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Entropic uncertainty relation at finite temperature

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The uncertainty relation associated with the measurements of a generic noncommutative pair of observables (A, B) in a normalized state $|\psi\rangle$ is usually expressed as

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle\psi|[A, B]|\psi\rangle|. \tag{1}$$

For a canonically conjugate pair, the position and momentum of a particle (X, P), this equation gives the original Heisenberg uncertainty relation $\Delta X \cdot \Delta P \geq \frac{1}{2}, (\hbar \equiv 1)$. On the other hand, if the commutator $[A, B]$ remains as a q-number, the r.h.s. depends on the state $|\psi\rangle$ and can be made arbitrarily small. For example, if $|\psi\rangle$ is chosen as an eigenstate of A, then Eq.(1) becomes trivial and no information can be extracted on ΔB. Thus this formulation of the uncertainty principle has no practical meanings in general.

To improve the situation, the information-theoretic formulation of the uncertainty principle has been repeatedly studied in the recent literature. Deutsch[1] and Partovi[2] discussed that the sum of entropies,

$$U[A, B : \psi] = S_A[\psi] + S_B[\psi], \tag{2}$$

has an irreducible lower bound independent of the choice of $|\psi\rangle$. Here, the information entropy associated with the measurement of A is defined by

$$S_A[\psi] = -\int_\alpha |\langle\alpha|\psi\rangle|^2 \ln |\langle\alpha|\psi\rangle|^2, \quad (A\,|\alpha\rangle = \alpha\,|\alpha\rangle), \tag{3}$$

where $\int_\alpha$ stands for the summation (integration) over the discrete (continuous) spectra. This is a quantity dependent on the choice of the representation $|\alpha\rangle$ in general and is not expressed as a quantum mechanical expectation value of a certain operator.

Prior to the authors of Refs.[1, 2], Bialynicki-Birula and Mycielski[3] discussed the sum (2) for the pair (X, P) and proved the optimal relation

$$U[X, P : \psi] \geq 1 + \ln \pi. \tag{4}$$

Here we discuss that how much information loses when a particle is in equilibrium with the thermal reservoir of temperature $T (= 1/\beta)$[4]. The universal temperature correction to the r.h.s. of Eq.(4) is determined.

For this purpose, it is convenient to employ the framework of *thermo field dynamics* (TFD) formulated by Takahashi and Umezawa[5]. This formulation of finite-temperature $(T \neq 0)$ quantum theory utilizes the doubled Hilbert space $\mathcal{H}\otimes\tilde{\mathcal{H}}$[6], the *normal* operator (A) acting on the objective space $\mathcal{H}$ and its corresponding *tildian* operator $(\tilde{A})$ on the fictitious space $\tilde{\mathcal{H}}$.

A thermal state $|\psi, \tilde{\psi}; \beta\rangle$ in $\mathcal{H} \otimes \tilde{\mathcal{H}}$ is not a physical state. The physical probability density associated with the measurement of the normal operator A is given by the reduced one

$$\rho_R(\alpha) = \int_{\tilde{\alpha}} |\langle\alpha, \tilde{\alpha}|\psi, \tilde{\psi}; \beta\rangle|^2, \tag{5}$$

where $|\alpha, \tilde{\alpha}\rangle$ is the complete eigenbasis of A and $\tilde{A}$. With this quantity, we define the information entropy at $T \neq 0$ as follows:

$$S_A[\psi, \tilde{\psi}; \beta] = -\int_\alpha \rho_R(\alpha) \ln \rho_R(\alpha). \tag{6}$$

Now we wish to find the stationary value of the functional

$$U[X, P : \psi, \tilde{\psi}; \beta] = S_X[\psi, \tilde{\psi}; \beta] + S_P[\psi, \tilde{\psi}; \beta], \tag{7}$$

at given T. In what follows, we propose a variational approach.

We are not concerned with the whole system including the tildian but only with the reduced one. Therefore, the minimum value of the functional U at given T can be determined completely within the reduced subsystem. This philosophy should be also respected by the variational operation itself. The operation proposed here is as follows:

$$|\psi,\tilde{\psi};\beta\rangle \to |\psi,\tilde{\psi};\beta\rangle + \epsilon|\xi,\tilde{\psi};\beta\rangle, \tag{8}$$

where ϵ and ξ denote an infinitesimal variation parameter and an arbitrary deformation of the $\mathcal{H}$ component, respectively. Under this operation, the functional U of the normalized thermal state $|\psi,\tilde{\psi};\beta\rangle$ varies as

