IDEAL PHOTON NUMBER AMPLIFIER AND DUPLICATOR

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ABSTRACT: The photon number-amplification and number-duplication mechanisms are analyzed in the ideal case. The search for unitary evolutions leads to consider also a number-deamplification mechanism, the symmetry between amplification and deamplification being broken by the integer-valued nature of the number operator. Both transformations—amplification and duplication—need an auxiliary field which, in the case of amplification, turns out to be amplified in the inverse way. Input-output energy conservation is accounted for using a classical pump or through frequency-conversion of the fields. Ignoring one of the fields is equivalent to consider the amplifier as an open system involving entropy production. The Hamiltonians of the ideal devices are given and compared with those of realistic systems.

1. INTRODUCTION

Squeezing and amplification are two intimately related concepts. A scaling of the quantum fluctuations $\langle \Delta \hat{O}^2 \rangle \rightarrow G^2 \langle \Delta \hat{O}^2 \rangle$, independently on the state of the field, corresponds to the amplification of the fluctuating observable $\hat{O} \rightarrow G\hat{O}$. Such kind of ideal quantum amplification rescales all the moments of $\hat{O}$ simultaneously, leaving the signal-to-noise ratio (SNR) unchanged when detecting $\hat{O}$.

Ideal quantum amplifiers are key-devices in quantum optical applications, where, depending on the particular circumstances, one would possibly change the levels of both signal and fluctuations without degrading the SNR. For example, in local-area network (LAN) communications, strongly subpoissonian fields with limited average number of photons are needed to exploit the ultimate channel capacity of the field (which is constrained in the total power and the bandwidth). On the other hand, a large signal is preferred just before the detection stage, in order to minimize all the subsequent sources of disturbance. In both cases an ideal photon number-amplifier ($\hat{O} \equiv \hat{n}$) would allow to change signal and fluctuations as desired, leading to significant improvements of the network performance.

Another point which should be considered in any quantum amplification process is the role played by the Heisenberg principle in defining the ideal behaviour of the amplifier. In fact, the amplification of the observable $\hat{O}$ affects the statistics of the observables which do not commute with $\hat{O}$. For a couple of conjugated variables $(\hat{O}_1, \hat{O}_2)$, analogous to the momentum and the

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position of a particle, the quantum fluctuations are constrained by the uncertainty relation

$$\langle \Delta \hat{O}_1^2 \rangle \langle \Delta \hat{O}_2^2 \rangle \geq \frac{1}{4} \langle [\hat{O}_1, \hat{O}_2] \rangle^2 \,, \tag{1}$$

according to which, when the $\hat{O}_1$ fluctuations are rescaled as $\langle \Delta \hat{O}_1^2 \rangle \to G^2 \langle \Delta \hat{O}_1^2 \rangle$, the corresponding $\hat{O}_2$ fluctuations become $G^{-2} \langle \Delta \hat{O}_2^2 \rangle$ or larger. An ideal $\hat{O}_1$ amplifier, namely an amplifier performing at best, should preserve the minimum uncertainty product and, as a consequence, it should simultaneously attain the two opposite amplifications

$$\hat{O}_1 \to G \hat{O}_1 \,, \quad \hat{O}_2 \to G^{-1} \hat{O}_2 \,. \tag{2}$$

Depending on the conjugated pair $(\hat{O}_1, \hat{O}_2)$ one has different kind of amplifiers and related different kind of squeezing. For example, when the conjugated variables are two quadrature of a field mode $(\hat{a}_1, \hat{a}_2)$—$\hat{a}_1 + i\hat{a}_2 = a$ being the annihilation operator—the rescaling (2) defines the phase sensitive linear amplifier (PSA). The ideal PSA (essentially a parametric amplifier) preserves the homodyne SNR= $\langle \hat{a}_1 \rangle / \sqrt{\langle \Delta \hat{a}_1^2 \rangle}$ and produces the squeezing in a quadrature of the field. In this case the transformation (2) is realized by the Yuen's[1] unitary evolution $\hat{U}^\dagger a \hat{U} = \mu a + \nu a^\dagger$, with $\mu = (G + G^{-1})/2$ and $\nu = (G - G^{-1})/2$, $\hat{U}$ representing the usual squeezing operator.

The photon number-amplifier (PNA) is another example of ideal amplifier, which would transform ideally $\hat{n}$ into $G\hat{n}$, preserving the direct detection SNR and the number-phase uncertainty product. The corresponding kind of squeezing is the number-phase squeezing[2] (or amplitude squeezing), in which the quantum noise is shared between the number $\hat{n}$ and the phase $\hat{\Phi}$. This kind of amplifier is a relatively new concept and is probably not simple to realize concretely: it has been proposed by Yuen[3-5], who also suggested physically viable approximate schemes based on resonance fluorescence. PNA's would be particularly useful in direct-detection receiver and transceiver in a LAN environment, where, as already mentioned, number states are preferred to coherent or squeezed states, in order to achieve the ultimate channel capacity of the field. Furthermore, a PNA (but also a PSA) can be used to realize a lossless optical tap, which, in a LAN would enable a very large number of users to obtain the same performance as the first user.[6]

In this paper the number-phase amplification mechanism is analyzed in detail, in order to find physical schemes for an ideal PNA. It is shown that, due to the peculiar role of the two conjugated variables $(\hat{n}, \hat{\Phi})$ in the Fock representation, the requirement of a unitary transformation leads to consider a second field in addition to the amplified one, the two fields being inversely amplified by the transformation. Input-output energy conservation can be accounted for either by adding a suited classical pump or by locking the frequencies of the two fields, attaining simultaneously number-amplification and frequency-conversion. The obvious constraint of integer gain G (preserving the integer-valued nature of $\hat{n}$) must be relaxed, to consider the deamplification case: as a consequence, the abstract amplifying transformation $|n\rangle \to |Gn\rangle$ is replaced by $|n\rangle \to |[Gn] + n_0\rangle$, $[x]$ denoting the integer part of x, and n_0 being a constant as a function of n and depending on the input state of the other field. The general Schrödinger evolution of the two fields is

$$\hat{U}_{(G^{-1})} |n, m\rangle = |[Gn] + G\langle G^{-1}m\rangle, [G^{-1}m] + G^{-1}\langle Gn\rangle\rangle \,, \tag{3}$$

$\langle x \rangle = x - [x]$ denoting the fractional part of x and the gain G being restricted to be either integer or the inverse of an integer. Eq.(3) can be attained by means of a unitary transformation involving a classical pump operating at the frequency $\Omega = G^{-1}\omega_a - \omega_b$, ω_a and ω_b being the frequencies of the G-amplified and G-deamplified fields respectively. In the case of simultaneous amplification/frequency-conversion one has the resonance condition $\omega_b = G^{-1}\omega_a$, and the two fields are intertwined in (3) in order to preserve the total input energy $E = (n + G^{-1}m)\omega_a$.

