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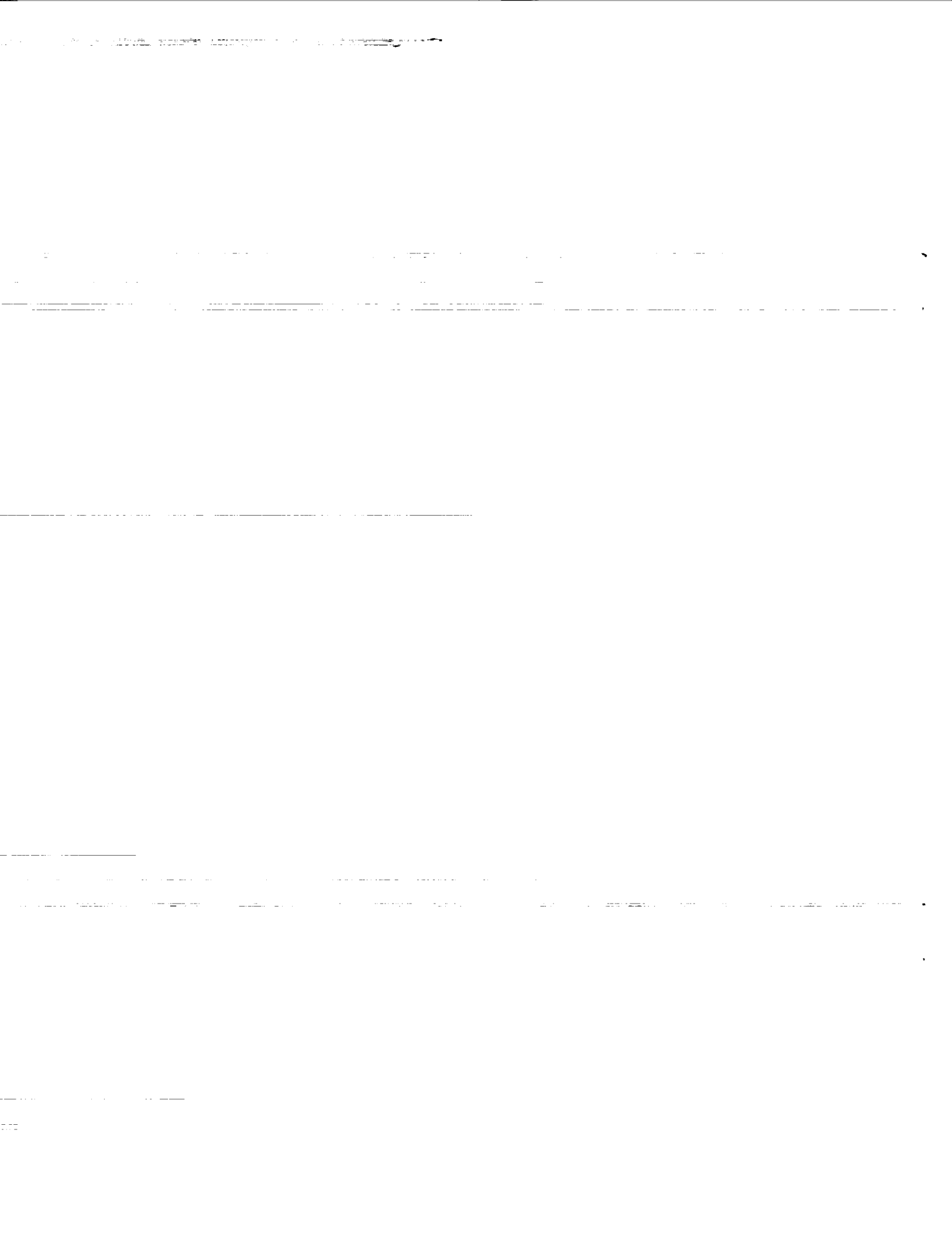
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# Renormalization Group Analysis of the Reynolds Stress Transport Equation

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## Abstract

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The pressure-velocity correlation and return to isotropy term in the Reynolds stress transport equation are analyzed using the Yakhot-Orszag renormalization group. The perturbation series for the relevant correlations, evaluated to lowest order in the  $\epsilon$ -expansion of the Yakhot-Orszag theory, are infinite series in tensor product powers of the mean velocity gradient and its transpose. Formal lowest order Padé approximations to the sums of these series produce a fast pressure strain model of the form proposed by Launder, Reece, and Rodi, and a return to isotropy model of the form proposed by Rotta. In both cases, the model constants are computed theoretically. The predicted Reynolds stress ratios in simple shear flows are evaluated and compared with experimental data. The possibility is discussed of deriving higher order nonlinear models by approximating the sums more accurately.

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## Introduction

The modelling of the pressure correlation and return to isotropy term in the Reynolds stress transport equation remains controversial.<sup>1,2,3</sup> Models will be derived here using the Yakhot-Orszag renormalization group<sup>4</sup> partially along the lines of our previous work<sup>5</sup>. The result is a model for the fast pressure-strain term of the form proposed by Launder, Reece and Rodi<sup>6</sup> (LRR) and a model for return to isotropy of the form proposed by Rotta<sup>7</sup> with theoretically computed constants in good agreement with accepted values. As usual in investigations of this sort, the priority of Yoshizawa in deriving a pressure strain model analytically<sup>8</sup> must be noted.

The analysis requires some new ideas in renormalization group theory recently introduced by Yakhot et al<sup>9</sup>. As Yakhot et al<sup>9</sup> emphasize, the application of the renormalization group mode elimination formalism to shear flow creates a double perturbation series in powers of  $\epsilon$ , the parameter of the isotropic theory, and in powers of a dimensionless strain rate,  $\eta = SK/\epsilon$ , where  $K$  denotes the turbulence kinetic energy,  $\epsilon$  denotes the dissipation rate, and  $S$  is a measure of the mean strain: in Ref. 9,  $S^2 = (\frac{\partial U_m}{\partial x_n} + \frac{\partial U_n}{\partial x_m}) \frac{\partial U_m}{\partial x_n}$ . The present analysis also leads to double expansions of this type, with the powers  $S^n$  replaced by tensors  $S^{(n)}$  homogeneous of degree  $n$  in the mean velocity gradient  $\nabla U$  and its transpose. It will be convenient to retain the terminology of Ref. 9 and call this expansion the  $\eta$ -expansion; when the distinction is pertinent, the expansion of Ref. 9 will be called a scalar  $\eta$ -expansion.

