N92-22059

QUADRATIC SQUEEZING: AN OVERVIEW*

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I. Introduction

The amplitude of the electric field of a mode of the electromagnetic field is not a fixed quantity, there are always quantum mechanical fluctuations. The amplitude, having both a magnitude and a phase, is a complex number and is described by the mode annihilation operator a. It is also possible to characterize the amplitude by its real and imaginary parts which correspond to the Hermitian and anti-Hermitian parts of a,

$$X_1=\frac{1}{2}(a^{+}+a) \qquad X_2=\frac{i}{2}(a^{+}-a) \ , \qquad (1.1)$$

respectively. These operators do not commute and, as a result, obey the uncertainty relation ($h=1$)

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4} \ . \qquad (1.2)$$

From this relation we see that the amplitude fluctuates within an "error box" in the complex plane whose area is at least 1/4. Coherent states, among them the vacuum state, are minimum uncertainty states with $\Delta X_1 = \Delta X_2 = 1/2$. A squeezed state, squeezed in the X_1 direction, has the property that $\Delta X_1 < 1/2$ (Refs.1-3). A squeezed state need not be a minimum uncertainty state, but those that are can be obtained by applying the squeeze operator

$$S(\zeta) = e^{\zeta^{*} a^{2} - \zeta a^{+2}} \ , \qquad (1.3)$$

to a coherent state(Ref.1). The phase of the complex parameter ζ determines the

*This work is supported by National Science Foundation under Grant No. PHY-900173 and by a grant from the City University of New York.

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direction of squeezing and its magnitude determines the extent of the squeezing.

Squeezed states are examples of nonclassical states, that is they cannot be described in terms of a nonnegative definite P representation(Ref.3). This means that a field in a squeezed state cannot be modeled as a classical stochastic field. It should be noted that even though a squeezed state is nonclassical it can have a large number of photons. In fact, a highly squeezed state must have a large number of photons(Ref.4). Thus we see that the usual association of large photon number with classical behavior is not correct.

It is possible to generalize the idea of squeezing by looking at fluctuations in variables more complicated than the mode amplitude. The simplest generalization involves variables quadratic, rather than linear, in the amplitude. In the case of a single mode the square of the amplitude, which corresponds to a^2, is one such observable. If one considers two modes with annihilation operators a and b, then products such as ab and a^+b can be considered. At first glance this procedure appears more mathematically than physically inspired. However, fluctuations in these quadratic quantities can be converted into fluctuations of a single mode amplitude by certain nonlinear optical processes after which they can be measured by standard techniques. We shall now disscuss the kinds of higher-order squeezing to which consideration of these quadratic variables leads and the properties they possess.

II. Amplitude-Squared Squeezing

This is perhaps the simplest example of quadratic squeezing, i.e. squeezing in a variable quadratic in the mode amplitudes. It describes the fluctuations in the square of the amplitude of a single mode, a^2 (Refs.5,6). Following the example of standard squeezing we break this variable into its real and imaginary parts

$$Y_1=\frac{1}{2}(a^{+2}+a^2) \qquad Y_2=\frac{i}{2}(a^{+2}-a^2) \quad . \qquad (2.1)$$

The commutator of these operators is $[Y_1, Y_2] = i(2N+1)$, where $N=a^+a$, and this leads to the uncertainty relation

$$\Delta Y_1 \Delta Y_2 \geq \langle N+\frac{1}{2}\rangle \quad . \qquad (2.2)$$

A state is amplitude-squared squeezed in the Y_1 direction if $(\Delta Y_1)^2 < \langle N+1/2\rangle$.

States with this property are nonclassical. This follows from the fact that $(\Delta Y_1)^2$ can be written as

$$<:(Y_1 - <Y_1>)^2:> = (\Delta Y_1)^2 - <N+\frac{1}{2}> \quad , \qquad (2.3)$$

where the double dots indicate normal ordering. For a classical state the normally ordered term is always nonnegative so one can see that the onset of amplitude-squared squeezing corresponds to the onset of nonclassical behavior.