In Sect.2 I derive the transformation (3) and the related Hamiltonian. Apart from the eventual classical pump, the ideal PNA in the present framework is a four-port nonlinear device (see Fig.1). However, it can be regarded as a two port device by fixing one input port state (for example,
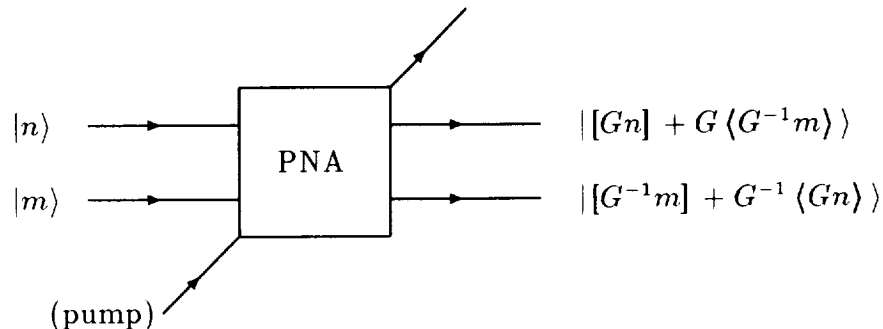

Figure 1: Scheme of the ideal PNA.

using the vacuum) or by totally ignoring one field. In the last case the PNA should be regarded as an open quantum system which changes the entropy of the input state of the field: the particular case of coherent inputs will be examined in this respect.

In Sect.3 another device analogous to the PNA is analyzed, namely the ideal photon number-duplicator (PND). Instead of amplifying the number of photons, the ideal PND produces two copies of the same input state for eigenstates of the number operator. Such a device would be extremely useful in LAN applications, because it provides a convenient realization of the quantum nondemolition measurement of the photon number, beside itself realizing lossless optical taps superior to the amplifier tap[6] (both applications make possible sharing of information in a LAN). Arguments related to unitarity—similar to those used for the ideal PNA—lead to the need of a third auxiliary field, whereas input-output energy conservation can be taken into account either by means of a classical pump or through frequency-conversion, in a way completely analogous to the case of the ideal PNA.

In the last section I make some preliminary comparisons between the Hamiltonians of the ideal devices and those of realistic systems, focusing attention on the gain-2 amplifier, in some respects very similar to the duplicating device.

2. IDEAL PHOTON NUMBER-AMPLIFIER

2.1 The unitary transformation

In the Heisenberg picture the ideal PNA corresponds to a multiplication of the number operator $\hat{n}$ by the amplification factor G

$$\hat{n} \longrightarrow G\hat{n} . \tag{4}$$

The requirement of an ideal—i.e. minimum-uncertainty preserving—behaviour reflects on the Heisenberg transformations for the phase operator $\hat{\Phi}$, which should be the inverse of (4), namely

$$\hat{\Phi} \longrightarrow G^{-1}\hat{\Phi} \ . \tag{5}$$

For highly excited states (i.e. states approximately orthogonal to the vacuum $|0\rangle$) and for small phase uncertainties $\langle \Delta\hat{\Phi}^2 \rangle \ll 1$ the following simple definition for the phase can be adopted[7]

$$\hat{E}_\pm = e^{\mp i\hat{\Phi}} \ , \tag{6}$$

$\hat{E}_\pm$ denoting the shift operators $\hat{E}_- = (a^\dagger a + 1)^{-1/2}a$, $\hat{E}_+ = (\hat{E}_-)^\dagger$ $(\hat{E}_\pm|n\rangle = |n \pm 1\rangle)$. Eq.(6) shows how the integer-valued nature of $\hat{n}$ reflects on the phase operator $\hat{\Phi}$: the amplification (5) can simply be attained for $G^{-1} = r$ integer, raising the shift operators to the power r, whereas, for the number operator, preservation of its integer-valued nature requires G itself to be integer. For noninteger G, the transformation (4) can be substituted with the following

$$\hat{n} \longrightarrow [G\hat{n}] \ , \tag{7}$$

which coincides with (4) for integer G. For the moment, I focus attention on the deamplification case ($G^{-1} = r$ integer), the integer-G case being naturally contained in the following framework.

Denoting by $\mathcal{S}_H^{(r)}$ the Heisenberg transformation corresponding to Eqs.(7) and (5), for integer $G^{-1} = r$ one has

$$\mathcal{S}_H^{(r)}(\hat{E}_\pm) = (\hat{E}_\pm)^r \ , \tag{8}$$

$(\hat{E}_\pm)^r$ now being represented on the Fock space as follows:

$$(\hat{E}_\pm)^r|n\rangle = |n \pm r\rangle \ . \tag{9}$$

From Eqs.(8) and (9) it turns out that the $\mathcal{S}_H^{(r)}$ acting on a generic operator $\hat{O}$ has the general form

$$\mathcal{S}_H^{(r)}(\hat{O}) = \sum_{\lambda=0}^{r-1}(\hat{S}_\lambda^{(r)})^\dagger \hat{O}\hat{S}_\lambda^{(r)} \ , \quad \hat{S}_\lambda^{(r)} = e^{i\phi_\lambda}\sum_{n=0}^{\infty}|n\rangle\langle nr + \lambda| \ , \tag{10}$$

and the phase factors, being totally ineffective in the action (10), will be dropped in the following. One can check that the Heisenberg transformation (8) attains the number-amplification (7)

$$\mathcal{S}_H^{(r)}(\hat{n}) = [\hat{n}/r] \ , \tag{11}$$

and, formally, $\mathcal{S}_H^{(r)}$ achieves the phase amplification (5) according to Eq.(6). The transformation (8) is not unitary and, as a consequence, there is no Hamiltonian producing it. The operators $\hat{S}_\lambda^{(r)}$ in the definition of the map (10) satisfy the following relations

$$\sum_{\lambda=0}^{r-1}(\hat{S}_\lambda^{(r)})^\dagger \hat{S}_\lambda^{(r)} = 1 \ , \tag{12}$$

$$\hat{S}_\lambda^{(r)}(\hat{S}_\mu^{(r)})^\dagger = \delta_{\lambda\mu} \ , \tag{13}$$

$$\hat{S}_\lambda^{(r)}\hat{S}_\mu^{(s)} = \hat{S}_{\lambda s+\mu}^{(rs)} \ . \tag{14}$$

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Despite the map $\mathcal{S}_H^{(r)}$ is not unitary, the completeness relation (12) and the orthogonality conditions (13) allow one to recover a unitary evolution on an enlarged quantum system. I postpone for the moment the construction of the corresponding unitary evolution and the related Hamiltonians, to continue the discussion on the properties of the map.