The heuristic program of evaluating all scalar amplitudes to lowest order in  $\epsilon$  has proven successful in the past: apparently, the  $\epsilon$ -expansion is an asymptotic series with sum given very nearly by its first term<sup>10</sup>. Unfortunately, there is no analogous basis for

truncating the  $\eta$ -expansion. There are fundamental reasons for this distinction between these expansions. The present  $\eta$ -expansion is tensorial: successively higher order terms do not introduce merely numerical corrections, but increasingly complex asymmetries into the theory. Truncation therefore imposes a possibly inappropriate symmetry or other constraint on the model. Thus, in Ref. 5 the  $\eta$ -expansion for the Reynolds stress  $\tau$  was truncated at second order as suggested by previous work of Yoshizawa<sup>11</sup> and Speziale<sup>12</sup>. Although this type of modelling permits unequal normal stresses in a simple shear flow, it is not maximally asymmetric: for example, in a flow with mean velocity components  $U_i(x_1, x_2)$ , a cubic model including a term  $\tau \sim \nabla U^2 \nabla U^T + \nabla U \nabla U^{T2}$  would permit nonzero  $\tau_{23}$  in the presence of vanishing  $\partial U_2 / \partial x_3$ , an effect which cannot be ruled out in advance.

Although generalizations<sup>13</sup> of the Cayley-Hamilton Theorem limit the number of independent tensors  $\mathbf{S}^{(n)}$ , anisotropy and asymmetry cannot exist at all without some terms of higher order in  $\eta$ ; indeed, truncation at lowest order in  $\eta$  just produces a theory of isotropic turbulence. But the series truncated at any higher order can be unsatisfactory in flow regions in which some components  $(\frac{\partial U_i}{\partial x_j})K/\varepsilon$  are large. In such regions, the truncated series is dominated by its highest order terms. For the quadratic stress models of Refs. 5, 11, 12, this domination can produce negative normal stresses in the buffer layers of wall bounded flows. Increasing the order of truncation obviously exacerbates this problem.

It follows that finite truncation of the  $\eta$ -expansion is theoretically unsatisfactory. Yakhot et al<sup>9</sup> therefore propose that this expansion must be summed, even if only approximately, and have suggested a prototype summation in a different context. It should be noted that the same issues arise naturally in Yoshizawa's formalism, which also generates infinite series in the mean velocity gradients (and in other quantities as well) for

correlations of interest in turbulence modeling. Yoshizawa has concluded independently that summation of this series is essential and has also derived a Reynolds stress transport model by introducing such summations<sup>14</sup>.

In this paper, the perturbation series which the Yakhot-Orszag renormalization group generates for the correlation

$$\Pi_{ij} = - \left\langle u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i} \right\rangle \quad (1)$$

is summed by a low order Padé approximation. Coefficients are evaluated to lowest order in the  $\epsilon$  expansion, but the summation includes effects of all orders in  $\eta$ . The result is essentially identical to the “model 1” proposed by Launder, Reece, and Rodi<sup>6</sup>. An entirely analogous treatment of return to isotropy yields a model of the form proposed by Rotta<sup>7</sup>. Combining these models leads to a preliminary Reynolds stress transport model. The problem of closing the Reynolds stress diffusion terms is addressed. This problem also leads to an infinite sum.

While it is encouraging that renormalization group methods can be used to derive familiar models, the goal of this investigation is not limited to providing theoretical justification for the LRR and Rotta models, which although widely applied are nevertheless deficient in several well-documented respects<sup>1,2,3</sup>. Instead, renormalization group methods together with approximate summation of the  $\eta$ -expansion can be used to derive higher order and nonlinear corrections to these models in a systematic fashion. Explicit development of such models is left to future investigations.

## I. Analysis of the Pressure Correlation

The analysis will follow Yakhot and Orszag's derivation of turbulence transport models by renormalization group methods<sup>4</sup>. The equation for velocity products is

$$\frac{\partial}{\partial t} u_i u_j + u_p \left( u_i \frac{\partial u_j}{\partial x_p} + u_j \frac{\partial u_i}{\partial x_p} \right) = - \left( u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i} \right) + \nu_0 \nabla^2 u_i u_j - 2\nu_0 \frac{\partial u_i}{\partial x_p} \frac{\partial u_j}{\partial x_p} \quad (2)$$

where  $\nu_0$  denotes the kinematic viscosity. The product  $-(u_i \partial p / \partial x_j + u_j \partial p / \partial x_i)$  on the right side of Eq. (2) will become the correlation  $\Pi_{ij}$  defined by Eq. (1) following elimination of all fluctuating modes. Its Fourier transform is

$$\begin{aligned} & \int u_j(\hat{k} - \hat{q}) i q_i q^{-2} (q_p - Q_p) u_q(\hat{q} - \hat{Q}) Q_q u_p(\hat{Q}) d\hat{q} d\hat{Q} / (2\pi)^{2d+2} \\ & + \int u_i(\hat{k} - \hat{q}) i q_j q^{-2} (q_p - Q_p) u_q(\hat{q} - \hat{Q}) Q_q u_p(\hat{Q}) d\hat{q} d\hat{Q} / (2\pi)^{2d+2} \end{aligned} \quad (3)$$

