.3195705E+01

C = 0.00 (continued)

ROW COL	5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.7960482E-01	-0.2777014E-22	0.7960482E-01	0.1461026E-24
2	-0.8061736E+00	-0.2384124E-21	-0.8061736E+00	0.1199700E-23
3	0.2468506E+00	-0.9222276E-22	0.2468506E+00	0.4812209E-24
4	0.1000000E+01	0.1413639E-25	0.1000000E+01	0.9138165E-26
5	0.4258956E+00	-0.1697927E-21	0.4258956E+00	0.8750992E-24
6	-0.4342541E+00	0.4665079E-22	-0.4342541E+00	-0.2358126E-24
7	0.3605847E-22	0.1432981E+00	0.1897984E-24	-0.1432981E+00
8	0.3462502E-21	-0.1451208E+01	0.1731329E-23	0.1451208E+01
9	0.1179011E-21	0.4443603E+00	0.6143775E-24	-0.4443603E+00
10	-0.2742540E-22	0.1800119E+01	-0.1169281E-24	-0.1800119E+01
11	0.2189405E-21	0.7666627E+00	0.1091998E-23	-0.7666627E+00
12	-0.9348757E-22	-0.7817090E+00	-0.4768633E-24	0.7817090E+00

SYSTEM EIGENVECTORS

ROW COL	7 REAL	7 IMAGINARY	8 REAL	8 IMAGINARY
1	0.1683205E+00	0.1125507E-26	0.1683205E+00	-0.1323918E-22
2	0.7862634E+00	0.1435964E-25	0.7862634E+00	-0.6347904E-21
3	0.5504385E+00	0.1097700E-26	0.5504385E+00	-0.2090125E-22
4	0.1592064E+00	0.2585699E-25	0.1592064E+00	-0.1302085E-20
5	0.1000000E+01	0.5656133E-27	0.1000000E+01	-0.9855082E-25
6	-0.7540265E+00	0.7494179E-28	-0.7540265E+00	-0.1113612E-21
7	0.1999565E-26	-0.2256697E+00	0.8926876E-22	0.2256697E+00
8	0.1170591E-25	-0.1054155E+01	0.4865417E-21	0.1054155E+01
9	0.4620358E-26	-0.7379808E+00	0.2465933E-21	0.7379808E+00
10	0.3189522E-25	-0.2134504E+00	0.1567947E-20	0.2134504E+00
11	0.1232516E-25	-0.1340714E+01	0.3921293E-21	0.1340714E+01
12	-0.8355615E-26	0.1010934E+01	-0.3798375E-21	-0.1010934E+01

SYSTEM EIGENVECTORS

ROW COL	9 REAL	9 IMAGINARY	10 REAL	10 IMAGINARY
1	0.1576819E+00	0.3317002E-24	0.1576819E+00	-0.4072416E-25
2	-0.6755755E+00	-0.2535076E-20	-0.6755755E+00	0.4547383E-22
3	0.5336430E+00	0.1982123E-24	0.5336430E+00	-0.9379241E-25
4	-0.7546734E+00	-0.3080662E-20	-0.7546734E+00	0.5538442E-22
5	0.1000000E+01	0.1165242E-24	0.1000000E+01	0.8305128E-26
6	-0.1674568E+00	-0.4503340E-20	-0.1674568E+00	0.8085720E-22
7	0.1108697E-24	-0.1481637E+00	0.5520764E-25	0.1481637E+00
8	-0.2230077E-20	0.6347957E+00	-0.3998230E-22	-0.6347957E+00
9	0.4162479E-22	-0.5014307E+00	0.6825982E-24	0.5014307E+00
10	-0.1735840E-20	0.7091191E+00	-0.3122213E-22	-0.7091191E+00
11	0.8568832E-22	-0.9396370E+00	0.1715425E-23	0.9396370E+00
12	-0.2169411E-20	0.1573486E+00	-0.3901747E-22	-0.1573486E+00

C = 0.00 (concluded)

ROW	COL 11 REAL	11 IMAGINARY	12 REAL	12 IMAGINARY
1	0.2129991E-01	0.9425783E-20	0.2129991E-01	-0.8705053E-20
2	0.3947968E+00	0.3412506E-18	0.3947968E+00	-0.3151563E-18
3	0.7420595E-01	0.7826565E-20	0.7420595E-01	-0.7228171E-20
4	0.7420656E+00	0.4384725E-18	0.7420656E+00	-0.4049437E-18
5	0.1425604E+00	-0.1148952E-19	0.1425604E+00	0.1061083E-19
6	0.1000000E+01	-0.4175283E-25	0.1000000E+01	-0.1048617E-24
7	0.7722762E-20	0.7390363E-02	0.7132210E-20	-0.7390363E-02
8	-0.3257859E-19	0.1369814E+00	-0.3008736E-19	-0.1369814E+00
9	0.4467487E-19	0.2574701E-01	0.4125867E-19	-0.2574701E-01
10	0.1002607E-18	0.2574722E+00	0.9259398E-19	-0.2574722E+00
11	0.1049185E-18	0.4946374E-01	0.9689564E-19	-0.4946374E-01
12	0.6169066E-18	0.3469670E+00	0.5697333E-18	-0.3469670E+00

C = .1

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.62898700E-02	0.15666400E+02	0.40148785E-01
-0.62898700E-02	-0.15666400E+02	0.40148785E-01
-0.33117715E-01	-0.63082618E+01	0.52498231E+00
-0.33117715E-01	0.63082618E+01	0.52498231E+00
-0.47024581E-02	0.17995853E+01	0.26130699E+00
-0.47024581E-02	-0.17995853E+01	0.26130699E+00
-0.31190537E-01	-0.13384467E+01	0.23297208E+01
-0.31190537E-01	0.13384467E+01	0.23297208E+01
-0.24363184E-01	-0.94100220E+00	0.25882004E+01
-0.24363184E-01	0.94100220E+00	0.25882004E+01
-0.33628621E-03	0.34697913E+00	0.96918236E-01
-0.33628629E-03	-0.34697913E+00	0.96918259E-01

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.5055093E-17	0.1000000E+01	-0.2368304E-17
2	-0.3809214E-05	-0.4676142E-07	-0.3809214E-05	0.4676142E-07
3	-0.7158054E+00	-0.4332292E-03	-0.7158054E+00	0.4332292E-03
4	0.9272907E-03	0.1213413E-04	0.9272907E-03	-0.1213413E-04
5	0.4505361E+00	0.6261089E-02	0.4505361E+00	-0.6261089E-02
6	-0.2257298E+00	-0.3136589E-02	-0.2257298E+00	0.3136589E-02
7	-0.6289870E-02	0.1566640E+02	-0.6289870E-02	-0.1566640E+02
8	0.7565426E-06	-0.5967637E-04	0.7565426E-06	0.5967637E-04
9	0.1128947E-01	-0.1121409E+02	0.1128947E-01	0.1121409E+02
10	-0.1959307E-03	0.1452723E-01	-0.1959307E-03	-0.1452723E-01
11	-0.1009225E+00	0.7058240E+01	-0.1009225E+00	-0.7058240E+01
12	0.5055886E-01	-0.3536354E+01	0.5055886E-01	0.3536354E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	-0.7237261E+00	-0.1663599E-01	-0.7237261E+00	0.1663599E-01
2	-0.3547530E-03	-0.7900585E-05	-0.3547530E-03	0.7900585E-05
3	-0.6962840E+00	-0.2517155E-01	-0.6962840E+00	0.2517155E-01
4	0.1341051E-01	0.1503608E-03	0.1341051E-01	-0.1503608E-03
5	0.1000000E+01	-0.3374579E-16	0.1000000E+01	0.2648164E-16
6	-0.5065323E+00	-0.7137615E-04	-0.5065323E+00	0.7137615E-04
7	-0.8097605E-01	0.4566005E+01	-0.8097605E-01	-0.4566005E+01
8	-0.3809035E-04	0.2238137E-02	-0.3809035E-04	-0.2238137E-02
9	-0.1357294E+00	0.4393176E+01	-0.1357294E+00	-0.4393176E+01
10	0.5043897E-03	-0.8460197E-01	0.5043897E-03	0.8460197E-01
11	-0.3311771E-01	-0.6308262E+01	-0.3311771E-01	0.6308262E+01
12	0.1632493E-01	0.3195341E+01	0.1632493E-01	-0.3195341E+01

SYSTEM EIGENVECTORS

C = .1 (continued)

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.7856006E-01	0.9365562E-02	0.7856006E-01	-0.9365562E-02
2	-0.8072873E+00	0.1103227E-01	-0.8072873E+00	-0.1103227E-01
3	0.2436459E+00	0.2889396E-01	0.2436459E+00	-0.2889396E-01
4	0.1000000E+01	0.1395910E-16	0.1000000E+01	0.2748453E-16
5	0.4204281E+00	0.4959143E-01	0.4204281E+00	-0.4959143E-01
6	-0.4311977E+00	-0.2795721E-01	-0.4311977E+00	0.2795721E-01
7	-0.1722355E-01	0.1413315E+00	-0.1722355E-01	-0.1413315E+00
8	-0.1605727E-01	-0.1452834E+01	-0.1605727E-01	0.1452834E+01
9	-0.5314288E-01	0.4383257E+00	-0.5314288E-01	-0.4383257E+00
10	-0.4702458E-02	0.1799585E+01	-0.4702458E-02	-0.1799585E+01
11	-0.9122105E-01	0.7563630E+00	-0.9122105E-01	-0.7563630E+00
12	0.5233908E-01	-0.7758456E+00	0.5233908E-01	0.7758456E+00