Eq.(14) leads to semigroup composition of the maps $\mathcal{S}_H^{(r)}$

$$\mathcal{S}_H^{(r)}\mathcal{S}_H^{(s)} = \mathcal{S}_H^{(rs)} \tag{15}$$

corresponding to the amplification of PNA's in series. On the other hand, as a consequence of the completeness and orthogonality relations (12) and (13), $\mathcal{S}_H^{(r)}$ preserves the operator products and the adjoint operation, thus transforming consistently the whole operator algebra. When applied on the particle operator a the transformation $\mathcal{S}_H^{(r)}$ gives the result

$$\mathcal{S}_H^{(r)}(a) = \sum_{n=1}^{\infty} |n-1\rangle\sqrt{[n/r]}\langle n| = \left[\frac{(1+[\hat{n}/r])\hat{n}!}{(\hat{n}+r)!}\right]^{1/2} a^r \equiv a_{(r)} , \tag{16}$$

and for the creation operator one has $\mathcal{S}_H^{(r)}(a^\dagger) = a_{(r)}^\dagger$. Eq.(16) shows that the transformed particle operator $\mathcal{S}_H^{(r)}(a)$ coincides with the r-boson operator $a_{(r)}$ introduced by Brandt and Greenberg:[10] $a_{(r)}$ and $a_{(r)}^\dagger$ annihilate and create r photons simultaneously and satisfy the commutation relations $[a_{(r)}, a_{(r)}^\dagger] = 1$, $[\hat{n}, a_{(r)}] = -ra_{(r)}$. The preservation of the operator product implies that the transformation $\mathcal{S}_H^{(r)}$ applied to a generic operator $\hat{O} = \hat{O}(a, a^\dagger)$ (Hermitian analytic function of a and $a^\dagger$) can simply be obtained substituting a and $a^\dagger$ with $a_{(r)}$ and $a_{(r)}^\dagger$, i.e. $\mathcal{S}_H^{(r)}(\hat{O}) = \hat{O}(a_{(r)}, a_{(r)}^\dagger)$. Therefore, $\mathcal{S}_H^{(r)}$ corresponds to the construction of the r-photon observables of D'Ariano.[11,12]

The completeness and orthogonality relations (12) and (13) are preserved by similarity transformations

$$\hat{S}_\lambda^{(r)\prime} = \hat{V}\hat{S}_\lambda^{(r)}\hat{W} , \tag{17}$$

$\hat{V}$ and $\hat{W}$ being unitary operators. A general transformation (17), however, would destroy, the ideal behaviour of the PNA: the only similarity transformations which preserve the Heisenberg evolutions (7) and (5) are the permutations of the λ's

$$\hat{S}_\lambda^{(r)\prime} = \hat{P}\hat{S}_\lambda^{(r)} = \hat{S}_{P(\lambda)}^{(r)} , \tag{18}$$

where $\hat{P}$ denotes the operator representing a permutation of the λ's, namely $\hat{P}|nr + \lambda\rangle = |nr + P(\lambda)\rangle$.

I return now to the construction of the unitary evolution corresponding to $\mathcal{S}_H^{(r)}$. From the definition (10) and the completeness relation (12) it follows that $\mathcal{S}_H^{(r)}$ is a *completely positive map* (shortly CP map):[8,9] this is physically relevant, because the subdynamics of the open systems are CP maps, the set of CP maps being closed under partial trace. Here I recall that a unit-preserving CP map has the general form $\mathcal{T}(\hat{O}) = \sum_\alpha \hat{V}_\alpha^\dagger \hat{O}\hat{V}_\alpha$, where $\sum_\alpha \hat{V}_\alpha^\dagger \hat{V}_\alpha = 1$. The space of the unit-preserving CP maps is closed under: i) convex combination $\sum_i p_i\mathcal{T}_i$; ii) composition $\mathcal{T}_1\mathcal{T}_2$; iii) tensor product $\mathcal{T}_1 \otimes \mathcal{T}_2$; iv) partial trace: namely, if $\mathcal{T}$ is CP on $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $\hat{\mu}$ is a

density operator on $\mathcal{F}_2$, then $\mathcal{T}_1(\hat{O}) = \text{Tr}_2[\hat{\mu}\mathcal{T}(\hat{O} \otimes \hat{1})]$ is CP on $\mathcal{F}_1$. The last point means that if we have a unitary evolution in a closed system and if subdynamics on a (open) subsystem can be defined—i.e. partial trace on the subsystem degrees of freedom—then these subdynamics are CP maps. However, the converse is not true in general (namely not every CP map corresponds to a unitary evolution on an enlarged system) and the additional orthogonality constraints (13) are essential in guaranteeing also the converse assertion.

The unitary evolution corresponding to $\mathcal{S}_H^{(r)}$ can be constructed by using two different photon fields in the amplification process and considering the following operator $\hat{U}_{(r)}$ acting on the Fock space of the composite system $\mathcal{F} \otimes \mathcal{F}$

$$\hat{U}_{(r)} = \sum_{\lambda=0}^{r-1} \hat{S}_\lambda^{(r)} \otimes (\hat{R}_\lambda^{(r)})^\dagger , \tag{19}$$

where $\hat{R}_\lambda^{(r)}$ are similar to $\hat{S}_\lambda^{(r)}$ in the sense of Eq.(17). $\hat{U}_{(r)}$-unitarity follows from Eqs.(12), (13) and (17). The subdynamics of the first photon field correspond to $\mathcal{S}_H^{(r)}$:

$$\langle \hat{U}_{(r)}^\dagger \hat{O}_1 \hat{U}_{(r)} \rangle = \text{Tr} \left[(\hat{\rho}_1 \otimes \hat{\rho}_2) \hat{U}_{(r)}^\dagger (\hat{O}_1 \otimes \hat{1}) \hat{U}_{(r)} \right] = \text{Tr}_1 \left[\hat{\rho}_1 \mathcal{S}_H^{(r)}(\hat{O}_1) \right] , \tag{20}$$

where the uncorrelated pair of states $(\hat{\rho}_1 \otimes \hat{\rho}_2)$ has been considered as the input of the amplifier. The semigroup property (14) reflects on the composition law for the operators $\hat{U}_{(r)}$

$$\hat{U}_{(r)} \hat{U}_{(s)} \simeq_P \hat{U}_{(rs)} , \tag{21}$$

the symbol $\simeq_P$ denoting similarity under permutations (18).

Among all operators $\hat{U}_{(r)}$ the case of $\hat{R}_\lambda^{(r)} = \hat{S}_\lambda^{(r)}$ is particularly interesting, because the second field undergoes the transposed transformation of $\mathcal{S}_H^{(r)}$

$$\hat{U}_{(r)} = \sum_{\lambda=0}^{r-1} \hat{S}_\lambda^{(r)} \otimes (\hat{S}_\lambda^{(r)})^\dagger . \tag{22}$$

One should notice, however, that $\langle \hat{U}_{(r)}^\dagger \hat{O}_2 \hat{U}_{(r)} \rangle$ depends on the first input state $\hat{\rho}_1$ in general. In fact, the action of the operator $\hat{U}_{(r)}$ in Eq.(22) on a number eigenstate is

$$\hat{U}_{(r)}|n,m\rangle = |\,[n/r], mr + \langle n/r \rangle\,\rangle , \tag{23}$$

and the second field undergoes an exact number-amplification only if the first field is in a r-photon state, (namely it contains only number of photons multiple of r), in particular if it is in the vacuum state. Eq.(23) can be rewritten in the following more symmetrical form

$$\hat{U}_{(G^{-1})}|n,m\rangle = |\,[Gn] + G\langle G^{-1}m \rangle, [G^{-1}m] + G^{-1}\langle Gn \rangle\,\rangle , \tag{24}$$

which coincides with Eq.(23) for $G^{-1} = r$ integer, whereas, for G integer, corresponds to (23) but with the roles of the two fields interchanged, as a consequence of the identity $\hat{U}_{(G)}^\dagger = \hat{U}_{(G)}^{-1} = \hat{U}_{(G^{-1})}$ (notice that Eq.(24) leads to integer valued number of photons only if either G or G^{-1} is integer).