Here the standard notation

$$\hat{k} = (\omega, \mathbf{k}) \quad k^2 = \mathbf{k} \cdot \mathbf{k}$$

is used and  $d = 3$  is the number of space dimensions. Introduce an ultraviolet cutoff  $\Lambda_d$  of the order of the inverse Kolmogorov scale  $(\nu_0^3/\epsilon)^{1/4}$ ; only inertial range scales with  $k \leq \Lambda_d$  will be treated explicitly. Introducing a parameter  $r$  initially near zero, partition wavenumber space  $0 \leq k \leq \Lambda_d$  into the two intervals  $0 \leq k \leq \Lambda_d e^{-r}$  and  $\Lambda_d e^{-r} \leq k \leq \Lambda_d$ . Denote velocity components with wavevectors in the first interval by the superscript  $<$  and those in the second by  $>$ . Introducing this decomposition into Eq. (3) produces eight terms; however, as in analogous calculations in Ref. 4, only three will contribute at the lowest order in  $\epsilon$ :

$$\begin{aligned}
I &= \int u_j^>(\hat{k} - \hat{q}) i q_i q^{-2} (q_p - Q_p) u_q^>(\hat{q} - \hat{Q}) Q_q u_p^<(\hat{Q}) d\hat{q} d\hat{Q} / (2\pi)^{2d+2} \\
II &= \int u_j^>(\hat{k} - \hat{q}) i q_i q^{-2} (q_p - Q_p) u_q^<(\hat{q} - \hat{Q}) Q_q u_p^>(\hat{Q}) d\hat{q} d\hat{Q} / (2\pi)^{2d+2} \\
III &= \int u_j^<(\hat{k} - \hat{q}) i q_i q^{-2} (q_p - Q_p) u_q^>(\hat{q} - \hat{Q}) Q_q u_p^>(\hat{Q}) d\hat{q} d\hat{Q} / (2\pi)^{2d+2}
\end{aligned}$$

The  $>$  modes are to be eliminated from these expressions by iterated use of the randomly forced Navier Stokes equations

$$(-i\omega + \nu_0 k^2) u_i(\hat{k}) = -\frac{i}{2} \lambda_0 P_{imn}(\mathbf{k}) \int u_m(\hat{k} - \hat{q}) u_n(\hat{q}) d\hat{q} / (2\pi)^{d+1} + f_i$$

where

$$P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$$

$$P_{in}(\mathbf{k}) = \delta_{in} - k_i k_n / k^2$$

and the Gaussian random force  $f_i$  is defined by its correlation function

$$\langle f_i(\hat{k}) f_j(\hat{k}') \rangle = 2 (2\pi)^{d+1} D_0 \delta(\omega + \omega') \delta(\mathbf{k} + \mathbf{k}') k^{-y}$$

The choice  $y = d$  generates a Kolmogorov inertial range. A detailed exposition of this procedure can be found in Ref. 4. It will suffice to note here that the result of the mode elimination is a series in powers of  $u^<$  and  $\lambda_0$ . At each order in  $u^<$ , perturbation theory will produce a finite number of types of terms with amplitudes given as series in  $\lambda_0$ . Under iterated mode elimination to the limit  $r \rightarrow \infty$ , the expansion in  $\lambda_0$  proves to be an expansion in powers of  $\epsilon = 4 + y - d$ . This is the  $\epsilon$ -expansion of the Yakhot-Orszag theory. Previous experience<sup>4</sup> and preliminary analysis<sup>10</sup> suggest that the amplitudes are best evaluated at lowest order in  $\epsilon$  with  $\epsilon$  set to zero. This procedure will be followed here.



Thus, the perturbation series will be written as

$$\Pi = T_0 + T_1 + \dots$$

where  $T_n$  is of order  $n$  in  $u^<$  and all amplitudes are evaluated to lowest order in  $\epsilon$ . To lowest order in  $\epsilon$  and  $SK/\epsilon$

$$III = \int \delta(\mathbf{q}) q_i q_p q^{-2} Q_q u_j(\hat{k}) u_q(-\hat{Q}) u_p(\hat{Q}) d\hat{q}/(2\pi)^{d+1} \equiv 0$$

since incompressibility implies  $Q_q u_q \equiv 0$ . Now term  $III$  is formally proportional to  $q_i u_j$  and since only indices  $i$  and  $j$  are uncontracted, the combination  $Q_p u_p$  or  $Q_q u_q$  must occur at all orders. Accordingly, term  $III$  vanishes identically.

Lowest order analysis of term  $I$  gives

$$\int q_i q_p q^{-2-y} \{ |G(\hat{q})|^2 P_{jq}(\mathbf{q}) 2D_o d\hat{q}/(2\pi)^{d+1} \} \cdot i k_q u_p(\hat{k})$$

Adding the corresponding term from  $II$  and the result of  $ij$  index interchange leads to

$$T_1 = \hat{T}_1 \left( \frac{\partial U_i^<}{\partial x_j} + \frac{\partial U_j^<}{\partial x_i} \right) \quad (4)$$

where  $\hat{T}_1$  satisfies the recursion relation

$$\frac{d\hat{T}_1}{dr} = \frac{6}{15} \frac{\mathcal{D}}{\nu \Lambda^2} \quad (5)$$

In Eq. (5),  $y$  has been set equal to  $d$  to obtain Kolmogorov scaling and

$$\mathcal{D} = 2D_0 S_d / (2\pi)^d$$

$$\Lambda = \Lambda_d e^{-r}$$

where  $S_d$  denotes the area of the  $d$ -dimensional sphere. Integrating the recursion relation (5) in the high Reynolds number asymptotic limit<sup>4</sup>  $r \rightarrow \infty$ ,

$$\hat{T}_1 = \frac{2}{5}K \quad (6)$$

Eqs. (4) and (6) give

$$T_1 = \frac{2}{5}K \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad (7)$$

in agreement with the analysis of Crow.<sup>16</sup>

At the next order in  $SK/\varepsilon$ ,

$$I = \int q_i q_p q^{-2} \left\{ P_{jrs}(-\mathbf{q}) G(-\hat{q}) |G(\hat{q} - \hat{Q})|^2 \times \right. \\ \left. P_{qs}(\mathbf{Q} - \mathbf{q}) |\mathbf{Q} - \mathbf{q}|^{-y} 2D_0 \cdot v_r(\hat{k} - \hat{Q}) \right\} d\hat{q}/(2\pi)^{d+1} \cdot iQ_q u_p(\hat{Q}) + (ij) \quad (8)$$