SYSTEM EIGENVECTORS

ROW COL	7		8	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1682317E+00	-0.1008734E-02	0.1682317E+00	0.1008734E-02
2	0.7772059E+00	0.7777795E-01	0.7772059E+00	-0.7777795E-01
3	0.5502990E+00	-0.1588694E-02	0.5502990E+00	0.1588694E-02
4	0.1563565E+00	0.8118889E-01	0.1563565E+00	-0.8118889E-01
5	0.1000000E+01	-0.1791644E-16	0.1000000E+01	0.2206351E-16
6	-0.7512228E+00	-0.4771140E-01	-0.7512228E+00	0.4771140E-01
7	-0.6597374E-02	-0.2251377E+00	-0.6597374E-02	0.2251377E+00
8	0.7986018E-01	-0.1042675E+01	0.7986018E-01	0.1042675E+01
9	-0.1929050E-01	-0.7364964E+00	-0.1929050E-01	0.7364964E+00
10	0.1037902E+00	-0.2118071E+00	0.1037902E+00	0.2118071E+00
11	-0.3119054E-01	-0.1338447E+01	-0.3119054E-01	0.1338447E+01
12	-0.4042812E-01	0.1006960E+01	-0.4042812E-01	-0.1006960E+01

SYSTEM EIGENVECTORS

ROW COL	9		10	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1577030E+00	-0.5132384E-03	0.1577030E+00	0.5132384E-03
2	-0.6685217E+00	0.8725714E-01	-0.6685217E+00	-0.8725714E-01
3	0.5336765E+00	-0.8123087E-03	0.5336765E+00	0.8123087E-03
4	-0.7494751E+00	0.6664837E-01	-0.7494751E+00	-0.6664837E-01
5	0.1000000E+01	0.1515173E-16	0.1000000E+01	-0.4171468E-16
6	-0.1702802E+00	-0.4730163E-01	-0.1702802E+00	0.4730163E-01
7	-0.4325106E-02	-0.1483864E+00	-0.4325106E-02	0.1483864E+00
8	0.9839648E-01	0.6269545E+00	0.9839648E-01	-0.6269545E+00
9	-0.1376644E-01	-0.5021710E+00	-0.1376644E-01	0.5021710E+00
10	0.8097586E-01	0.7036339E+00	0.8097586E-01	-0.7036339E+00
11	-0.2436318E-01	-0.9410022E+00	-0.2436318E-01	0.9410022E+00
12	-0.4036237E-01	0.1613864E+00	-0.4036237E-01	-0.1613864E+00

SYSTEM EIGENVECTORS

C = .1 (concluded)

ROW COL	11 REAL	11 IMAGINARY	12 REAL	12 IMAGINARY
1	0.2126892E-01	-0.8247214E-03	0.2126892E-01	0.8247214E-03
2	0.3948017E+00	0.1367405E-03	0.3948017E+00	-0.1367405E-03
3	0.7409794E-01	-0.2873877E-02	0.7409794E-01	0.2873877E-02
4	0.7420714E+00	0.1648839E-03	0.7420714E+00	-0.1648839E-03
5	0.1423528E+00	-0.5522199E-02	0.1423528E+00	0.5522199E-02
6	0.1000000E+01	0.1088946E-16	0.1000000E+01	-0.1133669E-16
7	0.2790087E-03	0.7380148E-02	0.2790087E-03	-0.7380148E-02
8	-0.1802125E-03	0.1369879E+00	-0.1802125E-03	-0.1369879E+00
9	0.9722571E-03	0.2571140E-01	0.9722571E-03	-0.2571140E-01
10	-0.3067597E-03	0.2574832E+00	-0.3067597E-03	-0.2574832E+00
11	0.1868216E-02	0.4939530E-01	0.1868216E-02	-0.4939530E-01
12	-0.3362862E-03	0.3469791E+00	-0.3362862E-03	-0.3469791E+00

C = .2

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.12573562E-01	-0.15666155E+02	0.80259375E-01
-0.12573562E-01	0.15666155E+02	0.80259375E-01
-0.66161739E-01	0.63060551E+01	0.10491202E+01
-0.66161739E-01	-0.63060551E+01	0.10491202E+01
-0.90785026E-02	-0.17980485E+01	0.50490222E+00
-0.90785026E-02	0.17980485E+01	0.50490222E+00
-0.62464502E-01	-0.13314316E+01	0.46863747E+01
-0.62464502E-01	0.13314316E+01	0.46863747E+01
-0.49051767E-01	0.94522781E+00	0.51824388E+01
-0.49051767E-01	-0.94522781E+00	0.51824388E+01
-0.67002982E-03	-0.34701547E+00	0.19308320E+00
-0.67002998E-03	0.34701547E+00	0.19308324E+00

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.9202166E-17	0.1000000E+01	-0.6860967E-18
2	-0.3807563E-05	0.9348586E-07	-0.3807563E-05	-0.9348586E-07
3	-0.7157892E+00	0.8660661E-03	-0.7157892E+00	-0.8660661E-03
4	0.9268317E-03	-0.2425712E-04	0.9268317E-03	0.2425712E-04
5	0.4502842E+00	-0.1251565E-01	0.4502842E+00	0.1251565E-01
6	-0.2256036E+00	0.6269909E-02	-0.2256036E+00	-0.6269909E-02
7	-0.1257356E-01	-0.1566615E+02	-0.1257356E-01	0.1566615E+02
8	0.1512439E-05	0.5964870E-04	0.1512439E-05	-0.5964870E-04
9	0.2256795E-01	0.1121365E+02	0.2256795E-01	-0.1121365E+02
10	-0.3916694E-03	-0.1451958E-01	-0.3916694E-03	0.1451958E-01
11	-0.2017338E+00	-0.7054065E+01	-0.2017338E+00	0.7054065E+01
12	0.1010620E+00	0.3534263E+01	0.1010620E+00	-0.3534263E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	-0.7233987E+00	0.3328313E-01	-0.7233987E+00	-0.3328313E-01
2	-0.3549465E-03	0.1580565E-04	-0.3549465E-03	-0.1580565E-04
3	-0.6958687E+00	0.5035656E-01	-0.6958687E+00	-0.5035656E-01
4	0.1341666E-01	-0.3006708E-03	0.1341666E-01	0.3006708E-03
5	0.1000000E+01	0.2848741E-16	0.1000000E+01	0.2003064E-16
6	-0.5065353E+00	0.1427249E-03	-0.5065353E+00	-0.1427249E-03
7	-0.1620239E+00	-0.4563994E+01	-0.1620239E+00	0.4563994E+01
8	-0.7618740E-04	-0.2239358E-02	-0.7618740E-04	0.2239358E-02
9	-0.2715114E+00	-0.4391518E+01	-0.2715114E+00	0.4391518E+01
10	0.1008377E-02	0.8462610E-01	0.1008377E-02	-0.8462610E-01
11	-0.6616174E-01	0.6306055E+01	-0.6616174E-01	-0.6306055E+01
12	0.3261323E-01	-0.3194249E+01	0.3261323E-01	0.3194249E+01

SYSTEM EIGENVECTORS

C = .2 (continued)

ROW	COL 5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.7554707E-01	-0.1810115E-01	0.7554707E-01	0.1810115E-01
2	-0.8105301E+00	-0.2146289E-01	-0.8105301E+00	0.2146289E-01
3	0.2343985E+00	-0.5586735E-01	0.2343985E+00	0.5586735E-01
4	0.1000000E+01	0.4626833E-16	0.1000000E+01	0.1589034E-17
5	0.4046413E+00	-0.9592703E-01	0.4046413E+00	0.9592703E-01
6	-0.4223660E+00	0.5411007E-01	-0.4223660E+00	-0.5411007E-01
7	-0.3323259E-01	-0.1356730E+00	-0.3323259E-01	0.1356730E+00
8	-0.3123293E-01	0.1457567E+01	-0.3123293E-01	-0.1457567E+01
9	-0.1025802E+00	-0.4209528E+00	-0.1025802E+00	0.4209528E+00
10	-0.9078503E-02	-0.1798049E+01	-0.9078503E-02	0.1798049E+01
11	-0.1761550E+00	-0.7266939E+00	-0.1761550E+00	0.7266939E+00
12	0.1011270E+00	0.7589433E+00	0.1011270E+00	-0.7589433E+00

SYSTEM EIGENVECTORS

ROW	COL 7 REAL	7 IMAGINARY	8 REAL	8 IMAGINARY
1	0.1679596E+00	-0.2005777E-02	0.1679596E+00	0.2005777E-02
2	0.7498341E+00	0.1569677E+00	0.7498341E+00	-0.1569677E+00
3	0.5498715E+00	-0.3159351E-02	0.5498715E+00	0.3159351E-02
4	0.1472464E+00	0.1610129E+00	0.1472464E+00	-0.1610129E+00
5	0.1000000E+01	-0.1841788E-16	0.1000000E+01	0.2424547E-16
6	-0.7425739E+00	-0.9525066E-01	-0.7425739E+00	0.9525066E-01
7	-0.1316207E-01	-0.2235014E+00	-0.1316207E-01	0.2235014E+00
8	0.1621537E+00	-0.1008158E+01	0.1621537E+00	0.1008158E+01
9	-0.3855391E-01	-0.7319189E+00	-0.3855391E-01	0.7319189E+00
10	0.2051799E+00	-0.2061061E+00	0.2051799E+00	0.2061061E+00
11	-0.6246450E-01	-0.1331432E+01	-0.6246450E-01	0.1331432E+01
12	-0.8043523E-01	0.9946361E+00	-0.8043523E-01	-0.9946361E+00