Amplitude-squared squeezing was first disscussed in a paper by Wodkiewicz and Eberly under the name SU(1,1) squeezing (Ref.7). The reason for this name is that commutation relations of the operators Y_1, Y_2, and N are closely related to those of the Lie algebra SU(1,1). In particular this Lie algebra is described by three operators K_1, K_2, and K_3, whose commutation relations are given by

$$[K_1, K_2] = -iK_3 \qquad [K_2, K_3] = iK_1 \qquad [K_3, K_1] = iK_2 \quad . \qquad (2.4)$$

If one makes the identification $K_1 = Y_1/2$, $K_2 = -Y_2/2$ and $K_3 = (N+1/2)/2$, the above commutation relations are satisfied. This means that the representations of SU(1,1) can be used to study higher-order squeezing and this has been done by a number of authors(Refs.8-10).

It is possible to find minimum uncertainty states for amplitude-squared squeezing, i.e. states for which the inequality in Eq.(2.2) is replaced by an equality(Ref.11). This is done by solving the eigenvalue equation

$$(Y_1 + i\lambda Y_2)|\Psi> = \beta|\Psi> \quad , \qquad (2.5)$$

where λ is real and positive, and β is complex. The states $|\Psi>$ which satisfy this equation have the property that

$$(\Delta Y_1)^2 = \lambda <\Psi| N+\frac{1}{2}|\Psi> \qquad (\Delta Y_2)^2 = \frac{1}{\lambda} <\Psi| N+\frac{1}{2}|\Psi> \quad . \qquad (2.6)$$

From these equations it is clear that λ plays the role of a squeezing parameter. If $0<\lambda<1$, then Y_1 is squeezed and if $\lambda>1$, then Y_2 is squeezed. The real and imaginary parts of β are related to the mean values of Y_1 and Y_2, respectively.

A particularly simple subset of these minimum uncertainty states occurs when β

and λ are related. If $\lambda \geq 1$ and $\beta = (\lambda^2 - 1)^{1/2}(m+1/2)$, where m is a nonnegative integer, then the minimum uncertainty states are of the form

$$|\Psi\rangle = C_m(\lambda)\, S(\zeta)\, H_m(\, i\gamma(\lambda)\, a^+)\, |0\rangle \quad . \qquad (2.7)$$

Here $C_m(\lambda)$ is a normalization constant, $S(\zeta)$ is a squeeze operator where the squeeze parameter ζ depends on λ, H_m is the m^{th} Hermite polynomial and $\gamma(\lambda) = [(\lambda^2 - 1)^{1/2}/2\lambda\,]^{1/2}$. The cases m=0 and m=1 correspond to the squeezed vacuum and squeezed one-photon states, respectively. Note that this implies that the squeezed vacuum state is a minimum uncertainty state for both normal squeezing and for amplitude-squared squeezing.

A second kind of minimum uncertainty state is the amplitude-squared squeezed vacuum $|0,\lambda\rangle$. These states satisfy Eq.(2.5) with $\beta=0$ which implies that they have the property that $\langle Y_1\rangle = \langle Y_2\rangle = 0$. Such states are superpositions of photon number states whose numbers are multiples of 4.

We now come to the conversion of fluctuations in a^2 into fluctuations of the mode amplitude of a second mode, b. This is accomplished by means of second harmonic generation(Ref.5). If the mode described by a has frequency ω and that described by b has frequency 2ω then the Hamiltonian which corresponds to this process is

$$H = \omega a^+a + 2\omega b^+b + k_2(a^{+2}b + a^2b^+) \quad . \qquad (2.8)$$

From this Hamiltonian, using perturbation theory, one can find how fluctuations are transferred from mode a to mode b. First define the slowly varying operators

$$A(t) = e^{i\omega t}a(t) \qquad B(t) = e^{2i\omega t}b(t) \quad , \qquad (2.9)$$

and

$$X_{1B}(t) = \frac{1}{2}[\, B^+(t) + B(t)\,] \qquad X_{2B}(t) = \frac{i}{2}[\, B^+(t) - B(t)\,]$$

$$Y_{1A}(t) = \frac{1}{2}[\, A^+(t)^2 + A(t)^2\,] \qquad Y_{2A}(t) = \frac{i}{2}[\, A^+(t)^2 - A(t)^2\,] \quad . \qquad (2.10)$$