where  $(ij)$  indicates the result of  $ij$  index interchange in the previous term. Adding the contribution from  $II$ ,

$$T_2 = \hat{T}_2 \frac{1}{105} \left[ 16 \left( \frac{\partial u_j^<}{\partial x_p} + \frac{\partial u_p^<}{\partial x_j} \right) \frac{\partial u_i^<}{\partial x_p} + 2 \left( \frac{\partial u_j^<}{\partial x_p} + \frac{\partial u_p^<}{\partial x_j} \right) \frac{\partial u_p^<}{\partial x_i} \right]^{(0)} + (ij)$$

where (0) indicates the deviatoric part: it is immediate that the  $ij$  index contraction of Eq. (8) vanishes. The amplitude  $\hat{T}_2$  satisfies

$$\frac{d\hat{T}_2}{dr} = -\frac{\mathcal{D}}{4\nu^2\Lambda^4}$$

In the high Reynolds number limit,

$$\begin{aligned}
T_2 &= -\frac{1}{105} \frac{1}{4} \frac{40}{3} \frac{3}{4} \nu \left[ 32 \left( \frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_j}{\partial x_p} + 4 \left( \frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_p}{\partial x_j} \right]^{(0)} + (ij) \\
&= -\frac{1}{21} \nu \left[ 16 \left( \frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_j}{\partial x_p} + 2 \left( \frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_p}{\partial x_j} \right]^{(0)} + (ij) \quad (9)
\end{aligned}$$

The next order will produce a term  $T_3$  containing cubic products of velocities  $u^<$ . In view of the form of the LRR model, it is reasonable to ask whether a term with only one gradient, proportional in the high Reynolds number limit to  $\tau \nabla U$  might occur at this order. Such terms do occur, but they cancel. Evaluation of  $T_3$  proves to require expansions of the projection operators to second order, leading instead to terms  $S^{(3)}$  homogeneous of degree three in the mean velocity gradient and its transpose. In general, the term  $T_n$  of order  $n$  has the form  $S^{(n)}(K/\varepsilon)^n$ . As noted in the Introduction, it will be imperative to include effects of all orders in  $SK/\varepsilon$  in the model, but because the terms  $T_n$  involve ever higher order derivatives of the transverse projection operators, they do not have an obvious law of formation. Therefore, an exact summation does not appear feasible.

It can be verified that the terms in braces in  $T_1$  and  $T_2$  are the lowest order terms in the expansion of the correlation spectrum tensor  $E_{mn}$  with the property

$$\overline{u_m u_n} = \int E_{mn}(\mathbf{q}) d\mathbf{q}$$

and that this identification holds to all orders. Thus, perturbation theory gives the standard result

$$\Pi_{ij} = \int q^{-2} [q_{ip} E_{jq} + q_{jp} E_{iq}] d\mathbf{q} \cdot \frac{\partial U_p}{\partial x_q}$$

but with a series for the right hand side which can be evaluated explicitly to any finite order. In fact, in view of Eqs. (7) and (9),

$$\begin{aligned} \Pi_{ij} = & \frac{2}{5} K \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \\ & - \frac{16}{21} \nu \left[ \left( \frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_j}{\partial x_p} + \left( \frac{\partial U_j}{\partial x_p} + \frac{\partial U_p}{\partial x_j} \right) \frac{\partial U_i}{\partial x_p} \right] \\ & - \frac{2}{21} \nu \left[ \left( \frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_p}{\partial x_j} + \left( \frac{\partial U_j}{\partial x_p} + \frac{\partial U_p}{\partial x_j} \right) \frac{\partial U_p}{\partial x_i} \right] + \varepsilon \sum_{n \geq 3} \mathbf{S}^{(n)} (K/\varepsilon)^n \end{aligned} \quad (10)$$

A simple approximate summation is obtained by introducing into Eq. (10) the perturbation series<sup>5</sup> for  $\overline{u_i u_j}^{(0)}$  in the form

$$\nu \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) = -\overline{u_i u_j}^{(0)} + \sum_{n \geq 2} \mathbf{S}^{(n)} (K/\varepsilon)^n$$

and dropping the quadratic terms. The resulting model,

$$\begin{aligned} \Pi_{ij} = & \frac{2}{5} K \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + C_{i1} \left[ \overline{u_i u_p}^{(0)} \frac{\partial U_j}{\partial x_p} + \overline{u_j u_p}^{(0)} \frac{\partial U_i}{\partial x_p} \right]^{(0)} \\ & + C_{i2} \left[ \overline{u_i u_p}^{(0)} \frac{\partial U_p}{\partial x_j} + \overline{u_j u_p}^{(0)} \frac{\partial U_p}{\partial x_i} \right]^{(0)} \end{aligned} \quad (11)$$

with

$$C_{i1} = \frac{16}{21} = .7619 \quad C_{i2} = \frac{2}{21} = .0952 \quad (12)$$

agrees with the perturbation series (10) to terms of order  $\mathbf{S}^{(3)}$ . However, unlike the explicit quadratic model which results from simply dropping the  $O(\mathbf{S}^{(3)})$  terms in Eq. (10), this model includes effects of all order in  $\mathbf{S}K/\varepsilon$ . The consequences of this fact will be discussed later. This type of summation has also been applied by Yoshizawa<sup>14</sup>. Eqs. (11) and (12)

can be compared with Launder, Reece and Rodi's "model 1", Eq. (11) with the empirically adjusted constants

$$C_{\tau 1} = .7636 \quad C_{\tau 2} = .1091 \quad (13)$$

The constants  $C_{\tau 1}$  and  $C_{\tau 2}$  are not chosen independently; instead, to insure certain consistency properties<sup>6,7</sup>, they are linear functions of a parameter  $C_2$ . The LRR model corresponds to the choice  $C_2 = .4$ ; Eqs. (11) and (12) correspond instead to the choice  $C_2 = 8/21 \sim .36$ .