SYSTEM EIGENVECTORS

ROW	COL 9 REAL	9 IMAGINARY	10 REAL	10 IMAGINARY
1	0.1577689E+00	0.1038491E-02	0.1577689E+00	-0.1038491E-02
2	-0.6469157E+00	-0.1744151E+00	-0.6469157E+00	0.1744151E+00
3	0.5337812E+00	0.1643578E-02	0.5337812E+00	-0.1643578E-02
4	-0.7335710E+00	-0.1334291E+00	-0.7335710E+00	0.1334291E+00
5	0.1000000E+01	0.9812030E-17	0.1000000E+01	-0.3718813E-16
6	-0.1789511E+00	0.9447302E-01	-0.1789511E+00	-0.9447302E-01
7	-0.8720455E-02	0.1490766E+00	-0.8720455E-02	-0.1490766E+00
8	0.1965944E+00	-0.6029273E+00	0.1965944E+00	0.6029273E+00
9	-0.2773646E-01	0.5044642E+00	-0.2773646E-01	-0.5044642E+00
10	0.1621039E+00	-0.6868468E+00	0.1621039E+00	0.6868468E+00
11	-0.4905177E-01	0.9452278E+00	-0.4905177E-01	-0.9452278E+00
12	-0.8052066E-01	-0.1737836E+00	-0.8052066E-01	0.1737836E+00

SYSTEM EIGENVECTORS

C = .2 (concluded)

ROW \ COL	11 REAL	11 IMAGINARY	12 REAL	12 IMAGINARY
1	0.2117643E-01	0.1642710E-02	0.2117643E-01	-0.1642710E-02
2	0.3948161E+00	-0.2724921E-03	0.3948161E+00	0.2724921E-03
3	0.7377557E-01	0.5724280E-02	0.7377557E-01	-0.5724280E-02
4	0.7420889E+00	-0.3285730E-03	0.7420889E+00	0.3285730E-03
5	0.1417332E+00	0.1099928E-01	0.1417332E+00	-0.1099928E-01
6	0.1000000E+01	0.5258381E-17	0.1000000E+01	-0.1660185E-16
7	0.5558568E-03	-0.7349649E-02	0.5558568E-03	0.7349649E-02
8	-0.3590976E-03	-0.1370071E+00	-0.3590976E-03	0.1370071E+00
9	0.1936982E-02	-0.2560510E-01	0.1936982E-02	0.2560510E-01
10	-0.6112416E-03	-0.2575161E+00	-0.6112416E-03	0.2575161E+00
11	0.3721953E-02	-0.4919100E-01	0.3721953E-02	0.4919100E-01
12	-0.6700299E-03	-0.3470155E+00	-0.6700299E-03	0.3470155E+00

C = .3

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.18845095E-01	0.15665746E+02	0.12029483E+00
-0.18845095E-01	-0.15665746E+02	0.12029483E+00
-0.99058318E-01	0.63023818E+01	0.15715659E+01
-0.99058318E-01	-0.63023818E+01	0.15715659E+01
-0.12855629E-01	0.17956874E+01	0.71589855E+00
-0.12855629E-01	-0.17956874E+01	0.71589855E+00
-0.93807180E-01	-0.13189978E+01	0.70940857E+01
-0.93807180E-01	0.13189978E+01	0.70940857E+01
-0.74435443E-01	0.95275050E+00	0.77889549E+01
-0.74435443E-01	-0.95275050E+00	0.77889549E+01
-0.99874304E-03	0.34707548E+00	0.28775842E+00
-0.99874328E-03	-0.34707548E+00	0.28775849E+00

SYSTEM EIGENVECTORS

ROW COL	1 REAL	1 IMAGINARY	2 REAL	2 IMAGINARY
1	0.1000000E+01	-0.9581637E-17	0.1000000E+01	0.3098108E-16
2	-0.3804816E-05	-0.1401363E-06	-0.3804816E-05	0.1401363E-06
3	-0.7157621E+00	-0.1298126E-02	-0.7157621E+00	0.1298126E-02
4	0.9260674E-03	0.3635784E-04	0.9260674E-03	-0.3635784E-04
5	0.4498650E+00	0.1875719E-01	0.4498650E+00	-0.1875719E-01
6	-0.2253936E+00	-0.9396705E-02	-0.2253936E+00	0.9396705E-02
7	-0.1884510E-01	0.1566575E+02	-0.1884510E-01	-0.1566575E+02
8	0.2267042E-05	-0.5960264E-04	0.2267042E-05	0.5960264E-04
9	0.3382471E-01	-0.1121292E+02	0.3382471E-01	0.1121292E+02
10	-0.5870246E-03	0.1450685E-01	-0.5870246E-03	-0.1450685E-01
11	-0.3023231E+00	0.7047117E+01	-0.3023231E+00	-0.7047117E+01
12	0.1514540E+00	-0.3530782E+01	0.1514540E+00	0.3530782E+01

SYSTEM EIGENVECTORS

ROW COL	3 REAL	3 IMAGINARY	4 REAL	4 IMAGINARY
1	-0.7228538E+00	0.4995251E-01	-0.7228538E+00	-0.4995251E-01
2	-0.3552705E-03	0.2371960E-04	-0.3552705E-03	-0.2371960E-04
3	-0.6951773E+00	0.7556851E-01	-0.6951773E+00	-0.7556851E-01
4	0.1342694E-01	-0.4508780E-03	0.1342694E-01	0.4508780E-03
5	0.1000000E+01	0.1221083E-16	0.1000000E+01	-0.3480289E-16
6	-0.5065403E+00	0.2140181E-03	-0.5065403E+00	-0.2140181E-03
7	-0.2432151E+00	-0.4560649E+01	-0.2432151E+00	0.4560649E+01
8	-0.1142975E-03	-0.2241400E-02	-0.1142975E-03	0.2241400E-02
9	-0.4073985E+00	-0.4388759E+01	-0.4073985E+00	0.4388759E+01
10	0.1511556E-02	0.8466636E-01	0.1511556E-02	-0.8466636E-01
11	-0.9905832E-01	0.6302382E+01	-0.9905832E-01	-0.6302382E+01
12	0.4882821E-01	-0.3192432E+01	0.4882821E-01	0.3192432E+01

SYSTEM EIGENVECTORS

C = .3 (continued)

ROW	COL 5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.7090664E-01	0.2567787E-01	0.7090664E-01	-0.2567787E-01
2	-0.8156148E+00	0.3075682E-01	-0.8156148E+00	-0.3075682E-01
3	0.2201406E+00	0.7930222E-01	0.2201406E+00	-0.7930222E-01
4	0.1000000E+01	-0.2225325E-16	0.1000000E+01	-0.1929880E-16
5	0.3802728E+00	0.1362540E+00	0.3802728E+00	-0.1362540E+00
6	-0.4087131E+00	-0.7692620E-01	-0.4087131E+00	0.7692620E-01
7	-0.4702098E-01	0.1269960E+00	-0.4702098E-01	-0.1269960E+00
8	-0.4474439E-01	-0.1464985E+01	-0.4474439E-01	0.1464985E+01
9	-0.1452320E+00	0.3942842E+00	-0.1452320E+00	-0.3942842E+00
10	-0.1285563E-01	0.1795687E+01	-0.1285563E-01	-0.1795687E+01
11	-0.2495582E+00	0.6810995E+00	-0.2495582E+00	-0.6810995E+00
12	0.1433897E+00	-0.7329319E+00	0.1433897E+00	0.7329319E+00

SYSTEM EIGENVECTORS

ROW	COL 7 REAL	7 IMAGINARY	8 REAL	8 IMAGINARY
1	0.1674867E+00	-0.2974251E-02	0.1674867E+00	0.2974251E-02
2	0.7033490E+00	0.2388106E+00	0.7033490E+00	-0.2388106E+00
3	0.5491282E+00	-0.4685801E-02	0.5491282E+00	0.4685801E-02
4	0.1301355E+00	0.2383035E+00	0.1301355E+00	-0.2383035E+00
5	0.1000000E+01	0.3084555E-16	0.1000000E+01	-0.6112190E-16
6	-0.7273081E+00	-0.1424926E+00	-0.7273081E+00	0.1424926E+00
7	-0.1963448E-01	-0.2206355E+00	-0.1963448E-01	0.2206355E+00
8	0.2490115E+00	-0.9501180E+00	0.2490115E+00	0.9501180E+00
9	-0.5769273E-01	-0.7238593E+00	-0.5769273E-01	0.7238593E+00
10	0.3021142E+00	-0.1940030E+00	0.3021142E+00	0.1940030E+00
11	-0.9380718E-01	-0.1318998E+01	-0.9380718E-01	0.1318998E+01
12	-0.1197207E+00	0.9726846E+00	-0.1197207E+00	-0.9726846E+00