2.2 *The Hamiltonian*

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I consider the operator $\hat{U}_{(G^{-1})}$ in Eq.(23) only for the case $G^{-1} = r$ integer, the integer-G case $\hat{U}_{(G)}$ corresponding to the inverse operator $\hat{U}_{(G^{-1})}^{\dagger}$. I denote by $a^{\dagger}$ and $b^{\dagger}$ the particle operators of the two fields, namely

$$|n, m\rangle = \frac{(a^{\dagger})^n}{\sqrt{n!}} \frac{(b^{\dagger})^m}{\sqrt{m!}} |0, 0\rangle .$$

(25)

Comparing the transformation (24) with the action of the multiboson operator of Eq. (16)

$$a_{(r)}^{\dagger}|n\rangle = \sqrt{[n/r] + 1}|n + r\rangle ,$$

(26)

one can see that the ideal amplification (23) can be attained by interchanging $a^{\dagger}$ with $b_{(r)}^{\dagger}$ and then permuting $a^{\dagger}$ with $b^{\dagger}$ modes. The operator permuting $a^{\dagger}$ and $b^{\dagger}$ (apart from a sign) has the form

$$\hat{V} = \exp\left[\frac{\pi}{2}\left(a^{\dagger}b - b^{\dagger}a\right)\right]$$

(27)

and, as a consequence, the operator $\hat{U}_{(r)}$ is given by

$$\hat{U}_{(r)} = \exp\left[\frac{\pi}{2}\left(a^{\dagger}b - b^{\dagger}a\right)\right]\exp\left[-\frac{\pi}{2}\left(a_{(r)}^{\dagger}b - b^{\dagger}a_{(r)}\right)\right] .$$

(28)

The representation (23) of the operator $\hat{U}_{(r)}$ in Eq. (28) can directly be checked using Eqs.(25) and (26). From Eq.(28) one can see that, apart from a permutation of the form (27) (which could be attained by means of beam splitters), the ideal PNA is described by an interaction Hamiltonian in the Dirac picture

$$\hat{H}_I^P = -ik\left(a_{(r)}^{\dagger}b - b^{\dagger}a_{(r)}\right) ,$$

(29)

and an interaction length L given by

$$kL = \frac{\pi}{2} ,$$

(30)

k being the interaction coupling per unit length. Using Eq.(16) the Hamiltonian (29) can be rewritten in terms of the a and b particle operators

$$\hat{H}_I^P = -ik\left((a^{\dagger})^r f_{(r)}(a^{\dagger}a)b - b^{\dagger}f_{(r)}(a^{\dagger}a)a^r\right) ,$$

(31)

$$f_{(r)}(a^{\dagger}a) = \left[\frac{1 + [a^{\dagger}a/r]}{(a^{\dagger}a + 1)\ldots(a^{\dagger}a + r)}\right]^{1/2} .$$

Regarding the energy conservation during the amplification process, one can now thinks to the four-port device as a globally inelastic process in the time domain (Dirac picture), with the free Hamiltonian

$$\hat{H}_0 = \omega_a a^{\dagger}a + \omega_b b^{\dagger}b .$$

(32)

The interaction Hamiltonian $\hat{H}_I^P$ has the form (29) when in the Schrödinger picture reads

$$\hat{H}_I = -ik \left(a_{(r)}^{\dagger} b e^{-i\Omega t} - b^{\dagger} a_{(r)} e^{i\Omega t} \right) \tag{33}$$

with $\Omega = r\omega_a - \omega_b$. As for the usual parametric frequency-converter the phase factor oscillating at frequency Ω can be attained by considering an additional classical (i.e. highly excited) pumping field. On the other hand, also the first permuting operator in (28) requires a classical pump (at frequency $\Omega' = \omega_a - \omega_b$) and it can be attained by means of beam splitters if $\omega_a = \omega_b$, otherwise it corresponds to a parametric frequency-converter.

The case $\omega_b = r\omega_a$ requires no pump at the second step in (28). In this case, the second operator in (28) can be reviewed as a PNA/frequency-converter (PNAFC) described by the equations

$$|n\rangle \rightarrow \big|\, [G^{-1}m] + G^{-1}\langle Gn\rangle \big\rangle , \qquad (\omega = \omega_a) ,$$

$$\tag{34}$$

$$|m\rangle \rightarrow \big|\, [Gn] + G\langle G^{-1}m\rangle \big\rangle , \qquad (\omega = \omega_b = r\omega_a) ,$$

The PNAFC conserves the total input-output energy $E = (n + rm)\omega_a$, as it follows from Eq.(34). On the other hand, for $\omega_b = r\omega_a$ the first permuting operator in Eq.(28) needs a pump at frequency $\Omega' = (r - 1)\omega_a$ and represents now a parametric frequency-converter (FC) intertwining the two amplified modes. In this fashion the ideal PNA can be viewed as the cascade of an ideal PNAFC (an energy preserving four-port device) followed by an ideal FC (a four port device with pump).

2.3 *The PNA as an open system: the amplification entropy*

In practical applications it is useful to consider the ideal PNA as a two-port device, actually ignoring not only the pumping field, but also one of the two amplified fields. This description is equivalent to consider the PNA as an open quantum system, which no longer preserves both the energy and the entropy of the input field. However, the amplification and the deamplification cases now become quite different. This follows from the unitary transformation (24) where, despite the apparent symmetrical roles of the two fields, the state of the amplified one depends on the state of the other, whereas the deamplified field is always independent on the amplified one. This strange unilateral dependence is due to the integer-valued nature of $\hat{n}$, that breaks the symmetry between amplification and deamplification. In the following I examine the two cases separately.