$$U[X,P:\psi,\tilde{\psi};\beta] \to U[X,P:\psi,\tilde{\psi};\beta] + \epsilon\Gamma + o(\epsilon^2), \tag{9}$$

$$\begin{aligned}\Gamma \equiv & [\int dx\rho_R(x)\ln\rho_R(x) + \int dp\rho_R(p)\ln\rho_R(p)]\langle\psi,\tilde{\psi};\beta|\xi,\tilde{\psi};\beta\rangle \\ & - \int\int dx\, d\tilde{x}\ln[\rho_R(x)]\langle\psi,\tilde{\psi};\beta|x,\tilde{x}\rangle\langle x,\tilde{x}|\xi,\tilde{\psi};\beta\rangle - \int\int dp\, d\tilde{p}\ln[\rho_R(p)]\langle\psi,\tilde{\psi};\beta|p,\tilde{p}\rangle\langle p,\tilde{p}|\xi,\tilde{\psi};\beta\rangle.\end{aligned} \tag{10}$$

We do not know how to solve generally the equation $\Gamma = 0$ with respect to the unknown state $|\psi,\tilde{\psi};\beta\rangle$. Here, instead, we examine the thermal coherent state (TCS)[7], which is the oscillator coherent state at $T \neq 0$. This is based on the following viewpoints; (i) the information entropy is the measure of uncertainty, and (ii) at $T = 0$, the coherent state saturates the Heisenberg uncertainty ($\Delta X \cdot \Delta P = \frac{1}{2}$).

Let us consider a harmonic oscillator with a frequency ω in TFD. The thermal vacuum state is generated from the $T = 0$ Fock vacuum state $|0,\tilde{0}\rangle$ by the Bogoliubov transformation

$$|0(\beta)\rangle = \exp(-iG)\,|0,\tilde{0}\rangle, \quad -iG(\beta) = \theta(\beta)(a^\dagger\tilde{a}^\dagger - \tilde{a}a), \tag{11}$$

$$\cosh\theta(\beta) = [1 - \exp(-\beta\omega)]^{-1/2}, \tag{12}$$

provided that the creation and annihilation operators satisfy $[a,a^\dagger] = [\tilde{a},\tilde{a}^\dagger] = 1$, $[a,\tilde{a}] = 0$, and so on. With this state, the TCS is defined as follows:

$$|z,\tilde{z};\beta\rangle = \exp[za^\dagger(\beta) - z^*a(\beta) + \tilde{z}^*\tilde{a}^\dagger(\beta) - \tilde{z}\tilde{a}(\beta)]\,|0(\beta)\rangle\,, \tag{13}$$

$$a(\beta)|z,\tilde{z};\beta\rangle = z|z,\tilde{z};\beta\rangle, \quad \tilde{a}(\beta)|z,\tilde{z};\beta\rangle = \tilde{z}^*|z,\tilde{z};\beta\rangle, \tag{14}$$

where the operators at $T \neq 0$ are given by

$$a(\beta) = \exp(-iG)a\exp(iG) = a\cosh\theta(\beta) - \tilde{a}^\dagger\sinh\theta(\beta), \tag{15a}$$

$$\tilde{a}(\beta) = \exp(-iG)\tilde{a}\exp(iG) = \tilde{a}\cosh\theta(\beta) - a^\dagger\sinh\theta(\beta), \tag{15b}$$

and so on. The self-tildian condition[7] states $z = \tilde{z}$.

One can find that the TCS actually gives the desired result $\Gamma^{\mathrm{TCS}} = 0$, and, therefore, Eq.(9) becomes

$$U[X,P:z,\tilde{z};\beta] \to 1 + \ln\pi + \ln[\cosh 2\theta(\beta)] + o(\epsilon^2). \tag{16}$$

Thus we have the thermal information-entropic uncertainty relation[4]

$$U[X,P:\psi,\tilde{\psi};\beta] \geq 1 + \ln\pi + \ln[\cosh 2\theta(\beta)]. \tag{17}$$

The third term in the r.h.s. determines the minimum loss of measurement information due to the thermal disturbance effects.