SYSTEM EIGENVECTORS

ROW	COL 9 REAL	9 IMAGINARY	10 REAL	10 IMAGINARY
1	0.1578884E+00	0.1589878E-02	0.1578884E+00	-0.1589878E-02
2	-0.6092681E+00	-0.2613393E+00	-0.6092681E+00	0.2613393E+00
3	0.5339708E+00	0.2516092E-02	0.5339708E+00	-0.2516092E-02
4	-0.7059264E+00	-0.2004835E+00	-0.7059264E+00	0.2004835E+00
5	0.1000000E+01	-0.1807907E-16	0.1000000E+01	-0.2429290E-17
6	-0.1941389E+00	0.1413734E+00	-0.1941389E+00	-0.1413734E+00
7	-0.1326725E-01	0.1503099E+00	-0.1326725E-01	-0.1503099E+00
8	0.2943423E+00	-0.5610276E+00	0.2943423E+00	0.5610276E+00
9	-0.4214356E-01	0.5085537E+00	-0.4214356E-01	-0.5085537E+00
10	0.2435567E+00	-0.6576487E+00	0.2435567E+00	0.6576487E+00
11	-0.7443544E-01	0.9527505E+00	-0.7443544E-01	-0.9527505E+00
12	-0.1202428E+00	-0.1954891E+00	-0.1202428E+00	0.1954891E+00

SYSTEM EIGENVECTORS

C = .3 (concluded)

ROW	COL 11 REAL	11 IMAGINARY	12 REAL	12 IMAGINARY
1	0.2102385E-01	-0.2447391E-02	0.2102385E-01	0.2447391E-02
2	0.3948400E+00	0.4062861E-03	0.3948400E+00	-0.4062861E-03
3	0.7324376E-01	-0.8528292E-02	0.7324376E-01	0.8528292E-02
4	0.7421177E+00	0.4898968E-03	0.7421177E+00	-0.4898968E-03
5	0.1407112E+00	-0.1638718E-01	0.1407112E+00	0.1638718E-01
6	0.1000000E+01	-0.1043545E-16	0.1000000E+01	-0.1699487E-16
7	0.8284320E-03	0.7299306E-02	0.8284320E-03	-0.7299306E-02
8	-0.5353557E-03	0.1370389E+00	-0.5353557E-03	-0.1370389E+00
9	0.2886809E-02	0.2542963E-01	0.2886809E-02	-0.2542963E-01
10	-0.9112162E-03	0.2575704E+00	-0.9112162E-03	-0.2575704E+00
11	0.5547053E-02	0.4885376E-01	0.5547053E-02	-0.4885376E-01
12	-0.9987432E-03	0.3470755E+00	-0.9987432E-03	-0.3470755E+00

C = .4

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.25097802E-01	-0.15665174E+02	0.16021380E+00
-0.25097802E-01	0.15665174E+02	0.16021380E+00
-0.13173314E+00	0.62972490E+01	0.20914582E+01
-0.13173314E+00	-0.62972490E+01	0.20914582E+01
-0.15858107E-01	0.17927596E+01	0.88452947E+00
-0.15858107E-01	-0.17927596E+01	0.88452946E+00
-0.12499314E+00	-0.12998665E+01	0.95716937E+01
-0.12499314E+00	0.12998665E+01	0.95716937E+01
-0.10099797E+00	0.96449645E+00	0.10414631E+02
-0.10099797E+00	-0.96449645E+00	0.10414631E+02
-0.13200455E-02	-0.34715832E+00	0.38024035E+00
-0.13200459E-02	0.34715832E+00	0.38024044E+00

SYSTEM EIGENVECTORS

ROW COL	1 REAL	1 IMAGINARY	2 REAL	2 IMAGINARY
1	0.1000000E+01	0.9025983E-17	0.1000000E+01	0.1409463E-16
2	-0.3800975E-05	0.1866764E-06	-0.3800975E-05	-0.1866764E-06
3	-0.7157242E+00	0.1729000E-02	-0.7157242E+00	-0.1729000E-02
4	0.9249993E-03	-0.4842527E-04	0.9249993E-03	0.4842527E-04
5	0.4492792E+00	-0.2497922E-01	0.4492792E+00	0.2497922E-01
6	-0.2251002E+00	0.1251373E-01	-0.2251002E+00	-0.1251373E-01
7	-0.2509780E-01	-0.1566517E+02	-0.2509780E-01	0.1566517E+02
8	0.3019714E-05	0.5953825E-04	0.3019714E-05	-0.5953825E-04
9	0.4504819E-01	0.1121190E+02	0.4504819E-01	-0.1121190E+02
10	-0.7818057E-03	-0.1448906E-01	-0.7818057E-03	0.1448906E-01
11	-0.4025797E+00	-0.7037410E+01	-0.4025797E+00	0.7037410E+01
12	0.2016792E+00	0.3525920E+01	0.2016792E+00	-0.3525920E+01

SYSTEM EIGENVECTORS

ROW COL	3 REAL	3 IMAGINARY	4 REAL	4 IMAGINARY
1	-0.7220926E+00	0.6665542E-01	-0.7220926E+00	-0.6665542E-01
2	-0.3557273E-03	0.3164666E-04	-0.3557273E-03	-0.3164666E-04
3	-0.6942110E+00	0.1008209E+00	-0.6942110E+00	-0.1008209E+00
4	0.1344137E-01	-0.6009251E-03	0.1344137E-01	0.6009251E-03
5	0.1000000E+01	-0.9920450E-17	0.1000000E+01	0.5551115E-16
6	-0.5065473E+00	0.2852252E-03	-0.5065473E+00	-0.2852252E-03
7	-0.3246223E+00	-0.4555977E+01	-0.3246223E+00	0.4555977E+01
8	-0.1524258E-03	-0.2244272E-02	-0.1524258E-03	0.2244272E-02
9	-0.5434439E+00	-0.4384901E+01	-0.5434439E+00	0.4384901E+01
10	0.2013502E-02	0.8472279E-01	0.2013502E-02	-0.8472279E-01
11	-0.1317331E+00	0.6297249E+01	-0.1317331E+00	-0.6297249E+01
12	0.6493293E-01	-0.3189892E+01	0.6493293E-01	0.3189892E+01

SYSTEM EIGENVECTORS

C = .4 (continued)

ROW COL	5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.6513201E-01	0.3174912E-01	0.6513201E-01	-0.3174912E-01
2	-0.8220986E+00	0.3851264E-01	-0.8220986E+00	-0.3851264E-01
3	0.2023713E+00	0.9812899E-01	0.2023713E+00	-0.9812899E-01
4	0.1000000E+01	-0.6071532E-17	0.1000000E+01	-0.1614445E-17
5	0.3498563E+00	0.1687363E+00	0.3498563E+00	-0.1687363E+00
6	-0.3916367E+00	-0.9537219E-01	-0.3916367E+00	0.9537219E-01
7	-0.5795140E-01	0.1162626E+00	-0.5795140E-01	-0.1162626E+00
8	-0.5600698E-01	-0.1474436E+01	-0.5600698E-01	0.1474436E+01
9	-0.1791309E+00	0.3612469E+00	-0.1791309E+00	-0.3612469E+00
10	-0.1585811E-01	0.1792760E+01	-0.1585811E-01	-0.1792760E+01
11	-0.3080516E+00	0.6245324E+00	-0.3080516E+00	-0.6245324E+00
12	0.1771900E+00	-0.7005981E+00	0.1771900E+00	0.7005981E+00

SYSTEM EIGENVECTORS

ROW COL	7 REAL	7 IMAGINARY	8 REAL	8 IMAGINARY
1	0.1667816E+00	-0.3886295E-02	0.1667816E+00	0.3886295E-02
2	0.6356043E+00	0.3242755E+00	0.6356043E+00	-0.3242755E+00
3	0.5480195E+00	-0.6124582E-02	0.5480195E+00	0.6124582E-02
4	0.1018152E+00	0.3122430E+00	0.1018152E+00	-0.3122430E+00
5	0.1000000E+01	0.5990217E-16	0.1000000E+01	-0.1687290E-17
6	-0.7038785E+00	-0.1894088E+00	-0.7038785E+00	0.1894088E+00
7	-0.2589823E-01	-0.2163081E+00	-0.2589823E-01	0.2163081E+00
8	0.3420687E+00	-0.8667329E+00	0.3420687E+00	0.8667329E+00
9	-0.7645982E-01	-0.7115867E+00	-0.7645982E-01	0.7115867E+00
10	0.3931481E+00	-0.1713744E+00	0.3931481E+00	0.1713744E+00
11	-0.1249931E+00	-0.1299867E+01	-0.1249931E+00	0.1299867E+01
12	-0.1582262E+00	0.9386229E+00	-0.1582262E+00	-0.9386229E+00

SYSTEM EIGENVECTORS

ROW COL	9 REAL	9 IMAGINARY	10 REAL	10 IMAGINARY
1	0.1580804E+00	0.2187002E-02	0.1580804E+00	-0.2187002E-02
2	-0.5524315E+00	-0.3477948E+00	-0.5524315E+00	0.3477948E+00
3	0.5342756E+00	0.3460739E-02	0.5342756E+00	-0.3460739E-02
4	-0.6643558E+00	-0.2679729E+00	-0.6643558E+00	0.2679729E+00
5	0.1000000E+01	-0.1409463E-17	0.1000000E+01	0.4209415E-16
6	-0.2172453E+00	0.1878309E+00	-0.2172453E+00	-0.1878309E+00
7	-0.1807516E-01	0.1522471E+00	-0.1807516E-01	-0.1522471E+00
8	0.3912413E+00	-0.4976917E+00	0.3912413E+00	0.4976917E+00
9	-0.5729862E-01	0.5149574E+00	-0.5729862E-01	-0.5149574E+00
10	0.3255575E+00	-0.6137041E+00	0.3255575E+00	0.6137041E+00
11	-0.1009980E+00	0.9644965E+00	-0.1009980E+00	-0.9644965E+00
12	-0.1592209E+00	-0.2285028E+00	-0.1592209E+00	0.2285028E+00