The approximate summation used to derive Eq. (11) can be systematically generalized to generate an infinite number of models for  $\Pi_{ij}$ . For example, suppose that the perturbation series for  $\tau$  is introduced into the cubic terms in the perturbation series (10) instead of in the quadratic terms as above. This substitution will produce a model which can be written symbolically in the form

$$\Pi \sim \mathbf{S}^{(1)} + \mathbf{S}^{(2)} + \tau(\mathbf{S}^{(1)'} + \mathbf{S}^{(2)'})$$

where  $\tau\mathbf{S}^{(i)'}$  denotes a sum of matrix products in all possible orders of  $\tau$  and terms  $\mathbf{S}^{(i)}$ . The requirement that the original series agree to order  $\mathbf{S}^{(4)}$  with the approximation when  $\tau$  is replaced by its perturbation series determines this approximation uniquely.

There is obviously a strong formal resemblance between this approximation scheme and Padé approximation. They differ in the introduction of  $\tau$  as an auxiliary quantity, but more fundamentally in the non-commutativity of all of the variables  $\tau$ ,  $\nabla U$ , and  $\nabla U^T$ . It will be evident in Sect. IV that introduction of these approximations into a Reynolds

stress transport equation will lead under suitable hypotheses to algebraic models for  $\tau$  of the form

$$\tau \sim K \left( \mathcal{S}^{(1)} + \dots + \mathcal{S}^{(n)} \right) + \tau \left( \mathcal{S}^{(1)'} + \dots + \mathcal{S}^{(n)'} \right)$$

which formally express  $\tau$  as a ratio of polynomials of degree  $n$  in  $\nabla U$  and  $\nabla U^T$ . It is noteworthy that the LRR model is the lowest order member of this series. Moreover, the analysis suggests that there is no unique optimal form for the closure of  $\Pi$  in terms of  $\tau$ ,  $\nabla U$ , and  $\nabla U^T$ ; instead, there is a series of approximations of (presumably) increasing accuracy.

The LRR model has been criticized by Shih and Lumley<sup>2</sup> because it fails to insure realizability at the limit of two component flow. Speziale<sup>1</sup> has found different limitations when this model is applied to homogeneous shear flow. It is not clear whether a different summation procedure would lead to models which could answer these criticisms, perhaps of the forms proposed in Refs. 1 and 2. However, these references also indicate that what conditions a good model should satisfy is itself a somewhat controversial question. Accordingly, the agreement of this theory with a plausible and often used pressure strain model is encouraging.

## II. The Return to Isotropy Model

The analytical description of return to isotropy is no less controversial than the modeling of the fast pressure strain term<sup>3</sup>. In the usual approach to turbulence modeling, in which correlations generated by Reynolds averaging are closed phenomenologically, this process is considered to result partly from the pressure correlation through a “slow” term

independent of the mean flow, and partly from the deviatoric part of the dissipative correlation  $\left\langle \nu_0 \frac{\partial u_i}{\partial x_p} \frac{\partial u_j}{\partial x_p} \right\rangle$ . From this viewpoint, the analysis in Sect. I is incomplete because it discloses only a term proportional to the mean velocity gradient, but no slow term. The return to isotropy will be derived here by renormalization group methods following a suggestion of Yakhot<sup>17</sup>.

From the renormalization group viewpoint, it is natural to investigate the return to isotropy, even independently of the stress transport equation, by writing the perturbation series for

$$u_i \frac{\partial u_j}{\partial t} + u_j \frac{\partial u_i}{\partial t} = \int u_i(\hat{k} - \hat{q})(-i\omega)u_j(\hat{q})d\hat{q} + (ij) \quad (14)$$

This perturbation series differs from the perturbation series for the Reynolds stresses previously reported<sup>5</sup> only in the occurrence of an additional factor  $-i\omega$  in all frequency integrals. It is therefore natural to identify the sum of this series as a functional of the Reynolds stresses which, like the slow term of turbulence modeling, is independent of the mean flow. By substituting the Navier-Stokes equations for the time derivatives in Eq. (14), one just recovers the equations of motion for velocity products, Eq. (2); thus, the quantity

$$\Pi'_{ij} = \left\langle u_i \frac{\partial u_j}{\partial t} + u_j \frac{\partial u_i}{\partial t} \right\rangle$$

which results from eliminating all fluctuating modes from Eq. (14) contains contributions from the pressure correlation through terms containing the transverse projection operator and contributions from the dissipative correlation through the terms containing  $\nu_0$ .

By evaluating these terms in the form of Eq. (14) and insisting that the sum be independent of the mean flow, we are extracting properties which were not disclosed in the analysis of Sect. I.

The analysis is straightforward. Only the deviatoric terms require attention because the part of the correlation proportional to  $\delta_{ij}$  contributes to the transport equation for  $K$  which has been analyzed by Yakhot and Smith<sup>15</sup>. The lowest order deviator appears at first order in  $\eta$ ; to lowest order in  $\epsilon$

$$T_1 = \hat{T}_1 \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad (15)$$

where

$$\frac{d\hat{T}_1}{dr} = \frac{1}{15} \frac{\mathcal{D}}{\nu \Lambda^2} \quad (16)$$

In view of the form of the Rotta model, it is reasonable to seek terms at the next order proportional to  $u_i u_j$ . As in Sect. I, such terms do appear, but cancel exactly. This apparently ubiquitous cancellation was also obtained by Smith and Reynolds<sup>18</sup> in an analysis of the  $\epsilon$  transport equation. Accordingly, the second order analysis in  $\eta$  produces quadratic terms in the velocity gradients. Finite truncation of this series violates the requirement that return to isotropy be independent of the mean flow. Therefore, we must seek a reasonable approximate summation. The form of the lowest order term given in Eqs. (15) and (16) suggests

$$\Pi'_{ij} = \frac{\hat{T}_1}{\nu} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \sim -\frac{\hat{T}_1}{\nu} \overline{u_i u_j}^{(0)}$$