We then find, if the b mode is initially in a coherent state, that after a time t

$$(\Delta X_{1B}(t))^2 = \frac{1}{4} + (k_2t)^2[\,(\Delta Y_{2A})^2 - <N_A + \frac{1}{2}>]$$

$$(\Delta X_{2B}(t))^2 = \frac{1}{4} + (k_2t)^2[\,(\Delta Y_{1A})^2 - <N_A + \frac{1}{2}>] \quad , \tag{2.11}$$

where quantities without a time argument, e.g. $(\Delta Y_{1A})^2$, are assumed to be evaluated at $t=0$. What these equations tell us is that if the a mode is initially amplitude-squared squeezed in the Y_2 direction then the b mode will become squeezed in the normal sense in the X_1 direction. Similarly, if the a mode is amplitude-squared squeezed in the Y_1 direction the b mode will become squeezed in the X_2 direction. Therefore, the second harmonic generation process converts amplitude-squared squeezing into normal squeezing.

Because normal squeezing can be measured via homodyne detection the preceding results suggests how amplitude-squared squeezing can be detected. One first sends the signal into a frequency doubler and then measures the squeezing of the second harmonic. If it is squeezed, then the original signal was amplitude-squared squeezed.

Finally, let us see how amplitude-squared squeezed states can be produced. The fact that the squeezed vacuum state is also amplitude-squared squeezed shows that a degenerate parametric amplifier can produce amplitude-squared squeezed states. As one of us (D. Yu) has shown, a degenerate parametric oscillator can as well(Ref.12). Well above threshold the field inside the cavity can reach a maximum level of amplitude-squared squeezing given by $(\Delta Y_1)^2 / <N + 1/2> = 1/2$, but just below threshold the amount of amplitude-squared squeezing in the output field can, in principle, be arbitrarily large. The fourth subharmonic generation process, which has been studied in connection with generalized squeezed states(Ref.13), can also produce amplitude-squared squeezing(Ref.6).

III. Sum Squeezing

Sum squeezing, as opposed to amplitude-squared squeezing, is a two mode effect(Ref.14). In fact, amplitude-squared squeezing is the degenerate limit of sum squeezing. Let us consider two modes with annihilation operators a and b and frequencies ω_a and ω_b. The variables involved in sum squeezing are the real and imaginary parts of the product ab, i.e.

$$V_1 = \frac{1}{2}(a^+b^+ + ab) \qquad V_2 = \frac{i}{2}(a^+b^+ - ab) \quad . \tag{3.1}$$

The commutator of these operators is $[V_1, V_2] = \frac{i}{2} (N_A + N_B + 1)$, where $N_A = a^+a$ and $N_B = b^+b$, which yields the uncertainty relation

$$\Delta V_1 \Delta V_2 \geq \frac{1}{4} < N_A + N_B + 1 > \quad . \qquad (3.2)$$

A state is said to be sum squeezed in the V_1 direction if

$$(\Delta V_1)^2 < \frac{1}{4} <N_A + N_B + 1> \quad . \qquad (3.3)$$

Such a state is nonclassical.

The commutation relations of the operators V_1, V_2 amd $N_A + N_B + 1$ are also closely related to those of the SU(1,1) Lie algebra. In fact if one sets $K_1 = V_1$, $K_2 = - V_2$ and $K_3 = \frac{1}{2}(N_A + N_B + 1)$ one obtains the SU(1,1) commutation relations given in Eq.(2.4).