SYSTEM EIGENVECTORS

C = .4 (concluded)

ROW	COL 11 REAL	11 IMAGINARY	12 REAL	12 IMAGINARY
1	0.2081346E-01	0.3232503E-02	0.2081346E-01	-0.3232503E-02
2	0.3948730E+00	-0.5371934E-03	0.3948730E+00	0.5371934E-03
3	0.7251047E-01	0.1126408E-01	0.7251047E-01	-0.1126408E-01
4	0.7421576E+00	-0.6477329E-03	0.7421576E+00	0.6477329E-03
5	0.1393019E+00	0.2164394E-01	0.1393019E+00	-0.2164394E-01
6	0.1000000E+01	-0.4691885E-16	0.1000000E+01	0.2789110E-16
7	0.1094716E-02	-0.7229833E-02	0.1094716E-02	0.7229833E-02
8	-0.7077416E-03	-0.1370827E+00	-0.7077416E-03	0.1370827E+00
9	0.3814703E-02	-0.2518748E-01	0.3814703E-02	0.2518748E-01
10	-0.1204548E-02	-0.2576453E+00	-0.1204548E-02	0.2576453E+00
11	0.7329990E-02	-0.4838838E-01	0.7329990E-02	0.4838838E-01
12	-0.1320046E-02	-0.3471583E+00	-0.1320046E-02	0.3471583E+00

C = .5

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.31325969E-01	-0.15664441E+02	0.19998101E+00
-0.31325969E-01	0.15664441E+02	0.19998101E+00
-0.16411189E+00	-0.62906666E+01	0.26079283E+01
-0.16411189E+00	0.62906666E+01	0.26079283E+01
-0.18026970E-01	0.17895488E+01	0.10072961E+01
-0.18026970E-01	-0.17895488E+01	0.10072961E+01
-0.15540295E+00	-0.12716557E+01	0.12130279E+02
-0.15540295E+00	0.12716557E+01	0.12130279E+02
-0.12950050E+00	-0.98242523E+00	0.13068666E+02
-0.12950050E+00	0.98242523E+00	0.13068666E+02
-0.16317135E-02	0.34726287E+00	0.46987331E+00
-0.16317139E-02	-0.34726287E+00	0.46987343E+00

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.9730714E-17	0.1000000E+01	0.8565197E-17
2	-0.3796046E-05	0.2330692E-06	-0.3796046E-05	-0.2330692E-06
3	-0.7156756E+00	0.2158315E-02	-0.7156756E+00	-0.2158315E-02
4	0.9236290E-03	-0.6044840E-04	0.9236290E-03	0.6044840E-04
5	0.4485280E+00	-0.3117533E-01	0.4485280E+00	0.3117533E-01
6	-0.2247239E+00	0.1561777E-01	-0.2247239E+00	-0.1561777E-01
7	-0.3132597E-01	-0.1566444E+02	-0.3132597E-01	0.1566444E+02
8	0.3769814E-05	0.5945563E-04	0.3769814E-05	-0.5945563E-04
9	0.5622803E-01	0.1121059E+02	0.5622803E-01	-0.1121059E+02
10	-0.9758239E-03	-0.1446624E-01	-0.9758239E-03	0.1446624E-01
11	-0.5023947E+00	-0.7024963E+01	-0.5023947E+00	0.7024963E+01
12	0.2516834E+00	0.3519685E+01	0.2516834E+00	-0.3519685E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	-0.7211166E+00	-0.8340300E-01	-0.7211166E+00	0.8340300E-01
2	-0.3563202E-03	-0.3959088E-04	-0.3563202E-03	0.3959088E-04
3	-0.6929715E+00	-0.1261274E+00	-0.6929715E+00	0.1261274E+00
4	0.1345998E-01	0.7507509E-03	0.1345998E-01	-0.7507509E-03
5	0.1000000E+01	0.2379824E-16	0.1000000E+01	0.2219904E-16
6	-0.5065563E+00	-0.3563138E-03	-0.5065563E+00	0.3563138E-03
7	-0.4063166E+00	0.4549991E+01	-0.4063166E+00	-0.4549991E+01
8	-0.1905767E-03	0.2247989E-02	-0.1905767E-03	-0.2247989E-02
9	-0.6797008E+00	0.4379952E+01	-0.6797008E+00	-0.4379952E+01
10	0.2513780E-02	-0.8479547E-01	0.2513780E-02	0.8479547E-01
11	-0.1641119E+00	-0.6290667E+01	-0.1641119E+00	0.6290667E+01
12	0.8089046E-01	0.3186635E+01	0.8089046E-01	-0.3186635E+01

SYSTEM EIGENVECTORS

C = .5 (continued)

ROW COL	5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.5877046E-01	0.3618999E-01	0.5877046E-01	-0.3618999E-01
2	-0.8294468E+00	0.4451666E-01	-0.8294468E+00	-0.4451666E-01
3	0.1827622E+00	0.1119503E+00	0.1827622E+00	-0.1119503E+00
4	0.1000000E+01	-0.2161628E-17	0.1000000E+01	0.3686287E-17
5	0.3162310E+00	0.1926707E+00	0.3162310E+00	-0.1926707E+00
6	-0.3727132E+00	-0.1090369E+00	-0.3727132E+00	0.1090369E+00
7	-0.6582321E-01	0.1045202E+00	-0.6582321E-01	-0.1045202E+00
8	-0.6471232E-01	-0.1485138E+01	-0.6471232E-01	0.1485138E+01
9	-0.2036352E+00	0.3250437E+00	-0.2036352E+00	-0.3250437E+00
10	-0.1802697E-01	0.1789549E+01	-0.1802697E-01	-0.1789549E+01
11	-0.3504943E+00	0.5624376E+00	-0.3504943E+00	-0.5624376E+00
12	0.2018458E+00	-0.6650228E+00	0.2018458E+00	0.6650228E+00

SYSTEM EIGENVECTORS

ROW COL	7 REAL	7 IMAGINARY	8 REAL	8 IMAGINARY
1	0.1657902E+00	-0.4693972E-02	0.1657902E+00	0.4693972E-02
2	0.5410758E+00	0.4141615E+00	0.5410758E+00	-0.4141615E+00
3	0.5464592E+00	-0.7400704E-02	0.5464592E+00	0.7400704E-02
4	0.5654809E-01	0.3824349E+00	0.5654809E-01	-0.3824349E+00
5	0.1000000E+01	-0.1477225E-17	0.1000000E+01	0.2523481E-16
6	-0.6692112E+00	-0.2361446E+00	-0.6692112E+00	0.2361446E+00
7	-0.3173340E-01	-0.2100986E+00	-0.3173340E-01	0.2100986E+00
8	0.4425861E+00	-0.7524240E+00	0.4425861E+00	0.7524240E+00
9	-0.9433252E-01	-0.6937578E+00	-0.9433252E-01	0.6937578E+00
10	0.4775377E+00	-0.1313412E+00	0.4775377E+00	0.1313412E+00
11	-0.1554030E+00	-0.1271656E+01	-0.1554030E+00	0.1271656E+01
12	-0.1962972E+00	0.8877037E+00	-0.1962972E+00	-0.8877037E+00

SYSTEM EIGENVECTORS

ROW COL	9 REAL	9 IMAGINARY	10 REAL	10 IMAGINARY
1	0.1583863E+00	-0.2862923E-02	0.1583863E+00	0.2862923E-02
2	-0.4697707E+00	0.4332587E+00	-0.4697707E+00	-0.4332587E+00
3	0.5347609E+00	-0.4529626E-02	0.5347609E+00	0.4529626E-02
4	-0.6042584E+00	0.3360869E+00	-0.6042584E+00	-0.3360869E+00
5	0.1000000E+01	0.3523657E-17	0.1000000E+01	0.6209768E-16
6	-0.2511911E+00	-0.2335790E+00	-0.2511911E+00	0.2335790E+00
7	-0.2332372E-01	-0.1552320E+00	-0.2332372E-01	0.1552320E+00
8	0.4864798E+00	0.4054074E+00	0.4864798E+00	-0.4054074E+00
9	-0.7370182E-01	-0.5247760E+00	-0.7370182E-01	0.5247760E+00
10	0.4084320E+00	0.5501153E+00	0.4084320E+00	-0.5501153E+00
11	-0.1295005E+00	-0.9824252E+00	-0.1295005E+00	0.9824252E+00
12	-0.1969445E+00	0.2770251E+00	-0.1969445E+00	-0.2770251E+00

SYSTEM EIGENVECTORS

C = .5 (continued)