Despite its triviality, this replacement does produce an approximate sum which agrees exactly with perturbation theory to lowest order. It therefore can be considered a type of Padé approximation. This lowest order summation yields



$$\Pi'_{ij} = -Z \overline{u_i u_j}^{(0)} \quad (17)$$

where

$$\frac{dZ}{dr} = \frac{1}{15} \frac{\mathcal{D}}{\nu^2 \Lambda^2} \quad (18)$$

At the infinite Reynolds number asymptotic limit, Eqs. (17) and (18) iterate to<sup>4</sup>

$$\Pi'_{ij} = -C_R \frac{\varepsilon}{K} \overline{u_i u_j}^{(0)} \quad (19)$$

where, in the Yakhot-Orszag theory,  $C_R = \mathcal{D}/\varepsilon \sim 1.6$ . Equation (19) is therefore simply the standard Rotta model with Rotta constant  $\sim 1.6$  in agreement with an earlier proposal of Yakhot<sup>17</sup>.

A preliminary discussion of higher order summation may be appropriate. By analyzing the spectral dynamics of the return to isotropy, Weinstock<sup>3</sup> concluded that the shear and normal stresses relax at different rates. Although this behavior is obviously not accommodated by the Rotta model, it is consistent with the present theory: the perturbation series for  $\Pi'$  is obtained from the series for  $\tau$  by multiplying the term of order  $n$  by the factor  $C_n \varepsilon / K$  for some constant  $C_n$ . The  $C_n$  are all unequal; therefore, the Rotta model is not exact. Now comparison with the series for  $\tau$  shows<sup>5</sup> that relaxation of the shear stress is governed by the linear term  $\mathbf{S}^{(1)}$ , whereas relaxation of the normal stresses is governed by the quadratic term  $\mathbf{S}^{(2)}$ . Since  $C_2 \neq C_1$ , these stresses relax at different rates. The difference is suppressed in the Rotta model, which arose in the present formalism by replacing all of the  $C_n$  by  $C_1$ .

### III. Reynolds Stress Transport Models

The renormalization group describes the effect of the universal small scales of turbulence on the large scales. Therefore, convection and production, which are determined entirely by the large scales, cannot be derived by the renormalization group and must be introduced instead by Reynolds averaging<sup>15</sup>. Combining the usual convection and production terms with the pressure correlation and return to isotropy models, Eqs. (11) and (19), gives the Reynolds stress transport equation

$$\begin{aligned} \frac{\partial \overline{u_i u_j}^{(0)}}{\partial t} + U_p \frac{\partial \overline{u_i u_j}^{(0)}}{\partial x_p} = & - \left( \overline{u_i u_p} \frac{\partial U_j}{\partial x_p} + \overline{u_j u_p} \frac{\partial U_i}{\partial x_p} \right)^{(0)} - C_R \frac{\varepsilon}{K} \overline{u_i u_j}^{(0)} \\ & + \frac{2}{5} K \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + C_{\tau 1} \left( \overline{u_i u_p}^{(0)} \frac{\partial U_j}{\partial x_p} + \overline{u_j u_p}^{(0)} \frac{\partial U_i}{\partial x_p} \right)^{(0)} \\ & + C_{\tau 2} \left( \overline{u_i u_p}^{(0)} \frac{\partial U_p}{\partial x_j} + \overline{u_j u_p}^{(0)} \frac{\partial U_p}{\partial x_i} \right)^{(0)} + \text{diff.} \end{aligned} \quad (20)$$

where “diff” denotes diffusion terms which will be discussed later. The predictions of models of the form (20) for homogeneous shear flow, in which the diffusion terms vanish, have been analyzed definitively by Speziale.<sup>19</sup> It will be useful to generalize this analysis somewhat and consider any simple shear flow with exactly one nonvanishing mean velocity gradient component  $S = \partial U_1 / \partial x_2$ , in which diffusion of all Reynolds stress components is negligible, and in which  $\overline{u_i u_j} / K$  is constant. These are the conditions under which Rodi's algebraic models<sup>20</sup> can be derived and include homogeneous shear flow as a special case. It follows from Eq. (20) that under these conditions,

$$(\overline{u_1 u_2} / K)^2 = \frac{4}{15} \frac{P/\varepsilon}{C_R + P/\varepsilon - 1} - \frac{2}{3} \left( \frac{P/\varepsilon}{C_R + P/\varepsilon - 1} \right)^2 \left[ (C_{\tau 1} - 1)^2 \right]$$

$$\begin{aligned}
& -4(C_{\dot{\tau}1} - 1)C_{\dot{\tau}2} + C_{\dot{\tau}2}^2 \Big] \\
\overline{u_1 u_1}^{(0)}/K &= \frac{P/\varepsilon}{C_R + P/\varepsilon - 1} \left[ -\frac{4}{3}(C_{\dot{\tau}1} - 1) + \frac{2}{3}C_{\dot{\tau}2} \right] \\
-\overline{u_2 u_2}^{(0)}/K &= \frac{P/\varepsilon}{C_R + P/\varepsilon - 1} \left[ -\frac{2}{3}(C_{\dot{\tau}1} - 1) + \frac{4}{3}C_{\dot{\tau}2} \right]
\end{aligned} \tag{21}$$

where  $P = -S\overline{u_1 u_2}$  is production. Note in particular that the ratio

$$-\overline{u_1 u_1}^{(0)}/\overline{u_2 u_2}^{(0)} = \frac{-\frac{4}{3}(C_{\dot{\tau}1} - 1) + \frac{2}{3}C_{\dot{\tau}2}}{-\frac{2}{3}(C_{\dot{\tau}1} - 1) + \frac{4}{3}C_{\dot{\tau}2}} \tag{22}$$

is independent of both  $P/\varepsilon$  and of the model constant  $C_R$ .

