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Nonclassical Depth of a Quantum State*

Ching Tsung Lee

Department of Physics, Alabama A & M University, Normal, Alabama 35762

A measure is defined for how nonclassical a quantum state is, with values ranging from 0 to 1. When it is applied to the photon-number states, the calculated value is 1, the maximum possible. For squeezed states, it is a monotonically increasing function of the squeeze parameter with values varying from 0 to 1/2. The physical meaning of the nonclassical depth is found to be just the number of thermal photons necessary to ruin the nonclassical nature of the quantum state.

In the coherent-state description of radiation fields, initiated by Glauber (Ref. 1) and Sudarshan (Ref. 2) in 1963, there are P and Q representations corresponding to the normal and antinormal ordering, respectively, of the creation and annihilation operators. Their distribution, or quasi-distribution, functions in the complex plane are related to each other through the following convolution transform (Ref. 3):

$$Q(z) = \int \frac{d^2w}{\pi} e^{-|z-w|^2} P(w), \tag{1}$$

where z and w are complex variables. We can introduce a continuous parameter τ and define a general distribution function as

$$R(z,\tau) = \frac{1}{\tau} \int \frac{d^2w}{\pi} \exp\left(-\frac{1}{\tau}|z-w|^2\right) P(w). \tag{2}$$

We shall call $R(z,\tau)$ the R function from now on. The original P and Q functions are two limiting cases of the R function with $\tau = 0$ and 1, respectively.

Our motivation for introducing the τ parameter is to define a measure of how nonclassical quantum states are. It is well known that the origin of the nonclassical effects is that the P functions of all pure quantum states are singular and not positive definite, as shown by Hillery (Ref. 4); hence it is called quasi-distribution function. On the other hand, the Q function is always a positive definite regular function. The smoothing effect of the convolution transform of Eq. (2) is enhanced as τ increases. If τ is large enough so that the R function becomes acceptable as a classical distribution function, *i.e.*, it is a positive definite regular function, then we say that the smoothing operation is complete. Let C denote the set of all the τ that will complete the smoothing of the P function of a quantum state and let the greatest lower bound, or infimum, of all the τ in C be denoted by

$$\tau_m \equiv \inf_{\tau \in C} (\tau). \tag{3}$$

We propose to adopt τ_m as the nonclassical depth of the quantum state.

According to this definition, we have $\tau_m = 0$ for an arbitrary coherent state $|\alpha\rangle$ since its P function is of the form of a delta function, $\pi\delta^2(z-\alpha)$. On the other hand, for $\tau = 1$ we have $R(z,1) = Q(z)$, which is always acceptable as a classical distribution function for any quantum state; hence, 1 is an upper bound for τ_m. Therefore, we can specify the range of τ_m to be

$$0 \leq \tau_m \leq 1. \tag{4}$$

We shall try this definition on two of the most familiar types of nonclassical radiation states: the photon-number (Fock) states and the squeezed states.

For a photon-number state $|n\rangle$, we obtain

$$R_n(z,\tau) = \frac{1}{\tau}\left(-\frac{1-\tau}{\tau}\right)^n \exp\left(-\frac{|z|^2}{\tau}\right) L_n\left(\frac{|z|^2}{\tau(1-\tau)}\right), \tag{5}$$

where L_n is the Laguerre polynomial. From Eq. (5), we see that, for $0 < \tau < 1$, $R_n(z,\tau)$ is not positive definite since the Laguerre polynomial has n real positive roots. However, for $\tau \geq 1$, the argument of the Laguerre polynomial is negative and $R_n(z,\tau)$ becomes positive definite. So we have $\tau_m = 1$, which reconfirms our belief that the photon-number states are the most nonclassical quantum states.