ROW COL	11 REAL	11 IMAGINARY	12 REAL	12 IMAGINARY
1	0.2054838E-01	-0.3992236E-02	0.2054838E-01	0.3992236E-02
2	0.3949147E+00	0.6643424E-03	0.3949147E+00	-0.6643424E-03
3	0.7158656E-01	-0.1391140E-01	0.7158656E-01	0.1391140E-01
4	0.7422079E+00	0.8010287E-03	0.7422079E+00	-0.8010287E-03
5	0.1375263E+00	-0.2673064E-01	0.1375263E+00	0.2673064E-01
6	0.1000000E+01	-0.3786576E-16	0.1000000E+01	0.3642919E-16
7	0.1352826E-02	0.7142202E-02	0.1352826E-02	-0.7142202E-02
8	-0.8750893E-03	0.1371381E+00	-0.8750893E-03	-0.1371381E+00
9	0.4714104E-02	0.2488205E-01	0.4714104E-02	-0.2488205E-01
10	-0.1489238E-02	0.2577399E+00	-0.1489238E-02	-0.2577399E+00
11	0.9058156E-02	0.4780138E-01	0.9058156E-02	-0.4780138E-01
12	-0.1631714E-02	0.3472629E+00	-0.1631714E-02	-0.3472629E+00

C = 1.0

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.61892497E-01	-0.15658408E+02	0.39526378E+00
-0.61892497E-01	0.15658408E+02	0.39526378E+00
-0.31892409E+00	-0.62365954E+01	0.51070798E+01
-0.31892409E+00	0.62365954E+01	0.51070798E+01
-0.19751550E-01	0.17758260E+01	0.11121770E+01
-0.19751551E-01	-0.17758260E+01	0.11121771E+01
-0.51263473E+00	-0.10174053E+01	0.44997261E+02
-0.51263473E+00	0.10174053E+01	0.44997261E+02
-0.83813675E-01	0.11262615E+01	0.74212386E+01
-0.83813675E-01	-0.11262615E+01	0.74212386E+01
-0.29834595E-02	-0.34805529E+00	0.85714826E+00
-0.29834601E-02	0.34805529E+00	0.85714846E+00

SYSTEM EIGENVECTORS

ROW COL	1 REAL	1 IMAGINARY	2 REAL	2 IMAGINARY
1	0.1000000E+01	0.2075231E-17	0.1000000E+01	-0.1492472E-17
2	-0.3755374E-05	0.4615726E-06	-0.3755374E-05	-0.4615726E-06
3	-0.7152753E+00	0.4268330E-02	-0.7152753E+00	-0.4268330E-02
4	0.9123443E-03	-0.1195251E-03	0.9123443E-03	0.1195251E-03
5	0.4423518E+00	-0.6154943E-01	0.4423518E+00	0.6154943E-01
6	-0.2216302E+00	0.3083425E-01	-0.2216302E+00	-0.3083425E-01
7	-0.6189250E-01	-0.1565841E+02	-0.6189250E-01	0.1565841E+02
8	0.7459922E-05	0.5877461E-04	0.7459922E-05	-0.5877461E-04
9	0.1111054E+00	0.1119981E+02	0.1111054E+00	-0.1119981E+02
10	-0.1928039E-02	-0.1427846E-01	-0.1928039E-02	0.1427846E-01
11	-0.9911443E+00	-0.6922715E+01	-0.9911443E+00	0.6922715E+01
12	0.4965324E+00	0.3468468E+01	0.4965324E+00	-0.3468468E+01

SYSTEM EIGENVECTORS

ROW` COL	3 REAL	3 IMAGINARY	4 REAL	4 IMAGINARY
1	-0.7131097E+00	-0.1682057E+00	-0.7131097E+00	0.1682057E+00
2	-0.3615050E-03	-0.7968353E-04	-0.3615050E-03	0.7968353E-04
3	-0.6827750E+00	-0.2539568E+00	-0.6827750E+00	0.2539568E+00
4	0.1361798E-01	0.1493761E-02	0.1361798E-01	-0.1493761E-02
5	0.1000000E+01	0.3699840E-16	0.1000000E+01	0.1795710E-17
6	-0.5066327E+00	-0.7085280E-03	-0.5066327E+00	0.7085280E-03
7	-0.8216029E+00	0.4501022E+01	-0.8216029E+00	-0.4501022E+01
8	-0.3816612E-03	0.2279974E-02	-0.3816612E-03	-0.2279974E-02
9	-0.1366072E+01	0.4339185E+01	-0.1366072E+01	-0.4339185E+01
10	0.4972881E-02	-0.8540625E-01	0.4972881E-02	0.8540625E-01
11	-0.3189241E+00	-0.6236595E+01	-0.3189241E+00	0.6236595E+01
12	0.1571586E+00	0.3159889E+01	0.1571586E+00	-0.3159889E+01

C = 1.0 (continued)

ROW	5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.3115362E-01	0.4020393E-01	0.3115362E-01	-0.4020393E-01
2	-0.8639792E+00	0.5255843E-01	-0.8639792E+00	-0.5255843E-01
3	0.9723323E-01	0.1248190E+00	0.9723323E-01	-0.1248190E+00
4	0.1000000E+01	-0.2710505E-18	0.1000000E+01	0.1002887E-16
5	0.1688589E+00	0.2156128E+00	0.1688589E+00	-0.2156128E+00
6	-0.2891885E+00	-0.1227091E+00	-0.2891885E+00	0.1227091E+00
7	-0.7201052E-01	0.5452933E-01	-0.7201052E-01	-0.5452933E-01
8	-0.7626970E-01	-0.1535315E+01	-0.7626970E-01	0.1535315E+01
9	-0.2235774E+00	0.1702039E+00	-0.2235774E+00	-0.1702039E+00
10	-0.1975155E-01	0.1775826E+01	-0.1975155E-01	-0.1775826E+01
11	-0.3862261E+00	0.2956053E+00	-0.3862261E+00	-0.2956053E+00
12	0.2236219E+00	-0.5111248E+00	0.2236219E+00	0.5111248E+00

SYSTEM EIGENVECTORS

ROW	7 REAL	7 IMAGINARY	8 REAL	8 IMAGINARY
1	0.1559523E+00	-0.1154731E-01	0.1559523E+00	0.1154731E-01
2	-0.1951235E-01	0.1781620E+00	-0.1951235E-01	-0.1781620E+00
3	0.5309572E+00	-0.1828779E-01	0.5309572E+00	0.1828779E-01
4	-0.2097983E+00	0.1983724E+00	-0.2097983E+00	-0.1983724E+00
5	0.1000000E+01	0.1219727E-16	0.1000000E+01	-0.2030169E-16
6	-0.4449782E+00	-0.1534676E+00	-0.4449782E+00	0.1534676E+00
7	-0.9169487E-01	-0.1527472E+00	-0.9169487E-01	0.1527472E+00
8	0.1912657E+00	-0.7148006E-01	0.1912657E+00	0.7148006E-01
9	-0.2907952E+00	-0.5308227E+00	-0.2907952E+00	0.5308227E+00
10	0.3093751E+00	0.1117573E+00	0.3093751E+00	-0.1117573E+00
11	-0.5126347E+00	-0.1017405E+01	-0.5126347E+00	0.1017405E+01
12	0.7197249E-01	0.5313960E+00	0.7197249E-01	-0.5313960E+00

SYSTEM EIGENVECTORS

ROW	9 REAL	9 IMAGINARY	10 REAL	10 IMAGINARY
1	0.1404719E-02	0.1021558E+00	0.1404719E-02	-0.1021558E+00
2	0.1000000E+01	0.2385245E-17	0.1000000E+01	0.2439455E-17
3	0.7102750E-02	0.3407863E+00	0.7102750E-02	-0.3407863E+00
4	0.7385597E+00	-0.1887922E+00	0.7385597E+00	0.1887922E+00
5	0.1716112E-01	0.6304836E+00	0.1716112E-01	-0.6304836E+00
6	-0.4901721E+00	-0.2788687E+00	-0.4901721E+00	0.2788687E+00
7	-0.1151719E+00	-0.6979974E-02	-0.1151719E+00	0.6979974E-02
8	-0.8381367E-01	0.1126262E+01	-0.8381367E-01	-0.1126262E+01
9	-0.3844098E+00	-0.2056300E-01	-0.3844098E+00	0.2056300E-01
10	0.1507280E+00	0.8476348E+00	0.1507280E+00	-0.8476348E+00
11	-0.7115277E+00	-0.3351524E-01	-0.7115277E+00	0.3351524E-01
12	0.3551622E+00	-0.5286889E+00	0.3551622E+00	0.5286889E+00

C = 1.0 (concluded)

ROW	COL 11 REAL	IMAGINARY	12 REAL	IMAGINARY
1	0.1855420E-01	0.7251692E-02	0.1855420E-01	-0.7251692E-02
2	0.3952321E+00	-0.1219082E-02	0.3952321E+00	0.1219082E-02
3	0.6463644E-01	0.2526831E-01	0.6463644E-01	-0.2526831E-01
4	0.7425909E+00	-0.1469667E-02	0.7425909E+00	0.1469667E-02
5	0.1241697E+00	0.4855120E-01	0.1241697E+00	-0.4855120E-01
6	0.1000000E+01	0.3138765E-16	0.1000000E+01	-0.3556183E-16
7	0.2468634E-02	-0.6479524E-02	0.2468634E-02	0.6479524E-02
8	-0.1603467E-02	-0.1375590E+00	-0.1603467E-02	0.1375590E+00
9	0.8601929E-02	-0.2257244E-01	0.8601929E-02	0.2257244E-01
10	-0.2727015E-02	-0.2584583E+00	-0.2727015E-02	0.2584583E+00
11	0.1652805E-01	-0.4336278E-01	0.1652805E-01	0.4336278E-01
12	-0.2983460E-02	-0.3480553E+00	-0.2983460E-02	0.3480553E+00