The number-amplification—ignoring the deamplified field—corresponds to the partial trace

$$\langle \hat{O}_2 \rangle = \mathrm{Tr}\left[(\hat{\rho}_1 \otimes \hat{\rho}_2)\hat{U}_{(r)}^{\dagger}(\hat{1} \otimes \hat{O}_2)\hat{U}_{(r)} \right] = \mathrm{Tr}_2 \left(\hat{\rho}_2 \sum_{\lambda=0}^{r-1} (\hat{V}_{\lambda}^{(r)})^{\dagger} \hat{O}_2 \hat{V}_{\lambda}^{(r)} \right) , \tag{35}$$

where

$$\hat{V}_{\lambda}^{(r)} = \left\{ \mathrm{Tr}_1 \left[\hat{\rho}_1 (\hat{S}_{\lambda}^{(r)})^{\dagger} \hat{S}_{\lambda}^{(r)} \right] \right\}^{1/2} (\hat{S}_{\lambda}^{(r)})^{\dagger} . \tag{36}$$

Therefore, the amplification corresponds to the CP map

$$\mathcal{S}_H^{(1/r)}(\hat{O}) = \sum_{\lambda=0}^{r-1} (\hat{V}_{\lambda}^{(r)})^{\dagger} \hat{O} \hat{V}_{\lambda}^{(r)} , \tag{37}$$

which, due to the form of operators $\hat{V}_\lambda^{(r)}$ in Eq.(36), depends on the state $\hat{\rho}_1$ of the other field, (namely on the 'temperature' of the PNA). The case of $\hat{\rho}_1$ equal to the vacuum state ('zero temperature') is particularly simple

$$\mathcal{S}_H^{(1/r)}(\hat{O}) = (\hat{V}_0^{(r)})^\dagger \hat{O} \hat{V}_0^{(r)} \,, \quad \hat{V}_0^{(r)} \equiv (\hat{S}_0^{(r)})^\dagger = \sum_{n=0}^\infty |rn\rangle\langle n| \,, \tag{38}$$

and corresponds to the exact number-amplification

$$\mathcal{S}_H^{(1/r)}(f(\hat{n})) = f(r\hat{n}) \,. \tag{39}$$

In the Schrödinger picture one has

$$\mathcal{S}_S^{(1/r)}(\hat{\rho}) = \hat{V}_0^{(r)} \hat{\rho} (\hat{V}_0^{(r)})^\dagger \,. \tag{40}$$

Despite the evolution (40) is not unitary (it is only an isometry), it preserves the Newmann-Shannon entropy

$$S(\hat{\rho}) = -\mathrm{Tr}\,\hat{\rho}\log\hat{\rho} \,. \tag{41}$$

The entropy conservation follows from the orthogonality conditions (13) which imply that $(\hat{V}_0^{(r)})^\dagger\hat{V}_0^{(r)} = 1$ (but $\hat{V}_0^{(r)}(\hat{V}_0^{(r)})^\dagger \neq 1$). Thus, the physical picture of the abstract number-amplification $|n\rangle \rightarrow |rn\rangle$ corresponds to an ideal PNA operating with the auxiliary field in the vacuum (namely a PNA at zero temperature). As long as the number-amplification is attained exactly, no entropy change of the field occurs.
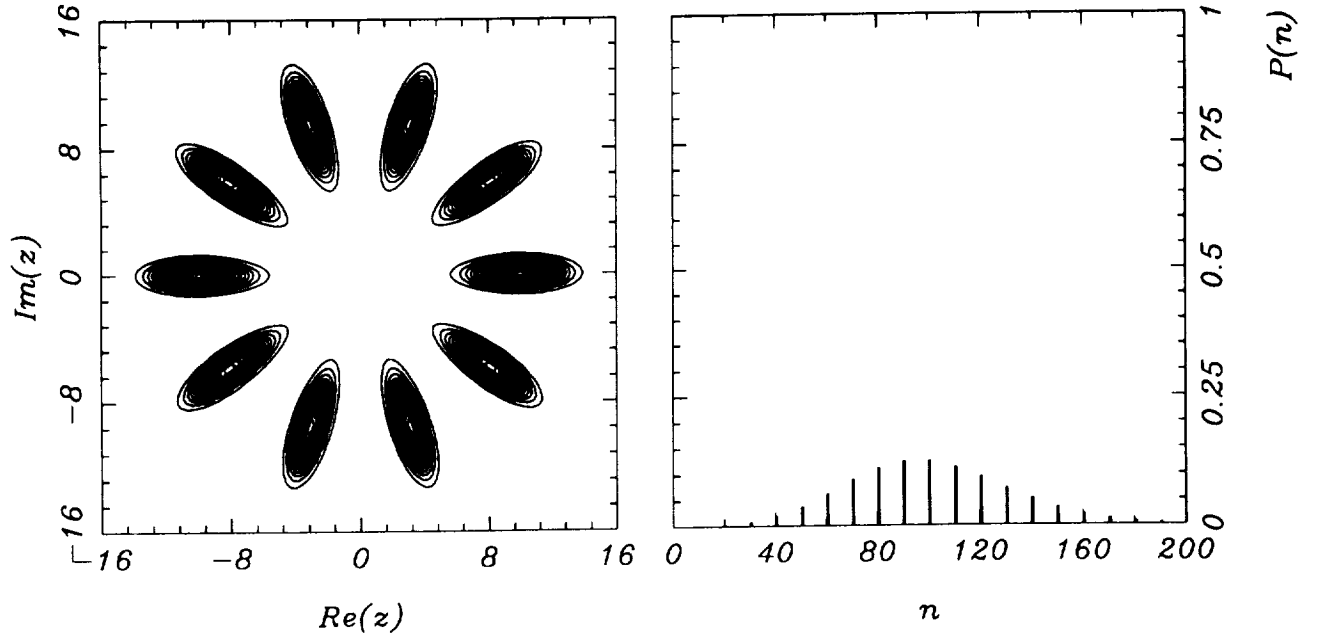

Figure 2: Q-function and number distribution of a coherent state having $\langle \hat{n} \rangle = 10$ photons after number-amplification with $G = 10$. The final moments are $\langle \hat{n} \rangle = 100$ and $\langle \Delta\hat{n}^2 \rangle^{1/2} = 31.52$.

In Fig.2 the effect of exact number-amplification on an input coherent state is illustrated in both the Q-function and number representations. The amplified coherent state

$$|\alpha\rangle_{(r)} = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|rn\rangle \tag{42}$$

has a Poisson distribution of multiples of r photons. In the phase space the exact number-amplification corresponds to a symmetrical split of the quantum distribution into r identical replicas having fluctuations enhanced in the number and reduced in the phase (the quantum distribution becomes longer in the radial direction and narrower in the tangential one).