For the squeezed state generated from the vacuum state by the well-known squeeze operator $S(\zeta)$ with the complex parameter $\zeta \equiv re^{i\theta}$, we obtain the Gaussian function

$$R_\zeta(z,\tau) = \frac{(\operatorname{sech} r)}{\sqrt{D}} \exp\left\{-\frac{1}{D}[ax^2 + 2bxy + bx^2]\right\} \tag{6}$$

with

$$\begin{aligned} a &= \tau + (1-\tau)\tanh^2 r - \cos\theta \tanh r, \qquad & b &= \sin\theta \tanh r, \\ c &= \tau + (1-\tau)\tanh^2 r + \cos\theta \tanh r, \qquad & D &= \tau^2 - (1-\tau)^2 \tanh^2 r. \end{aligned} \tag{7}$$

For $R_\zeta(z,\tau)$ to be normalizable we must have

$$ac - b^2 > 0 \quad \text{and} \quad D > 0. \tag{8}$$

Both conditions lead to the same conclusion that

$$\tau_m = \tanh r/(1+\tanh r). \tag{9}$$

This nonclassical depth can be expressed as a function of the squeeze parameter, $s \equiv e^r$, as follows:

$$\tau_m(s) = (s^2-1)/2s^2. \tag{10}$$

From Eq. (10) we see that τ_m is a monotonically increasing function of s; it varies from 0 to $1/2$ as s varies from 1 to ∞.

In the first example, τ_m is determined by the requirement that $R_n(z,\tau)$ be positive definite; while in the second example, it is determined by the condition that $R_\zeta(z,\tau)$ be normalizable. Is there a more systematic way to determine τ_m? To answer this question, we need to study more examples.

As a by-product of such calculations, we will also obtain new expressions for the P functions as follows:

$$P(z) = \lim_{\tau\to 0} R(z,\tau). \tag{11}$$

Since the P function of a quantum state is typically highly singular, it is usually very difficulty to visualize in its original form. Now we can visualize it as the limit of the regular R function as $\tau \to 0$.

On the other hand, we consider the superposition of two quantum states with $P_1(z)$ and $P_2(z)$ as their P functions. According to Glauber (Ref. 5) the P function for the superposed state is the convolution product of $P_1(z)$ and $P_2(z)$; namely,

$$P_{su}(z) = \int \frac{d^2 z}{\pi} P_1(z-w)P_2(w) \tag{12}$$

It is well known that the P function for a single-mode thermal radiation is (Ref. 5)

$$P_{th}(z) = \frac{1}{\langle n_{th}\rangle} \exp\left(-|z|^2/\langle n_{th}\rangle\right), \tag{13}$$

where $\langle n_{th}\rangle \equiv \left(e^{\hbar\omega/kT}-1\right)^{-1}$ is the *average photon number* in the thermal radiation.

We now consider the superposition of the thermal radiation with an arbitrary state of single-mode radiation with $P(z)$ as its P function. Then the P function of this quantum state with thermal noise can be expressed as

$$P_{su}(z) = \frac{1}{\langle n_{th}\rangle} \int \frac{d^2 w}{\pi} \exp\left(-|z-w|^2/\langle n_{th}\rangle\right) P(w) \tag{14}$$

Comparing Eqs. (1) and (14) we see that the superposed P function, $P_{su}(z)$, is identical to the Q function when $\langle n_{th}\rangle = 1$.The implication of this coincidence can be stated as follows: *One thermal photon is always sufficient to destroy whatever nonclassical effects any single-mode radiation might have.*

The R function for the superposed state of Eq. (14) can be obtained as

$$R_{su}(z,\tau) = \frac{1}{\tau + \langle n_{th}\rangle} \int \frac{d^2 w}{\pi} \exp\left[-|z-w|^2/(\tau + \langle n_{th}\rangle)\right] P(w). \tag{15}$$

Therefore, we have

$$\tau_m^{th} = \tau_m - \langle n_{th}\rangle; \tag{16}$$

which means that the reduction in the nonclassical depth of a quantum state in the presence of thermal noise is exactly equal to the average number of thermal photons present. This also gives the following physical meaning to the nonclassical depth we have defined previously: *The nonclassical depth of a quantum state is the minimum number of thermal photons necessary to destroy any of its nonclassical characteristics.*

We have previously calculated the nonclassical depth of a Fock state to be exactly 1, so it takes one thermal photon to ruin the nonclassical nature of a Fock state. We have also calculated the nonclassical depth of a squeezed state to varies from 0 to 1/2 as s varies from 1 to ∞; so it never takes more than 1/2 of a thermal photon to ruin a squeezed state.

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