C = 1.50

SYSTEM EIGENVALUES		DAMPING
REAL	IMAGINARY	PERCENT CRITICAL
-0.90992143E-01	0.15648651E+02	0.58145975E+00
-0.90992143E-01	-0.15648651E+02	0.58145975E+00
-0.45491753E+00	-0.61499463E+01	0.73769429E+01
-0.45491753E+00	0.61499463E+01	0.73769429E+01
-0.16464412E-01	0.17689761E+01	0.93069084E+00
-0.16464412E-01	-0.17689761E+01	0.93069084E+00
-0.88305042E+00	0.74630418E+00	0.76376673E+02
-0.88305042E+00	-0.74630418E+00	0.76376673E+02
-0.50672102E-01	-0.11273850E+01	0.44901249E+01
-0.50672102E-01	0.11273850E+01	0.44901249E+01
-0.39033895E-02	0.34911932E+00	0.11179976E+01
-0.39033903E-02	-0.34911932E+00	0.11179978E+01

SYSTEM EIGENVECTORS

ROW COL	1		2	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1000000E+01	0.8741380E-18	0.1000000E+01	-0.4377466E-16
2	-0.3689166E-05	-0.6811762E-06	-0.3689166E-05	0.6811762E-06
3	-0.7146264E+00	-0.6284735E-02	-0.7146264E+00	0.6284735E-02
4	0.8940552E-03	0.1759447E-03	0.8940552E-03	-0.1759447E-03
5	0.4323820E+00	0.9037963E-01	0.4323820E+00	-0.9037963E-01
6	-0.2166362E+00	-0.4527745E-01	-0.2166362E+00	0.4527745E-01
7	-0.9099214E-01	0.1564865E+02	-0.9099214E-01	-0.1564865E+02
8	0.1099517E-04	-0.5766848E-04	0.1099517E-04	0.5766848E-04
9	0.1633730E+00	-0.1118237E+02	0.1633730E+00	0.1118237E+02
10	-0.2834649E-02	0.1397475E-01	-0.2834649E-02	-0.1397475E-01
11	-0.1453663E+01	0.6757972E+01	-0.1453663E+01	-0.6757972E+01
12	0.7282431E+00	-0.3385944E+01	0.7282431E+00	0.3385944E+01

SYSTEM EIGENVECTORS

ROW COL	3		4	
	REAL	IMAGINARY	REAL	IMAGINARY
1	-0.7003134E+00	-0.2558198E+00	-0.7003134E+00	0.2558198E+00
2	-0.3711115E-03	-0.1205074E-03	-0.3711115E-03	0.1205074E-03
3	-0.6663783E+00	-0.3852480E+00	-0.6663783E+00	0.3852480E+00
4	0.1389138E-01	0.2215325E-02	0.1389138E-01	-0.2215325E-02
5	0.1000000E+01	-0.3258028E-16	0.1000000E+01	-0.7480995E-17
6	-0.5067647E+00	-0.1049694E-02	-0.5067647E+00	0.1049694E-02
7	-0.1254693E+01	0.4423267E+01	-0.1254693E+01	-0.4423267E+01
8	-0.5722886E-03	0.2337136E-02	-0.5722886E-03	-0.2337136E-02
9	-0.2066107E+01	0.4273447E+01	-0.2066107E+01	-0.4273447E+01
10	0.7304697E-02	-0.8643906E-01	0.7304697E-02	0.8643906E-01
11	-0.4549175E+00	-0.6149946E+01	-0.4549175E+00	0.6149946E+01
12	0.2240806E+00	0.3117053E+01	0.2240806E+00	-0.3117053E+01

SYSTEM EIGENVECTORS

C = 1.50 (continued)

ROW COL	5 REAL	5 IMAGINARY	6 REAL	6 IMAGINARY
1	0.1703675E-01	0.3379409E-01	0.1703675E-01	-0.3379409E-01
2	-0.8833838E+00	0.4557761E-01	-0.8833838E+00	-0.4557761E-01
3	0.5326868E-01	0.1051073E+00	0.5326868E-01	-0.1051073E+00
4	0.1000000E+01	0.2629190E-17	0.1000000E+01	0.6894684E-16
5	0.9267497E-01	0.1818933E+00	0.9267497E-01	-0.1818933E+00
6	-0.2456216E+00	-0.1038279E+00	-0.2456216E+00	0.1038279E+00
7	-0.6006143E-01	0.2958121E-01	-0.6006143E-01	-0.2958121E-01
8	-0.6608132E-01	-0.1563435E+01	-0.6608132E-01	0.1563435E+01
9	-0.1868093E+00	0.9250049E-01	-0.1868093E+00	-0.9250049E-01
10	-0.1646441E-01	0.1768976E+01	-0.1646441E-01	-0.1768976E+01
11	-0.3232907E+00	0.1609451E+00	-0.3232907E+00	-0.1609451E+00
12	0.1877131E+00	-0.4327893E+00	0.1877131E+00	0.4327893E+00

SYSTEM EIGENVECTORS

ROW COL	7 REAL	7 IMAGINARY	8 REAL	8 IMAGINARY
1	0.1451437E+00	0.1344744E-01	0.1451437E+00	-0.1344744E-01
2	-0.3668040E-01	-0.5817737E-01	-0.3668040E-01	0.5817737E-01
3	0.5138148E+00	0.2142018E-01	0.5138148E+00	-0.2142018E-01
4	-0.1582141E+00	-0.8097058E-01	-0.1582141E+00	0.8097058E-01
5	0.1000000E+01	0.9595189E-17	0.1000000E+01	-0.4179599E-16
6	-0.4217223E+00	0.8672913E-01	-0.4217223E+00	-0.8672913E-01
7	-0.1382051E+00	0.9644655E-01	-0.1382051E+00	-0.9644655E-01
8	0.7580866E-01	0.2399881E-01	0.7580866E-01	-0.2399881E-01
9	-0.4697103E+00	0.3645470E+00	-0.4697103E+00	-0.3645470E+00
10	0.2001397E+00	-0.4657473E-01	0.2001397E+00	0.4657473E-01
11	-0.8830504E+00	0.7463042E+00	-0.8830504E+00	-0.7463042E+00
12	0.3076757E+00	-0.3913193E+00	0.3076757E+00	0.3913193E+00

SYSTEM EIGENVECTORS

ROW COL	9 REAL	9 IMAGINARY	10 REAL	10 IMAGINARY
1	0.5017824E-03	-0.6129145E-01	0.5017824E-03	0.6129145E-01
2	0.1000000E+01	-0.6491661E-17	0.1000000E+01	0.3882799E-17
3	0.2550694E-02	-0.2044287E+00	0.2550694E-02	0.2044287E+00
4	0.7315707E+00	0.1142539E+00	0.7315707E+00	-0.1142539E+00
5	0.6177524E-02	-0.3781517E+00	0.6177524E-02	0.3781517E+00
6	-0.4778583E+00	0.1671697E+00	-0.4778583E+00	-0.1671697E+00
7	-0.6912449E-01	0.2540065E-02	-0.6912449E-01	-0.2540065E-02
8	-0.5067210E-01	-0.1127385E+01	-0.5067210E-01	0.1127385E+01
9	-0.2305991E+00	0.7483220E-02	-0.2305991E+00	-0.7483220E-02
10	0.9173795E-01	-0.8305513E+00	0.9173795E-01	0.8305513E+00
11	-0.4266355E+00	0.1219729E-01	-0.4266355E+00	-0.1219729E-01
12	0.2126787E+00	0.5302595E+00	0.2126787E+00	-0.5302595E+00

SYSTEM EIGENVECTORS

C = 1.50 (concluded)

ROW	COL 11 REAL	11 IMAGINARY	12 REAL	12 IMAGINARY
1	0.1591786E-01	-0.9404970E-02	0.1591786E-01	0.9404970E-02
2	0.3956620E+00	0.1602719E-02	0.3956620E+00	-0.1602719E-02
3	0.5544906E-01	-0.3276955E-01	0.5544906E-01	0.3276955E-01
4	0.7431094E+00	0.1931741E-02	0.7431094E+00	-0.1931741E-02
5	0.1065151E+00	-0.6296137E-01	0.1065151E+00	0.6296137E-01
6	0.1000000E+01	-0.4163336E-16	0.1000000E+01	-0.1406752E-16
7	0.3221323E-02	0.5593942E-02	0.3221323E-02	-0.5593942E-02
8	-0.2103963E-02	0.1381270E+00	-0.2103963E-02	-0.1381270E+00
9	0.1122404E-01	0.1948625E-01	0.1122404E-01	-0.1948625E-01
10	-0.3575054E-02	0.2594263E+00	-0.3575054E-02	-0.2594263E+00
11	0.2156526E-01	0.3743224E-01	0.2156526E-01	-0.3743224E-01
12	-0.3903390E-02	0.3491193E+00	-0.3903390E-02	-0.3491193E+00