The name, sum squeezing, comes from the fact that this kind of squeezing is converted into normal squeezing by the process of sum frequency generation. Sum frequency generation is a three-mode process which is described by the Hamiltonian

$$H = \omega_a a^+a + \omega_b b^+b + \omega_c c^+c + k_s (ca^+b^+ + c^+ab) \quad , \qquad (3.4)$$

where $\omega_c = \omega_a + \omega_b$. As before we define the slowly varying operator $A(t) = e^{i\omega_a t}a(t)$, and similarly for $B(t)$ and $C(t)$. We also define

$$V_1(t) = \frac{1}{2}(A^+B^+ + AB) \qquad X_{C2} = \frac{i}{2}(C^+ - C) \quad . \qquad (3.5)$$

If the c mode is initially in a coherent state then to second order in k_s we find

$$(\Delta X_{C2}(t))^2 = \frac{1}{4} + (k_s t)^2 [(\Delta V_1)^2 - \frac{1}{4} <N_A + N_B + 1>] \quad , \qquad (3.6)$$

where, as before, quantities without a time argument are evaluated at $t=0$. Comparing this equation to Eq.(3.3) we see that the c mode will be squeezed in the X_{C2} direction if the a and b modes are sum squeezed in the V_1 direction.

If the a and b modes are uncorrelated, then there is a connection between squeezing in the individual modes and sum squeezing. In particular, if neither mode is squeezed, then the state is not sum squeezed. If one of the two modes is squeezed and the other is in a coherent state, then the state is sum squeezed. Finally, if both modes are squeezed, then the resulting state may or may not be sum squeezed.

This connection disappears if the modes are correlated. This can be seen by considering the state produced from the vacuum by a parametric amplifier. This system is described by the Hamiltonian

$$H = \omega_a a^+ a + \omega_b b^+ b + g\,(e^{-i\omega_c t} a^+ b^+ + e^{i\omega_c t} ab) \quad , \qquad (3.7)$$

where, again, $\omega_c = \omega_a + \omega_b$. This Hamiltonian is an approximation to that in Eq.(3.4) when the c mode is in a highly excited coherent state. Using this Hamiltonian we find that if both the a and b modes are originally in the vacuum state, then

$$(\Delta V_1(t))^2 - \frac{1}{4} <N_A(t) + N_B(t) + 1> = -\frac{1}{2}\sinh^2(gt) \quad . \qquad (3.8)$$

Therefore, the amount of sum squeezing increases with time and this device is a possible source of sum squeezed light. A further calculation shows that neither of the two modes is squeezed in the normal sense. Therefore, for correlated modes normal squeezing is not a prerequisite for sum squeezing.

IV. Difference Squeezing

Difference squeezing is also a two-mode effect(Ref.14). Its name comes from its close connection to difference-frequency generation. We again describe the modes by annihilation operators a and b , and we assume that $\omega_b > \omega_a$. The observables which describe it are

$$W_1 = \frac{1}{2}(ab^+ + a^+ b) \qquad W_2 = \frac{i}{2}(ab^+ - a^+ b) \quad . \qquad (4.1)$$

Their commutator is given by

$$[W_1, W_2] = \frac{i}{2}(N_A - N_B) \quad , \qquad (4.2)$$

which yields the uncertainty relation

$$\Delta W_1 \, \Delta W_2 \geq \frac{1}{4}|<N_A - N_B>| \quad . \qquad (4.3)$$

A state is said to be difference squeezed in the W_1 direction if

$$(\Delta W_1)^2 < \frac{1}{4}<N_A - N_B> \quad . \tag{4.4}$$

Note that for a state to be difference squeezed we must have $<N_A> \; > \; <N_B>$.

Difference squeezed states are nonclassical but there is a difference in this regard between them and sum or amplitude-squared squeezed states. For both of the latter, the condition for squeezing and the condition for being nonclassical are the same. For difference squeezing this is not true. A state is nonclassical if

$$(\Delta W_1)^2 < \frac{1}{4}<N_A + N_B> \quad , \tag{4.5}$$

which is not the same as the squeezing condition Eq.(4.4). Therefore, difference squeezed states are well within the nonclassical regime.

Difference squeezing is also related to a Lie algebra but this time it is SU(2) instead of SU(1,1). In fact, the operators which describe difference squeezing are those used in the Schwinger representation of the angular momentum operators (Ref.15). The SU(2) Lie algebra consists of three operators J_1, J_2 and J_3 whose commutation relations are

$$[J_k, J_l] = i\varepsilon_{klm}J_m \quad , \tag{4.6}$$

where all indices run from 1 to 3 and ε_{klm} is the completely antisymmetric tensor of rank 3.

