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THREE-DIMENSIONAL UNSTRUCTURED GRID GENERATION VIA INCREMENTAL INSERTION AND LOCAL OPTIMIZATION

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ABSTRACT

Algorithms for the generation of three-dimensional unstructured surface and volume grids are discussed. These algorithms are based on incremental insertion and local optimization. The present algorithms are very general and permit local grid optimization based on various measures of grid quality. This is very important; unlike the two-dimensional Delaunay triangulation, the 3-D Delaunay triangulation appears not to have a lexicographic characterization of angularity. (The Delaunay triangulation is known to minimize that maximum containment sphere, but unfortunately this is not true lexicographically). Consequently, Delaunay triangulations in three-space can result in poorly shaped tetrahedral elements. Using the present algorithms, three-dimensional meshes can be constructed which optimize a certain angle measure, albeit locally. We also discuss the combinatorial aspects of the algorithm as well as implementational details.

INTRODUCTION

Unstructured grids offer great advantages in the automatic generation of computational grids about complex geometries. In the present work we consider triangulation methods in two and three space dimensions. In order to minimize the amount of required user interaction, incremental triangulation algorithms are employed which permit automatic (adaptive) point placement. Several incremental algorithms exist for the triangulation of two and three dimensional domains. Most of these algorithms are designed to produce the Delaunay triangulation of specified sites which form the vertices of the mesh. In the present work, we have chosen an incremental triangulation algorithm based on point insertion and local edge swapping. The advantage of this algorithm is that the edge swapping can be used to optimize user specified grid quality measures. (This algorithm can be used to generate a Delaunay triangulation by proper choice of measure.) In addition, we produce Steiner triangulations by inserting new sites at triangle circumcenters of the existing triangulation to minimize the maximum triangle aspect ratio (or any other user specified criteria). An objective of the current research is to extend this two-dimensional algorithm to include the generation of surface grids or volume grids in three-dimensions. The algorithm based on edge swapping is well known in two dimensions [1] but the extension to surface grids and volume grids in three dimensions in not straightforward. In the present work, we show the extension of edge swapping to include the generation of surface grids and three dimensional volume grids. Differing edge swapping optimization criterion are considered. Steiner triangulations in three dimensions are produced by placing new sites at circumsphere centers of the existing triangulation.

Motivation:

- Develop fast automatic three-dimensional unstructured grid generation capability for complex geometries.

Objectives:

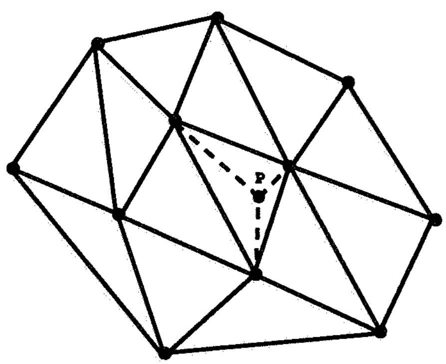
- Generalize a 2-D unstructured grid generation algorithm based on incremental insertion and edge swapping.
  - Generation of surface grids on 2-manifolds in 3-D.
  - Automatic adaptation to maintain surface fidelity in 3-D.
- Extend incremental insertion and edge swapping to 3-D volume mesh generation.
  - Investigation of differing edge swapping optimization criterion.
  - Automatic point insertion (Steiner triangulations).

Edge Swapping or Watson's Algorithm?