I now examine the case of number-deamplification. Ignoring the amplified field corresponds to the partial trace

$$\langle \hat{O}_1 \rangle = \mathrm{Tr}\left[(\hat{\rho}_1 \otimes \hat{\rho}_2)\hat{U}_{(r)}^{\dagger}(\hat{O}_1 \otimes \hat{1})\hat{U}_{(r)}\right] = \mathrm{Tr}_1\left(\hat{\rho}_1 \sum_{\lambda=0}^{r-1}(\hat{S}_{\lambda}^{(r)})^{\dagger}\hat{O}_1\hat{S}_{\lambda}^{(r)}\right) . \tag{43}$$

The deamplification thus corresponds to the CP map (10)

$$\mathcal{S}_H^{(r)}(\hat{O}) = \sum_{\lambda=0}^{r-1}(\hat{S}_{\lambda}^{(r)})^{\dagger}\hat{O}\hat{S}_{\lambda}^{(r)} , \tag{44}$$

and is totally independent on the state of the amplified field. In the Schrödinger picture one has

$$\mathcal{S}_S^{(r)}(\hat{\rho}) = \sum_{\lambda=0}^{r-1}\hat{S}_{\lambda}^{(r)}\hat{\rho}(\hat{S}_{\lambda}^{(r)})^{\dagger} , \tag{45}$$

namely, the number-deamplification corresponds to an isometric evolution which does not preserve the entropy (41) in general. The entropy change depends only on the gain $G^{-1} = r$ and on the input state of the deamplified field (and not on the other field). It is worth noticing that the entropy during deamplification can either increase or decrease as a function of r. As the photon number-deamplification leads to the vacuum state for $r = G^{-1} \to \infty$, the entropy is asymptotically a decreasing function of r for large r. On the other hand, the evolution (45) would in general transform a pure state into a mixed one (the only state which are left pure being the number eigenstates and the r-photon states), and thus leads to an increase of entropy in this case. Therefore, when a pure state is number-deamplified, the entropy exhibits at least one maximum as a function of the inverse gain r. In Fig.3 the Newmann-Shannon entropy (41) is plotted as a function of r, for two different input coherent states. One can see that for small average number of input photons the entropy has only one maximum, whereas for intense input fields several maxima appear and local very low minima can occur (corresponding to almost pure states). As a rule, for coherent inputs the maxima are located approximately at $r \simeq |\alpha|^2/l$—$|\alpha|^2$ being the average number of input photons and $l = 1, 2, \ldots$—the maxima weakening for increasing l and the entropy S being always smaller than $\log 2$, which is the entropy of two pure states mixing.

In Fig.4 the Q-function and the number distribution of a strongly deamplified coherent state are reported, in order to illustrate the number-phase squeezing related to number-deamplification. In a fashion which is exactly the opposite of that depicted in Fig.2 the number-deamplification leads to spreading in the phase and narrowing in the number, thus converting highly excited
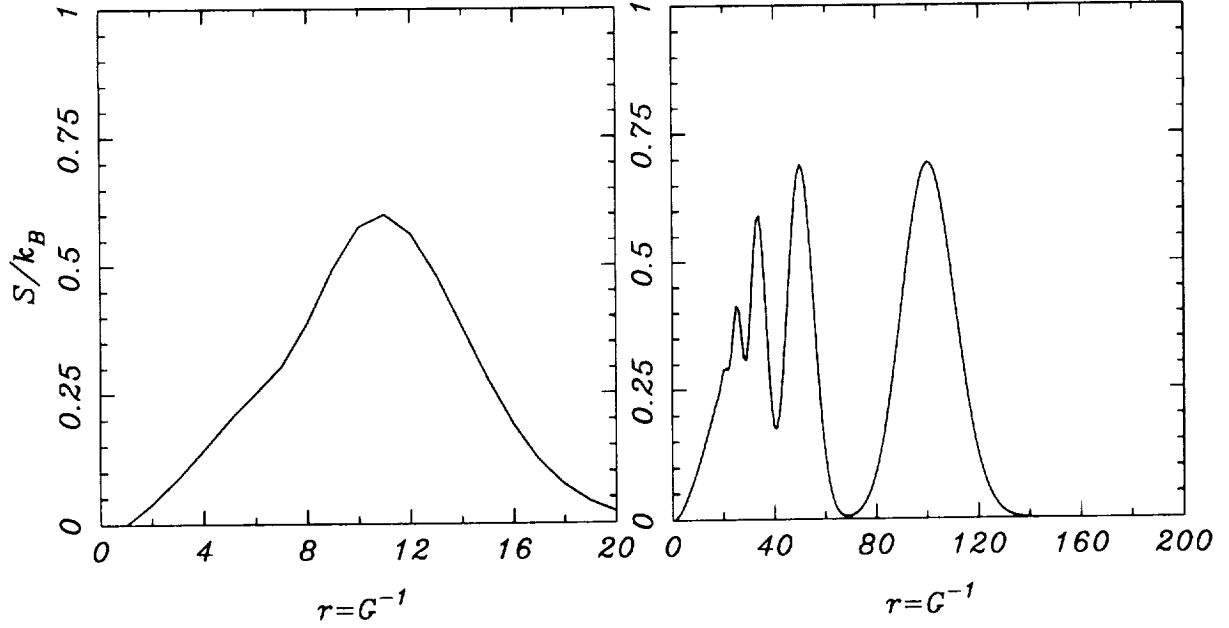
320

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Figure 3: Deamplification Newmann-Shannon entropy versus the inverse gain $r = G^{-1}$ for a coherent state with $\langle \hat{n} \rangle = 10$ photons (figure on the left) and $\langle \hat{n} \rangle = 100$ photons.

states into nearly Fock states. Asymptotic evaluations[12] for $G \to 0$, keeping constant the average number of output photons, leads to $\langle \hat{n} \rangle \sim G|\alpha|^2$ and $\langle \Delta \hat{n}^2 \rangle \sim G^2|\alpha|^2$. Therefore, the gain G corresponds to the Fano factor $F = \langle \hat{n} \rangle / \langle \Delta \hat{n}^2 \rangle$ of the output distribution, as long as the input state is excited before amplification in order to keep constant the intensity at the output: in this way the PNA works as a device converting coherent states in nearly-number eigenstates.

3. IDEAL PHOTON NUMBER-DUPLICATOR

A photon number-duplicator (PND) is a device which, upon acting on a input field in a certain number eigenstate, produces two output fields both of which are in the same number eigenstate as the input. Such a device can be realized in principle, whereas a 'cloning' device producing multiple copies of a (generally not orthogonal) input set of states would violate unitarity.[4,13] For the ideal PNA the unitary transformation has been obtained starting from the amplifying CP map defined by the relation

$$\mathcal{S}_H^{(r)}(\hat{E}_\pm) = (\hat{E}_\pm)^r \ . \tag{46}$$

The case of the ideal PND can be obtained in strict analogy with Eq.(46) by means of the duplicating CP map

$$\mathcal{S}_H(\hat{E}_\pm) = \hat{E}_\pm \otimes \hat{E}_\pm \ . \tag{47}$$

The general transformation attaining the duplication of the shift operators (47) has the form