These results can be applied to model calibration. Rotta noted<sup>7</sup> that certain consistency conditions require

$$8C_{\dot{\tau}1} - C_{\dot{\tau}2} = 6 \tag{23}$$

In view of Eqs. (22) and (23), the model proposed here, which is defined by the values of  $C_{\dot{\tau}1}$  and  $C_{\dot{\tau}2}$  of Eq. (12) is the unique model of its form for which

$$-\overline{u_1 u_1}^{(0)}/\overline{u_2 u_2}^{(0)} = 4/3 \tag{24}$$

The validity of Eq. (24) can be assessed from the experimental data summarized in Table II. Other Reynolds stress ratios can also be readily evaluated. Substituting Eq. (23) into Eqs. (21)

$$\begin{aligned}
\overline{u_1 u_1}^{(0)}/K &= C^* \left( 4C_{\dot{\tau}1} - \frac{8}{3} \right) \\
-\overline{u_2 u_2}^{(0)}/K &= C^* \left( 10C_{\dot{\tau}1} - \frac{22}{3} \right) \\
-\overline{u_3 u_3}^{(0)}/K &= C^* \left( -6C_{\dot{\tau}1} - \frac{14}{3} \right)
\end{aligned}$$

$$(\overline{u_1 u_2}/K)^2 = \frac{4}{15}C^* - \frac{2}{3}(C^*)^2 (33C_{\dot{\tau}1}^2 - 42C_{\dot{\tau}1} + 13) \quad (25)$$

where

$$C^* = \frac{P/\varepsilon}{C_R + P/\varepsilon - 1}$$

Simple shear flow data suggest that  $\overline{u_1 u_1}^{(0)}/K$ ,  $-\overline{u_2 u_2}^{(0)}/K$ , and  $-\overline{u_3 u_3}^{(0)}/K$  are all positive. In view of Eq. (25), this forces  $C_{\dot{\tau}1}$  to be in the narrow interval

$$\frac{11}{15} \leq C_{\dot{\tau}1} \leq \frac{7}{9}$$

from about .73 to .78. Values of the Reynolds stress ratios for various values of  $C_{\dot{\tau}1}$  appear in Table I. No value of  $C_{\dot{\tau}1}$  gives entirely satisfactory values for all of the ratios; in particular, the ratio  $\overline{u_1 u_1}^{(0)}/K \sim .4$  in homogeneous shear flow cannot be obtained from a model of this form.

Phenomenological modelling has suggested numerous modifications of the original LRR form. The simplest modification retains the form of Eq. (11), but drops the constraint expressed by Eq. (23). Thus, the constants  $C_{\dot{\tau}1}$  and  $C_{\dot{\tau}2}$  are considered independent, as in Launder, Reece and Rodi's "model 2"<sup>6</sup>, or in the model of Speziale, Sarkar, and Gatski<sup>1</sup>. Justification of this step requires that the idea of modelling each individual correlation in the exact Reynolds stress transport equation be abandoned; instead, the model is proposed for the entire equation at once. A reasonable model of this type is Eq. (11) with the constants

$$C_{\dot{\tau}1} = \frac{43}{63} \quad C_{\dot{\tau}2} = \frac{8}{63}$$

chosen so that the Reynolds stress transport equation reduces to the nonlinear model of Ref. 5 when convection and diffusion of Reynolds stress are negligible. The values of the Reynolds stress ratios for this model in Table I are in excellent agreement with shear flow data, but the fundamental validity of such modelling is unclear. An alternative procedure is to explore the higher order summations described in Sect. I.

A point emphasized by both Speziale<sup>19</sup> and Reynolds<sup>21</sup> can be noted here, that while the elementary eddy viscosity formula and its nonlinear generalization<sup>5,11,12</sup> applied to simple shear flow give

$$\overline{u_1 u_2}/K \sim SK/\varepsilon \quad \overline{u_i u_i^{(0)}}/K \sim (SK/\varepsilon)^2 \quad (26)$$

for any value of  $SK/\varepsilon$ , models of the form of Eq. (20) give (26) for moderate  $SK/\varepsilon$ , but

$$\overline{u_1 u_2}/K \sim \text{const.} \quad \overline{u_i u_i^{(0)}}/K \sim \text{const.} \quad (27)$$

in the rapid distortion limit  $SK/\varepsilon \rightarrow \infty$ . The finite limit expressed by Eq. (27) also holds for algebraic Reynolds stress models derived following Rodi's original suggestion<sup>20</sup>, for example for the model proposed in Ref. 22,

$$\nu_T = \frac{2}{15} \frac{K^2/\varepsilon}{C_R + (P/\varepsilon - 1)} \quad (28)$$

The behavior (26) is certainly incorrect when  $SK/\varepsilon \rightarrow \infty$ ; this reflects the derivation, for example of the relation  $\nu_T \sim K^2/\varepsilon$  from Kolmogorov scaling. This derivation assumes quasi-static spectral evolution in which the Kolmogorov spectrum instantaneously adjusts to local conditions in both space and time. The finite limit (27) at large  $SK/\varepsilon$  can only

occur if effects of all order in  $SK/\varepsilon$  are included. No finite truncation of the  $\eta$ -expansion will have this behavior.