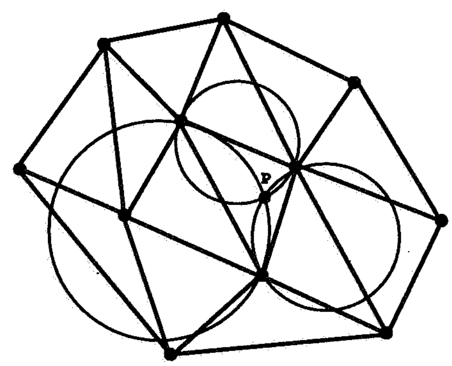
Conceptually speaking, the incremental algorithm based on edge swapping differs significantly from the incremental Watson algorithm [4] for the Delaunay triangulation. However, when edge swapping is used to generate a Delaunay triangulation, both methods have similar implementations. Both methods begin by inserting a new site P in the interior of an existing triangulation. The edge swapping algorithm begins by connecting site P to the existing triangulation. The union of all triangles incident to P forms a polygon surrounding this new site. The edges of this polygon must each pass the circumsphere test. Edges failing this test must be swapped. The process continues until all edges of the polygon pass. The algorithm is trivial to implement using recursive programming. Conceptually, Watson's algorithm finds all triangles with circumcircles containing site P. This identifies invalid edges that when removed reveal a polygon with vertices visible from P. The vertices of this polygon are then connected to P. The actual implementation of Watson's algorithm is similar to the edge swapping code. The first step is to find *any* triangle with its circumcircle containing site P. Once this triangle is found, edges are reconnected and neighboring triangles tested. This can also be easily programmed recursively. The resulting code is very similar to the edge swapping implementation.

Edge Swapping

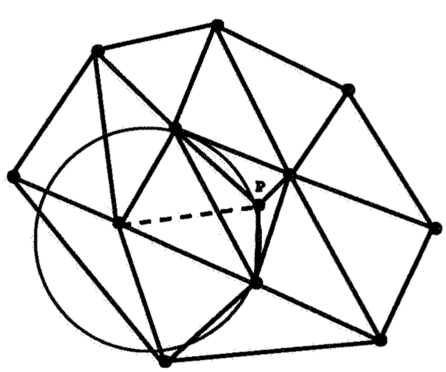
Watson's Algorithm



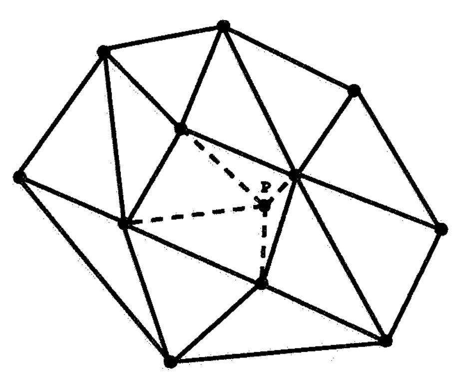
Connect Site P



Identify Circumcircles Containing Site P



Recursive Edge Swapping on Suspect Edges

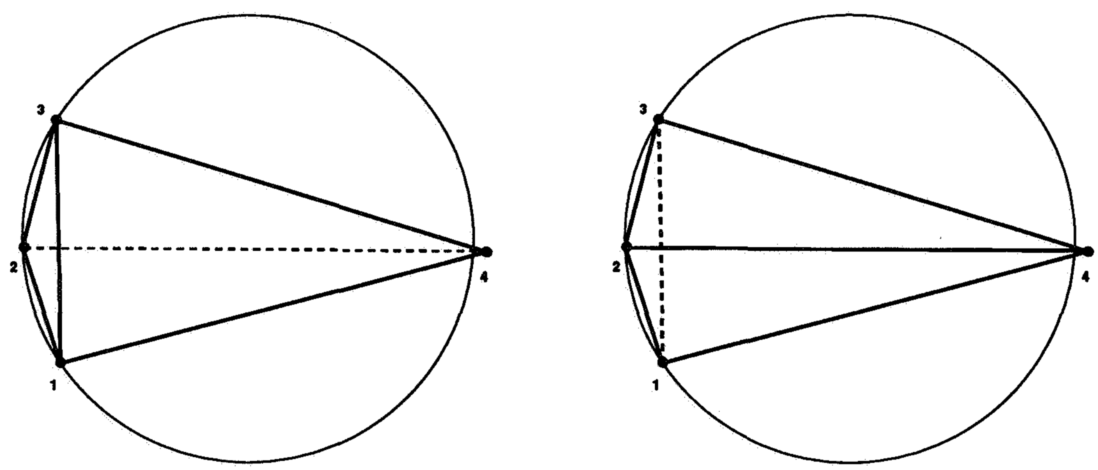


Remove Invalid Edges and Reconnect

Min-Max and Max-Min Triangulations

The edge swapping algorithm easily extends to allow edge swapping based on criteria other than the circumsphere test. The circumsphere test is known to be equivalent to the angularity optimization that chooses the diagonal position that maximizes the minimum interior angle (Max-Min triangulation). Another popular method chooses a diagonal position for a given convex triangle pair that minimizes the maximum interior angle (the Min-Max triangulation). If the diagonal position is swapped using this measure then the four neighboring edges must be checked (the Max-Min version of this algorithm only requires that two edges be checked). It should also be noted that the Min-Max triangulation can produce triangulations that locally minimize the maximum angle but are not globally optimal with respect to the same measure.