C = 2.0

SYSTEM EIGENVALUES REAL	IMAGINARY	DAMPING PERCENT CRITICAL
-0.11801539E+00	0.15635587E+02	0.75476556E+00
-0.11801539E+00	-0.15635587E+02	0.75476556E+00
-0.56283407E+00	-0.60371410E+01	0.92826048E+01
-0.56283407E+00	0.60371410E+01	0.92826048E+01
-0.13452656E-01	-0.17657839E+01	0.76182966E+00
-0.13452656E-01	0.17657839E+01	0.76182966E+00
-0.17299727E+01	0.63409056E-10	0.10000000E+03
-0.36949523E-01	0.11277039E+01	0.32747696E+01
-0.36949523E-01	-0.11277039E+01	0.32747696E+01
-0.79872751E+00	-0.11243424E-14	0.10000000E+03
-0.43982814E-02	-0.35022652E+00	0.12557400E+01
-0.43982814E-02	0.35022652E+00	0.12557400E+01

SYSTEM EIGENVECTORS

ROW	COL 1 REAL	1 IMAGINARY	2 REAL	2 IMAGINARY
1	0.1000000E+01	0.4309704E-17	0.1000000E+01	0.3935654E-16
2	-0.3599685E-05	-0.8879952E-06	-0.3599685E-05	0.8879952E-06
3	-0.7137543E+00	-0.8167904E-02	-0.7137543E+00	0.8167904E-02
4	0.8694940E-03	0.2285858E-03	0.8694940E-03	-0.2285858E-03
5	0.4190707E+00	0.1170330E+00	0.4190707E+00	-0.1170330E+00
6	-0.2099683E+00	-0.5863034E-01	-0.2099683E+00	0.5863034E-01
7	-0.1180154E+00	0.1563559E+02	-0.1180154E+00	-0.1563559E+02
8	0.1430914E-04	-0.5617839E-04	0.1430914E-04	0.5617839E-04
9	0.2119440E+00	-0.1115900E+02	0.2119440E+00	0.1115900E+02
10	-0.3676687E-02	0.1356807E-01	-0.3676687E-02	-0.1356807E-01
11	-0.1879336E+01	0.6538604E+01	-0.1879336E+01	-0.6538604E+01
12	0.9414993E+00	-0.3276058E+01	0.9414993E+00	0.3276058E+01

SYSTEM EIGENVECTORS

ROW	COL 3 REAL	3 IMAGINARY	4 REAL	4 IMAGINARY
1	-0.6837042E+00	-0.3475890E+00	-0.6837042E+00	0.3475890E+00
2	-0.3864901E-03	-0.1616318E-03	-0.3864901E-03	0.1616318E-03
3	-0.6449168E+00	-0.5217492E+00	-0.6449168E+00	0.5217492E+00
4	0.1428944E-01	0.2890024E-02	0.1428944E-01	-0.2890024E-02
5	0.1000000E+01	0.1764539E-16	0.1000000E+01	0.1170938E-16
6	-0.5069562E+00	-0.1367328E-02	-0.5069562E+00	0.1367328E-02
7	-0.1713632E+01	0.4323253E+01	-0.1713632E+01	-0.4323253E+01
8	-0.7582642E-03	0.2424267E-02	-0.7582642E-03	-0.2424267E-02
9	-0.2786892E+01	0.4187112E+01	-0.2786892E+01	-0.4187112E+01
10	0.9404902E-02	-0.8789396E-01	0.9404902E-02	0.8789396E-01
11	-0.5628341E+00	-0.6037141E+01	-0.5628341E+00	0.6037141E+01
12	0.2770775E+00	0.3061336E+01	0.2770775E+00	-0.3061336E+01

C = 2.0 (continued)

SYSTEM EIGENVECTORS

ROW COL	5		6	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1036344E-01	-0.2773020E-01	0.1036344E-01	0.2773020E-01
2	-0.8929918E+00	-0.3795372E-01	-0.8929918E+00	0.3795372E-01
3	0.3243021E-01	-0.8631915E-01	0.3243021E-01	0.8631915E-01
4	0.1000000E+01	-0.1900064E-16	0.1000000E+01	-0.4100995E-16
5	0.5646788E-01	-0.1495054E+00	0.5646788E-01	0.1495054E+00
6	-0.2248202E+00	0.8546268E-01	-0.2248202E+00	-0.8546268E-01
7	-0.4910496E-01	-0.1792655E-01	-0.4910496E-01	0.1792655E-01
8	-0.5500495E-01	0.1577341E+01	-0.5500495E-01	-0.1577341E+01
9	-0.1528572E+00	-0.5610353E-01	-0.1528572E+00	0.5610353E-01
10	-0.1345266E-01	-0.1765784E+01	-0.1345266E-01	0.1765784E+01
11	-0.2647539E+00	-0.9769884E-01	-0.2647539E+00	0.9769884E-01
12	0.1539331E+00	0.3958342E+00	0.1539331E+00	-0.3958342E+00

SYSTEM EIGENVECTORS

ROW COL	7		8	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.1203451E+00	-0.1651375E-15	0.2659510E-03	0.4459968E-01
2	-0.1845592E-01	0.2697800E-17	0.1000000E+01	0.9690057E-18
3	0.4739241E+00	-0.1695828E-15	0.1352447E-02	0.1487484E+00
4	-0.9214683E-01	0.1402348E-16	0.7296491E+00	-0.8333625E-01
5	0.1000000E+01	-0.1637145E-16	0.3276075E-02	0.2751424E+00
6	-0.4416152E+00	-0.3260738E-16	-0.4745571E+00	-0.1216124E+00
7	-0.2081937E+00	-0.4868047E-14	-0.5030506E-01	-0.1348023E-02
8	0.3192824E-01	-0.5676815E-17	-0.3694952E-01	0.1127704E+01
9	-0.8198757E+00	0.2834836E-14	-0.1677941E+00	-0.3971022E-02
10	0.1594115E+00	-0.1818072E-16	0.6701843E-01	0.8259074E+00
11	-0.1729973E+01	-0.1562552E-14	-0.3104002E+00	-0.6471936E-02
12	0.7639823E+00	0.8151574E-15	0.1546775E+00	-0.5306664E+00

SYSTEM EIGENVECTORS

ROW COL	9		10	
	REAL	IMAGINARY	REAL	IMAGINARY
1	0.2659510E-03	-0.4459968E-01	0.1416727E+00	0.7304812E-17
2	0.1000000E+01	-0.9676504E-17	-0.5840039E-01	0.4336809E-17
3	0.1352447E-02	-0.1487484E+00	0.5081991E+00	-0.4583465E-16
4	0.7296491E+00	0.8333625E-01	-0.1540582E+00	0.4743385E-18
5	0.3276075E-02	-0.2751424E+00	0.1000000E+01	0.1897354E-18
6	-0.4745571E+00	0.1216124E+00	-0.3479999E+00	-0.5468445E-16
7	-0.5030506E-01	0.1348023E-02	-0.1131579E+00	-0.3152938E-14
8	-0.3694952E-01	-0.1127704E+01	0.4664600E-01	0.2389311E-16
9	-0.1677941E+00	0.3971022E-02	-0.4059126E+00	0.1725196E-14
10	0.6701843E-01	-0.8259074E+00	0.1230505E+00	0.2708473E-16
11	-0.3104002E+00	0.6471936E-02	-0.7987275E+00	-0.9102961E-15
12	0.1546775E+00	0.5306664E+00	0.2779571E+00	0.4483447E-15

C = 2.0 (concluded)

SYSTEM EIGENVECTORS

ROW COL	11 REAL	11 IMAGINARY	12 REAL	12 IMAGINARY
1	0.1322368E-01	0.1050203E-01	0.1322368E-01	-0.1050203E-01
2	0.3961138E+00	-0.1815051E-02	0.3961138E+00	0.1815051E-02
3	0.4606119E-01	0.3658992E-01	0.4606119E-01	-0.3658992E-01
4	0.7436541E+00	-0.2187163E-02	0.7436541E+00	0.2187163E-02
5	0.8847686E-01	0.7029820E-01	0.8847686E-01	-0.7029820E-01
6	0.1000000E+01	0.2986977E-16	0.1000000E+01	0.5827587E-18
7	0.3619930E-02	-0.4677474E-02	0.3619930E-02	0.4677474E-02
8	-0.2377899E-02	-0.1387216E+00	-0.2377899E-02	0.1387216E+00
9	0.1261217E-01	-0.1629278E-01	0.1261217E-01	0.1629278E-01
10	-0.4036803E-02	-0.2604378E+00	-0.4036803E-02	0.2604378E+00
11	0.2423115E-01	-0.3129613E-01	0.2423115E-01	0.3129613E-01
12	-0.4398281E-02	-0.3502265E+00	-0.4398281E-02	0.3502265E+00

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13. ABSTRACT *(Maximum 200 words)*

Engineering preliminary design methods for approximating and predicting the effects of viscous or equivalent viscous-type damping treatments on the free and forced vibration of lightly damped aircraft-type structures are developed. Similar developments are presented for dynamic hysteresis-viscoelastic-type damping treatments. It is shown by both engineering analysis and numerical illustrations that the intermodal coupling of the undamped modes arising from the introduction of damping may be neglected in applying these preliminary design methods, except when dissimilar modes of these lightly damped, complex aircraft-type structures have identical or nearly identical natural frequencies. In such cases it is shown that a relatively simple, additional interaction calculation between pairs of modes exhibiting this "modal resonance" phenomenon suffices in the prediction of interacting modal damping fractions. The accuracy of the methods is shown to be very good to excellent, depending on the normal natural frequency separation of the system modes, thereby permitting a relatively simple preliminary design approach. This approach is shown to be a natural precursor to elaborate finite element, digital computer design computations in evaluating the type, quantity, and location of damping treatments. It is expected that in many instances these simplified computations will supplant the more elaborate ones.

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