#### IV. Algebraic Reynolds Stress Models

The approximation, due to Rodi<sup>20</sup>, of the Reynolds stress transport equation by an algebraic model under the conditions of semi-homogeneous flow (negligible diffusion of  $\tau$  and  $\tau/K$  approximately constant) takes the form

$$\frac{P - \varepsilon}{K} \overline{u_i u_j}^{(0)} = - \left( \overline{u_i u_p} \frac{\partial U_j}{\partial x_p} + \overline{u_j u_p} \frac{\partial U_i}{\partial x_p} \right)^{(0)} + \Pi'_{ij} + \Pi_{ij} \quad (29)$$

where  $\Pi$  and  $\Pi'$  depend on  $\tau$  and  $\nabla U$ . Explicit solutions for  $\tau$  can be obtained, at least in principle, for any such approximation<sup>23</sup>. Briefly, one introduces a basis for polynomials in  $\nabla U$ , and  $\nabla U^T$ . The basis contains 11 terms of homogeneity order  $n \leq 5$ . Writing  $\tau$  as a sum of these terms with unknown coefficients and substituting in Eq. (29) leads to the explicit expression

$$\tau/K = \sum H_i^{(m)} S_i^{(n)} (\nabla U, \nabla U^T) \quad (30)$$

where  $H_i^{(m)}$  is a scalar function of  $\nabla U$  and  $\nabla U^T$  such that

$$H_i^{(m)} \sim |\nabla U|^m$$

when  $|\nabla U| \rightarrow \infty$ . The assumptions made on the approximate summations require  $m + n = 0$ ; thus,  $\tau/K$  is bounded when  $SK/\varepsilon \rightarrow \infty$ . For example, the familiar eddy viscosity formula is replaced in Eq. (30) by a term

$$\tau \sim \frac{K^2}{\varepsilon} H^{(-1)} (\nabla U, \nabla U^T) (\nabla U + \nabla U^T)$$

This is the type of eddy viscosity modification sought by Horiuti<sup>24</sup>, but with the boundedness property expressed by Eq. (27).

Pope observed<sup>23</sup> that the coefficients  $H^{(-n)}$  in Eq. (30) would certainly be intractably complex; although they could be explicitly exhibited by symbolic computation, the result would only pertain to the particular implicit equation for the Reynolds stresses assumed initially in Eq. (29). Therefore, it is equally reasonable just to postulate simple forms for the functions  $H^{(-n)}$ ; for example, Eq. (28) suggests the possibility

$$H^{(-1)} = \frac{2}{15} \frac{1}{C_R - 1 + \sqrt{C_\nu} SK/\varepsilon}$$

based on the identification  $\tau/K \sim \sqrt{C_\nu}$ . This type of modeling could be particularly interesting when applied to the coefficients of the quadratically nonlinear models of Refs. 5, 11, and 12.

Some modifications of the summation procedures used here suggest themselves. First, it is perhaps closer to the spirit of Padé approximation to substitute  $\Pi$  itself into its perturbation series instead of  $\tau$ . The procedure would lead to fast pressure strain models which are explicit function of  $\nabla U$  and  $\nabla U^T$ . Although previous experience strongly suggests the appearance of  $\tau$  in the model, this unconventional procedure may deserve further consideration. A second related possibility is to apply the Padé method to the perturbation series<sup>5</sup> for  $\tau$  directly. This also will produce a family of implicit models linear in  $\tau$  and of all orders in  $\nabla U$  and  $\nabla U^T$ , and to explicit models of the form (30). The lowest order such model, analogous to the LRR model, would have the form

$$\begin{aligned}\overline{u_i u_j} = & \frac{2}{3} K \delta_{ij} - \nu \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + C_1 \frac{K}{\varepsilon} \left( \overline{u_i u_p}^{(0)} \frac{\partial U_j}{\partial x_p} + \overline{u_j u_p}^{(0)} \frac{\partial U_p}{\partial x_i} \right) \\ & + C_2 \frac{K}{\varepsilon} \left( \overline{u_i u_p}^{(0)} \frac{\partial U_p}{\partial x_j} + \overline{u_j u_p}^{(0)} \frac{\partial U_i}{\partial x_p} \right)\end{aligned}$$

with model constants  $C_1$  and  $C_2$ . This is actually the procedure followed by Yoshizawa<sup>14</sup>, although Yoshizawa's two-scale perturbation theory leads naturally to a transport model rather than to an algebraic model.

## V. The Diffusion Term

The diffusion of Reynolds stress arises from the triple correlation

$$\frac{\partial}{\partial x_p} \langle u_i u_j u_p \rangle$$

Lowest order analysis of this term will lead to a diffusive term

$$\frac{\partial}{\partial x_p} \alpha_\tau \nu \frac{\partial \overline{u_i u_j}}{\partial x_p}$$

where  $\alpha_\tau \sim 1.4$  in the high Reynolds number limit<sup>4</sup>, otherwise expressed, to an isotropic diffusivity for Reynolds stress  $\kappa = \alpha_\tau \nu$ .

In Ref. 22, we analyzed the diffusion of a passive scalar to second order in the  $\eta$ -expansion, and found, as in a similar analysis by Yoshizawa<sup>25</sup>, corrections leading to

$$\kappa_{ij} = \kappa \delta_{ij} + \kappa' \frac{K}{\varepsilon} \frac{\partial U_i}{\partial x_j} \quad (31)$$



Unlike the isotropic diffusivity, this model permits a nonzero diffusivity  $\kappa_{12}$ . However, it rules out inequality of the normal diffusivities. At the next order, perturbation theory will correct Eq. (31) with terms quadratic in the velocity gradients. In this theory, unequal normal diffusivities are possible. But now the discussion in the Introduction applies again: it is necessary to sum this  $\eta$ -expansion. The type of Padé approximation used in Sects. I-II may lead to a diffusivity dependent on the Reynolds stresses, as in the passive scalar models of Rogers et al.<sup>26</sup> This type of diffusivity has also been proposed by Launder, Reece and Rodi<sup>6</sup> for the Reynolds stresses. The details are considerably more elaborate than for the pressure correlation, and this possibility will be left for future investigation.

## VI. Conclusions

The present analysis of the Reynolds stress transport equation, based on the Yakhot-Orszag renormalization group and (tensorial)  $\eta$ -expansion summation as suggested by Yakhot et al.<sup>9</sup>, has led to a model transport equation incorporating the well-known LRR and Rotta models. The analysis gives theoretical support both to these models and to the constants sometimes used with them. More significantly, it exhibits the LRR and Rotta models as lowest order approximations, and therefore also supports their replacement with higher order nonlinear models which would be deduced by more accurate approximate summations. The consistency of the analysis with higher order effects like the unequal relaxation rates of shear and normal stresses has been discussed.

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