- Advantage of incremental edge swapping is that it permits other types of local optimization.
  - Requires more sophisticated recursion implementation and usually only produces a locally optimum triangulation.
  - Typical local optimizations minimize the maximum interior angle (Min-Max) or maximize the minimum interior angle (Max-Min,Delaunay triangulation).



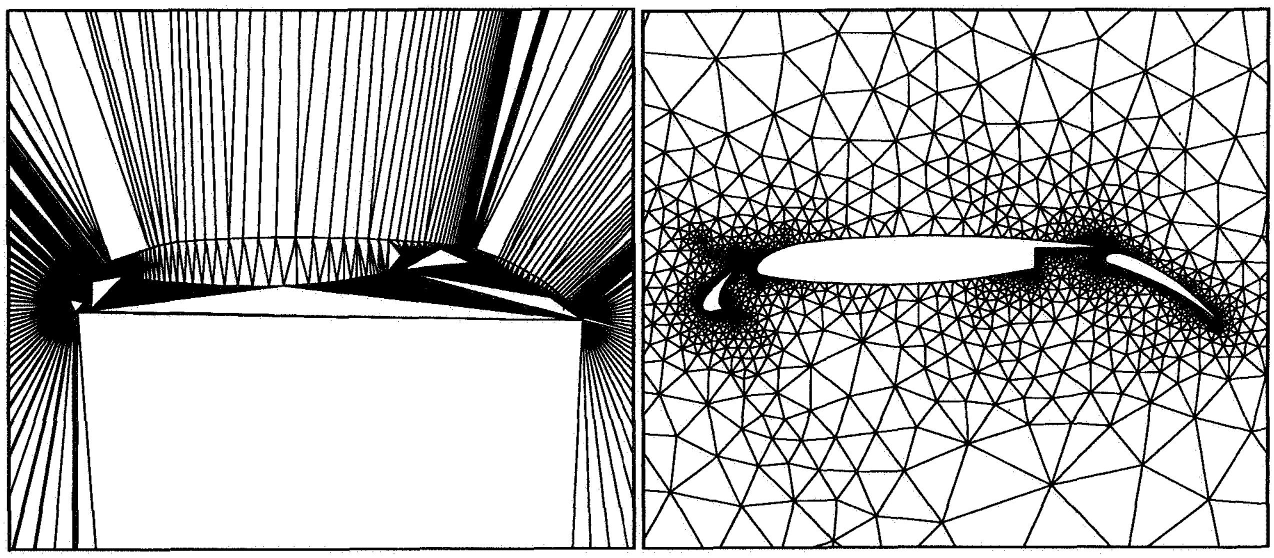
Delaunay (Max-Min) Configuration Min-Max Configuration

- Min-Max triangulation favors less obtuse triangles (important for highly stretched meshes).

Steiner Triangulations

A Steiner triangulation is any triangulation that adds additional sites into an existing triangulation to improve a specified measure of grid quality. For the present application, Delaunay triangulations are constructed which refine a given triangle by adding a new site at the circumcenter of the triangle. This technique is well known (Homes and Snyder [3], Warren et al [4]). The choice of circumcenter guarantees that no other point in the triangulation lies closer than the radius of this circle. The initial triangulation consists of a Delaunay triangulation of the boundary points (possibly edge swapped to maintain the boundary curve definitions). Points are inserted to minimize the maximum triangle aspect ratio using a dynamic heap data structure.

- Steiner Triangulation - insert additional sites into an existing triangulation to improve a specified measure of grid quality.