The Heisenberg uncertainty relation at $T \neq 0$ can be derived from Eq.(17). To see this, let us find the maximum value of the concave entropy functional S_X with fixing the variance $\langle(X - \langle X\rangle)^2\rangle = (\Delta X)^2$. ($\langle\cdot\rangle$ denotes the expectation value with respect to the normalized probability density $\rho_R(x)/\langle\psi,\tilde{\psi};\beta|\psi,\tilde{\psi};\beta\rangle$.) This is just the constrained variational problem characterized by the functional

$$\Phi[\psi,\tilde{\psi};\beta] = S_X[\psi,\tilde{\psi};\beta] - \lambda[\langle(X - \langle X\rangle)^2\rangle - (\Delta X)^2], \tag{18}$$

where λ is Lagrange's multiplier. Applying again the variational operation (8), we can find the maximum value

$$S_X{}^{\max}[\psi,\tilde{\psi};\beta] = \frac{1}{2}\ln[2\pi e(\Delta X)^2]. \tag{19}$$

Therefore we have an inequality

$$S_X[\psi,\tilde{\psi};\beta] \leq \frac{1}{2}\ln[2\pi e(\Delta X)^2]. \tag{20}$$

Repeating a similar discussion for the momentum P, we also get

$$S_P[\psi, \tilde{\psi}; \beta] \leq \frac{1}{2} \ln [2\pi e(\Delta P)^2]. \tag{21}$$

The combination of Eqs.(20) and (21) leads to

$$\begin{aligned} 2(\Delta P)^2 &\geq \exp(-1 - \ln \pi + 2S_P[\psi, \tilde{\psi}; \beta]) \\ &\geq \exp(1 + \ln \pi + 2\ln\{\cosh[2\theta(\beta)]\} - 2S_X[\psi, \tilde{\psi}; \beta]) \\ &\geq \frac{1}{2}\cosh^2[2\theta(\beta)](\Delta X)^{-2}. \end{aligned} \tag{22}$$

Thus we obtain the thermal Heisenberg uncertainty relation

$$\Delta X \cdot \Delta P \geq \frac{1}{2} \cosh [2\theta(\beta)]. \tag{23}$$

We have used Eq.(17) in the second inequality of Eq.(22). This shows that the information-entropic uncertainty relation is *stronger* than Heisenberg uncertainty relation[3].

Finally, we comment on squeezing of the thermal uncertainty relation. The thermal squeezed state is defined by

$$\begin{aligned} |z, \tilde{z} : \eta, \tilde{\eta}; \beta\rangle &= \exp[z a^\dagger(\beta) - z^* a(\beta) + \tilde{z}^* \tilde{a}^\dagger(\beta) - \tilde{z}\tilde{a}(\beta)] \\ &\times \exp[\frac{1}{2}\{\eta a^{\dagger 2}(\beta) - \eta^* a^2(\beta) + \tilde{\eta}^* \tilde{a}^{\dagger 2}(\beta) - \tilde{\eta}\tilde{a}^2(\beta)\}] \, |0(\beta)\rangle . \end{aligned} \tag{24}$$

Straightforward calculation gives

$$S_X[z, \tilde{z} : \eta, \tilde{\eta}; \beta] = \frac{1}{2}(1 + \ln \pi + \ln\{\cosh[2\theta(\beta)]\} + \ln[\cosh(2r) + \sinh(2r)\cos(\varphi)]), \tag{25a}$$

$$S_P[z, \tilde{z} : \eta, \tilde{\eta}; \beta] = \frac{1}{2}(1 + \ln \pi + \ln\{\cosh[2\theta(\beta)]\} + \ln[\cosh(2r) - \sinh(2r)\cos(\varphi)]), \tag{25b}$$

$$U[X, P : z, \tilde{z} : \eta, \tilde{\eta}; \beta] = 1 + \ln \pi + \ln\{\cosh[2\theta(\beta)]\} + \ln[1 + \sinh^2(2r)\sin^2(\varphi)]^{\frac{1}{2}}, \tag{26}$$

$$\Delta X = \{\frac{1}{2}\cosh[2\theta(\beta)](\cosh(2r) + \sinh(2r)\cos(\varphi))\}^{\frac{1}{2}}, \tag{27a}$$

$$\Delta P = \{\frac{1}{2}\cosh[2\theta(\beta)](\cosh(2r) - \sinh(2r)\cos(\varphi))\}^{\frac{1}{2}}, \tag{27b}$$

$$\Delta X \cdot \Delta P = \frac{1}{2}\cosh[2\theta(\beta)](1 + \sinh^2(2r)\sin^2(\varphi))^{\frac{1}{2}}, \tag{28}$$

where we have employed the self-tildian condition for a squeeze factor (i.e., $\eta = \tilde{\eta}$), and $\eta \equiv r\exp(i\varphi)$. These results describe how the thermal disturbance effects in S_X or S_P (ΔX or ΔP) can be suppressed by squeezing with keeping $U = S_X + S_P$ ($\Delta X \cdot \Delta P$) its minimum value.

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