If the modes are uncorrelated, then at least one of them must be squeezed for difference squeezing to be present. If the b mode is squeezed and the a mode is in a coherent state $|\alpha>$, the state wil be difference squeezed but only if $|\alpha|^2$ is large enough. A necessary, but not sufficient condition is that $<N_B> < \frac{1}{2}|\alpha|^2$. If the modes are correlated, then it is no longer true that squeezing in the individual modes is required for difference squeezing.

Finally, as might be suspected from the name, difference squeezing is turned into normal squeezing by difference frequency generation. The Hamiltonian describing this process is

$$H = \omega_a a^+ a + \omega_b b^+ b + \omega_c c^+ c + k_d(a^+ b c^+ + a b^+ c) \quad , \tag{4.7}$$

where $\omega_c = \omega_b - \omega_a$. We define slowly varying $A(t)$, $B(t)$, and $C(t)$ as in the previous section and then set

$$W_1(t) = \frac{1}{2}(A(t)B^+(t) + A^+(t)B(t)) \qquad X_{C2} = \frac{i}{2}(C^+(t) - C(t)) \quad . \tag{4.8}$$

Using perturbation theory we find that if the c mode is originally in a coherent state, then

$$(\Delta X_{C2}(t))^2 = \frac{1}{4} + (k_d t)^2[(\Delta W_1)^2 - \frac{1}{4}<N_A - N_B>] \quad . \tag{4.9}$$

This equation shows us that X_{C2} becomes squeezed if W_1 is difference squeezed. Therefore, difference frequency generation can be used to detect difference squeezed light.

V. Amplification of Higher-Order Squeezing

An amplifier consists of a collection of two-level atoms N_1 of which are in their ground states and N_2 of which are in their excited states where N_2 is greater than N_1. We shall assume that we are in the linear regime of this system. An input signal is put into the amplifier at $t=0$ and emerges at the output at time t. The signal amplitudes at the input and the output are related by $<a(t)> = G <a(0)>$ where G is the amplitude gain. This system was analyzed rather thoroughly by Carusotto(Ref.16).

Hong, Friberg, and Mandel examined the effect of amplification on sub-Poissonian photon statistics and normal squeezing(Ref.17). They found that both of these effects disappear at the output, no matter what the input state is, if the intensity gain, $|G|^2$, is greater than two. The gain $|G|^2 = 2$ is known as the photon cloning limit because one gets two photons out for every one that goes in. This gain has stood as an upper limit for the amplification of nonclassical behavior.

Recently two of us looked at the situation for amplitude-squared squeezing (Ref.18). We found that it can survive amplification for gains slightly greater than two. In particular, amplitude-squared squeezing will be present at the output if

$$|G|^2 < 2 + \frac{1}{<N_0> + 1/2} \quad , \tag{5.1}$$

where $<N_0>$ is the photon number of the input state. Because the right-hand side is greater than two this suggests that there are states which will still be amplitude-squared squeezed at the output if $|G|^2$ is slightly greater than two. Further investigation shows that the amplitude-squared squeezed vacuum, $|0,\lambda>$, with $\lambda<<1$ is such a state. Therefore, the photon cloning limit does not, at least in principle, represent a barrier to nonclassical behavior. It would be of considerable interest to know if there are nonclassical states which can remain nonclassical when they are amplified at gains substantially larger than two.

VI. Conclusion

Quadratic squeezing represents a new class of nonclassical effects. States with this property have fluctuations smaller than is possible for classical light in a variable which is quadratic in mode creation and annihilation operators. As we have seen, quadratic squeezing can be converted into normal squeezing by $\chi^{(2)}$ type nonlinear interactions.

A direction for further investigations into quadratic squeezing is its connection to interferometry. Interferometers, both with and without nonlinear elements, can be described in a natural fashion in terms of the variables which describe quadratic squeezing(Ref.19). This suggests that interferometers can be used to measure quadratic squeeezing and that quadratic squeezed states may be of use in interferometric measurements. We are currently studying these issues.

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