$$\mathcal{S}_H(\hat{O}) = \sum_{\lambda = -\infty}^{\infty} \hat{s}_\lambda^\dagger \hat{O} \hat{s}_\lambda \ , \tag{48}$$
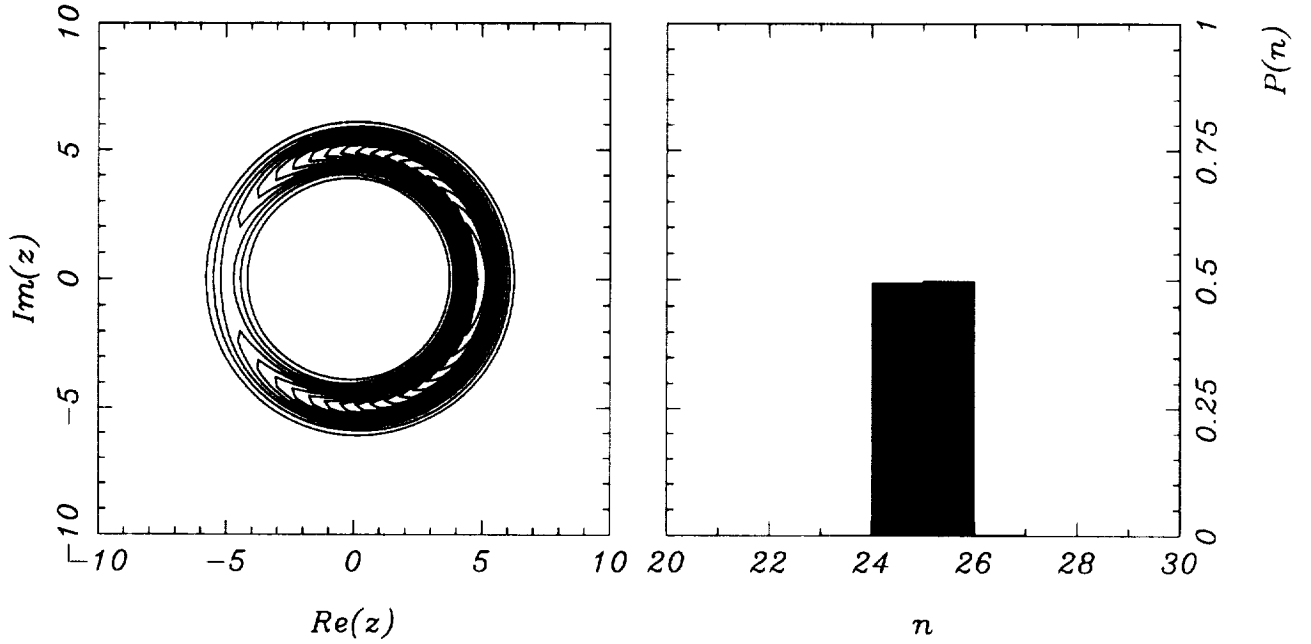
Figure 4: Q-function and number distribution of a coherent state having $\langle \hat{n} \rangle = 5000$ photons, after strong number-deamplification $G = .005$. The final moments are $\langle \hat{n} \rangle = 25$ and $\langle \Delta \hat{n}^2 \rangle^{1/2} = 1.16$.

where the nonunitary operators $\hat{S}_\lambda$ $(\hat{S}_\lambda : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F})$ are given by

$$\hat{S}_\lambda = \sum_{n,m=0}^{\infty} \delta_{m,n+\lambda} |\min\{n,m\}\rangle\langle m,n| , \tag{49}$$

and satisfy the orthogonality and completeness relations

$$\hat{S}_\lambda \hat{S}_\mu^\dagger = \delta_{\lambda\mu} \hat{1} , \tag{50}$$

$$\sum_{\lambda=-\infty}^{\infty} \hat{S}_\lambda^\dagger \hat{S}_\lambda = \hat{1} \otimes \hat{1} . \tag{51}$$

By adding a third photon field we can write a unitary operator $\hat{U}$ $(\hat{U} : \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F})$ as follows

$$\hat{U} = \sum_{\lambda=-\infty}^{\infty} \hat{S}_\lambda \otimes \hat{S}_\lambda^\dagger = \sum_{\lambda=-\infty}^{\infty} \sum_{n_1,n_2,m_1,m_2=0}^{\infty} \delta_{\{m_l\},\{n_l+\lambda\}} \times$$
$$|\min\{n_1,m_1\}\rangle\langle\min\{n_2,m_2\}| \otimes |n_2\rangle\langle n_1| \otimes |m_2\rangle\langle m_1| . \tag{52}$$

The operator $\hat{U}$ is involutive (i.e. $\hat{U}^2 = 1$) and produces the intertwining

$$\hat{U}(\hat{E}_\pm \otimes \hat{1} \otimes \hat{1})\hat{U} = \hat{1} \otimes \hat{E}_\pm \otimes \hat{E}_\pm , \tag{53}$$

$$\hat{U}(\hat{1} \otimes \hat{E}_\pm \otimes \hat{E}_\pm)\hat{U} = \hat{E}_\pm \otimes \hat{1} \otimes \hat{1} ,$$

which corresponds to the Fock representation

$$\hat{U}|l,m,n\rangle = \begin{cases} |m,l,l+n-m\rangle & n \geq m , \\ |m,l-n+m,l\rangle & n \leq m . \end{cases} \tag{54}$$

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In particular, one has $\hat{U}|l,n,n\rangle = |n,l,l\rangle$, and for the practically interesting case of the second and third fields in the vacuum state one obtains

$$\hat{U}|n,0,0\rangle = |0,n,n\rangle ,$$ (55)

which is the required duplication. The scheme of the ideal PND is depicted in Fig.(5). As the ideal PNA corresponds to intertwining the one-particle operator $a^{\dagger}$ with the r-particle operator $b^{\dagger}_{(r)}$, the PND performs a change between the one-mode operator $a^{\dagger}$ and the two-mode operator $b^{\dagger}_{(1,1)}$

$$b^{\dagger}_{(1,1)} = b^{\dagger}c^{\dagger}\left(\max\{b^{\dagger}b, c^{\dagger}c\} + 1\right)^{-1/2} ,$$ (56)

which satisfies the commutation relations $[b_{(1,1)}, b^{\dagger}_{(1,1)}] = 1$ and $[b^{\dagger}b + c^{\dagger}c, b_{(1,1)}] = -2b_{(1,1)}$. It follows that the Hamiltonian in the Dirac picture is

$$\hat{H}^{D}_{I} = -ik\left(a^{\dagger}b_{(1,1)} - b^{\dagger}_{(1,1)}a\right) ,$$ (57)

with the same interaction length as in (30). Conservation of energy now requires a classical pump at frequency $\Omega = \omega_a - \omega_b - \omega_c$, apart from the case of frequency matching $\omega_a = \omega_b + \omega_c$, which preserves the input energy $E = \omega_a l + \omega_b m + \omega_c n$. The PND described in the present context is more precisely a PNDFC (frequency-converter): in order to keep the frequency constant during the duplication one can choose $\omega_b = \omega_c$ and put a parametric frequency-converter acting on the input field a.

In conclusion of this section I remark that the ideal PND produces the same effect of a gain-2 PNA when the total number of photons of the two replica outputs is detected. In fact, one has

$$_{(1,1)}\langle\alpha|f(b^{\dagger}b + c^{\dagger}c)|\alpha\rangle_{(1,1)} = {}_{(2)}\langle\alpha|f(a^{\dagger}a)|\alpha\rangle_{(2)} = \langle\alpha|f(2a^{\dagger}a)|\alpha\rangle ,$$ (58)

where

$$|\alpha\rangle_{(2)} = \sum_{n=0}^{\infty}\alpha_n|2n\rangle$$ (59)

denotes the gain-2 PNA output state and

$$|\alpha\rangle_{(1,1)} = \sum_{n=0}^{\infty}\alpha_n|n,n\rangle$$ (60)

the PND output state corresponding to input

$$|\alpha\rangle = \sum_{n=0}^{\infty}\alpha_n|n\rangle , \qquad \sum_{n=0}^{\infty}|\alpha_n|^2 = 1 .$$ (61)

Moreover, one has

$$_{(1,1)}\langle\alpha|f(b^{\dagger}b)|\alpha\rangle_{(1,1)} = {}_{(1,1)}\langle\alpha|f(c^{\dagger}c)|\alpha\rangle_{(1,1)} = \langle\alpha|f(a^{\dagger}a)|\alpha\rangle ,$$ (62)
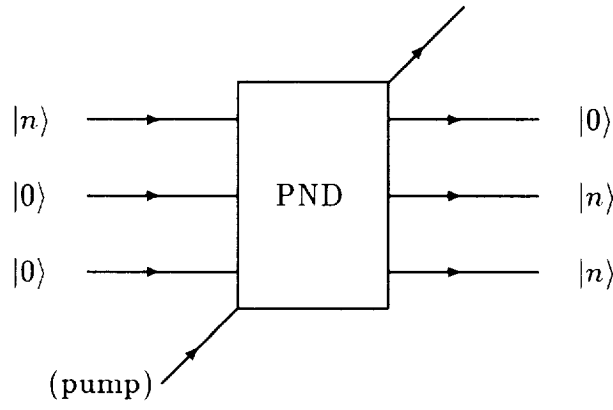
Figure 5: Scheme of the ideal PND.

and the SNR in detecting $b^\dagger b$, $c^\dagger c$ (or their sum) remains the same as at the input of the PND: in this sense the PND can be regarded an ideal amplifier.

4. COMPARISON WITH REAL SYSTEMS

The Hamiltonians (31) and (57) are quite complicated, due to the presence of an interaction strength which depends on the input number of photons of one field. For high average number of photons $\langle a^\dagger a \rangle \gg r$ the interaction strength in (31) behaves asymptotically as follows

$$f_{(r)}(a^\dagger a) \sim \left(a^\dagger a\right)^{-\frac{1}{2}(r-1)} . \tag{63}$$

Alternatively, one can look directly at the asymptotic behaviour of the multiboson operator $a^\dagger_{(r)}$:

$$a^\dagger_{(r)} = \left\{ \frac{[a^\dagger a/r](\hat{n} - r)!}{\hat{n}!} \right\}^{1/2} (a^\dagger)^r =$$

$$[a^\dagger a/r]^{1/2}(\hat{n} - r + 1)^{-1/2} r^{\frac{1}{2}} \kappa_r(\hat{\Phi}) a^\dagger \sim \kappa_r(\hat{\Phi}) a^\dagger , \tag{64}$$

where $\kappa_r(\hat{\Phi})$ denotes the function of the phase

$$\kappa_r(\hat{\Phi}) = r^{-\frac{1}{2}} e^{-i(r-1)\hat{\Phi}} . \tag{65}$$

Taking into account also the pumping field, the phase-number amplifier would require a medium with a $\chi^{(2)}$ susceptibility and an interaction Hamiltonian of the form

$$\hat{H}_I \sim \lambda \kappa_r(\hat{\Phi}_a) a^\dagger bc + \text{h.c.} , \tag{66}$$

c denoting the annihilator of the pumping field. From Eq.(66) it follows that in order to attain phase-number amplification one should use a $\chi^{(2)}$ medium having polarization which depends on the phase of the field according to (65) only in a limited frequency range. The amplifier gain r is involved only in the phase factor (65), and the interaction length has to be tuned at the complete conversion value $L = \pi/[2\lambda I_c^{1/2}]$, I_c being the average power flux of the (classical undepleted)

pump, and $\lambda \propto \chi^{(2)}$. For $r = 2$ one has $\kappa_2(\hat{\Phi}) = 2^{-\frac{1}{2}}e^{-i\hat{\Phi}} = (2\hat{n})^{-\frac{1}{2}}a^{\dagger}$. This case is approximated by the usual degenerate four-wave mixing medium having Hamiltonian

$$\hat{H}_I \propto \chi^{(3)}(a^{\dagger})^2 bc + \text{h.c.} \,. \tag{67}$$

For $\lambda \propto \chi^{(3)}I_a^{1/2}$ this medium attains gain-2 number amplification approximately in the average values.

The ideal PND is quite similar to the gain-2 ideal PNA, the main difference being that the field in the phase-dependent frequency range now splits into the two nondegenerate modes bearing the replica-states (actually one can analogously define G-'multiplicator' devices, which then compares to gain-G PNA's). When operating on two vacuua as in Fig.5, one can substitute the function $\max\{b^{\dagger}b, c^{\dagger}c\}$ in the Hamiltonian with either $b^{\dagger}b$ or $c^{\dagger}c$, without changing the output.

In conclusion, some remarks are in order, regarding the possibility of attaining the amplifying CP maps (37) and (44) through interaction with atomic—instead of electromagnetic—fields. In this case the nonunitary operators in Eqs.(37) and (44) should be regarded as partial trace of the interaction over the atomic degrees of freedom. The relations (12-14) have no faithful representation on a finite-dimensional Hilbert space, and one cannot realize them using atoms with a finite number of levels. However, some similarities can be recognized between this case and the PNA mechanism. For example, in the *high-Q micromaser Fock state generation*, the role of the auxiliary field is played by an inverted two-level atom entering the cavity with a well defined velocity:[14] the nonunitary reduction of the signal field is obtained by means of nonselective measurements of the atomic variables. The CP map experimented by the electromagnetic field is

$$\mathcal{S}_S^{(r)}(\hat{\rho}) = \sum_{\alpha=0}^{r-1} \hat{V}_\alpha^{(r)}\hat{\rho}(\hat{V}_\alpha^{(r)})^{\dagger} \,, \tag{68}$$

$$\hat{V}_\alpha^{\dagger} = \langle\alpha| \exp\left(-i\hat{H}_I^P t\right) |\uparrow\rangle \,, \tag{69}$$

where $r = 2$, $\hat{H}_I^P$ denotes the usual Jaynes-Cummings Hamiltonian, and $|\uparrow\rangle$ represents the inverted state of the atom. As a matter of fact, the high-Q micromaser works as a 'number-phase squeezer' (in this fashion the micromaser is a sort of PNA and the successive atomic passages correspond to gain-2 open PNA's in series). One should notice, however, that the number-phase squeezing in the micromaser strongly depends on the initial state of the field (which should have less photons then the asymptotic 'trapping' state), and this feature does not depend on the particular form of the interaction Hamiltonian $\hat{H}_I^P$, as long as a finite number of the atomic levels is concerned.

5. ACKNOWLEDGMENTS

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