Initial Triangulation

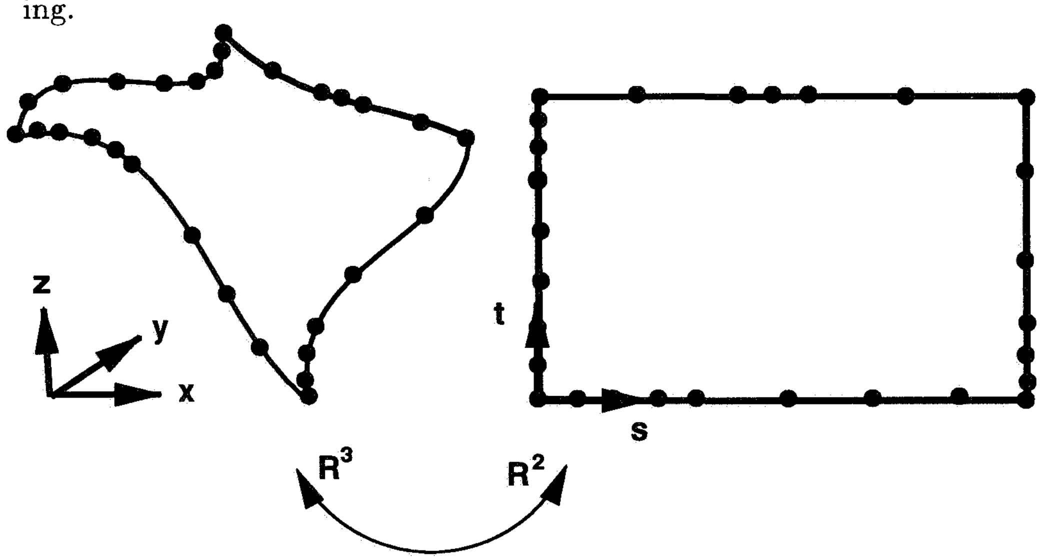
After Steiner Point Insertion and Interior Edge Removal

- Insert sites at circumcenters to minimize triangle aspect ratio.

3-D Surface Patch Projection

Three-dimensional surface grids are constructed from rectangular surface patches (assumed at least C^0 smooth) using a generalization of the 2-D Steiner triangulation scheme. Points are placed on the perimeter of each patch using an adaptive refinement strategy based on absolute error and curvature measures. The surface patches are projected onto the plane. Simple stretching of the rectangular patches permits the user to produce preferentially stretched meshes. (This is useful near the leading edge of a wing for example.)

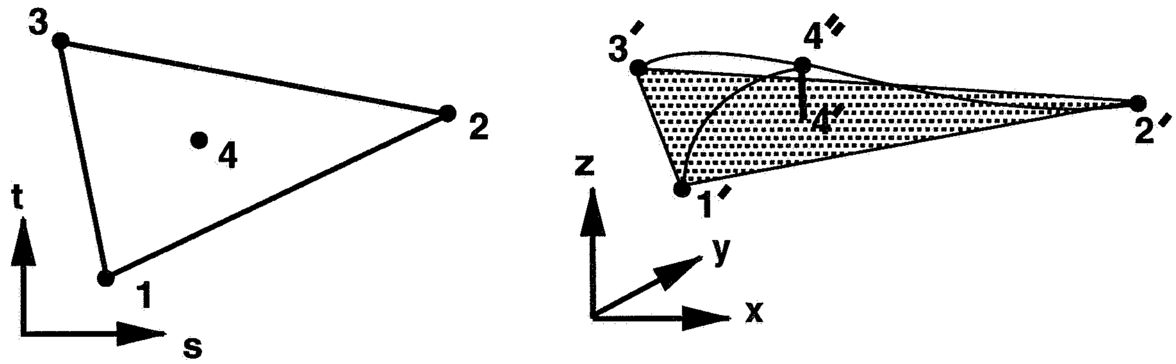
- Sites are distributed on the perimeter of each patch using 1-D adaptive subdivision.
- Curve adaptation criterion:
  - Absolute tolerance
  - Curvature tolerance
- Patches projected into 2-space for triangulation.
  - Mapping to rectangle permits simple stretching which results in directional biasing.



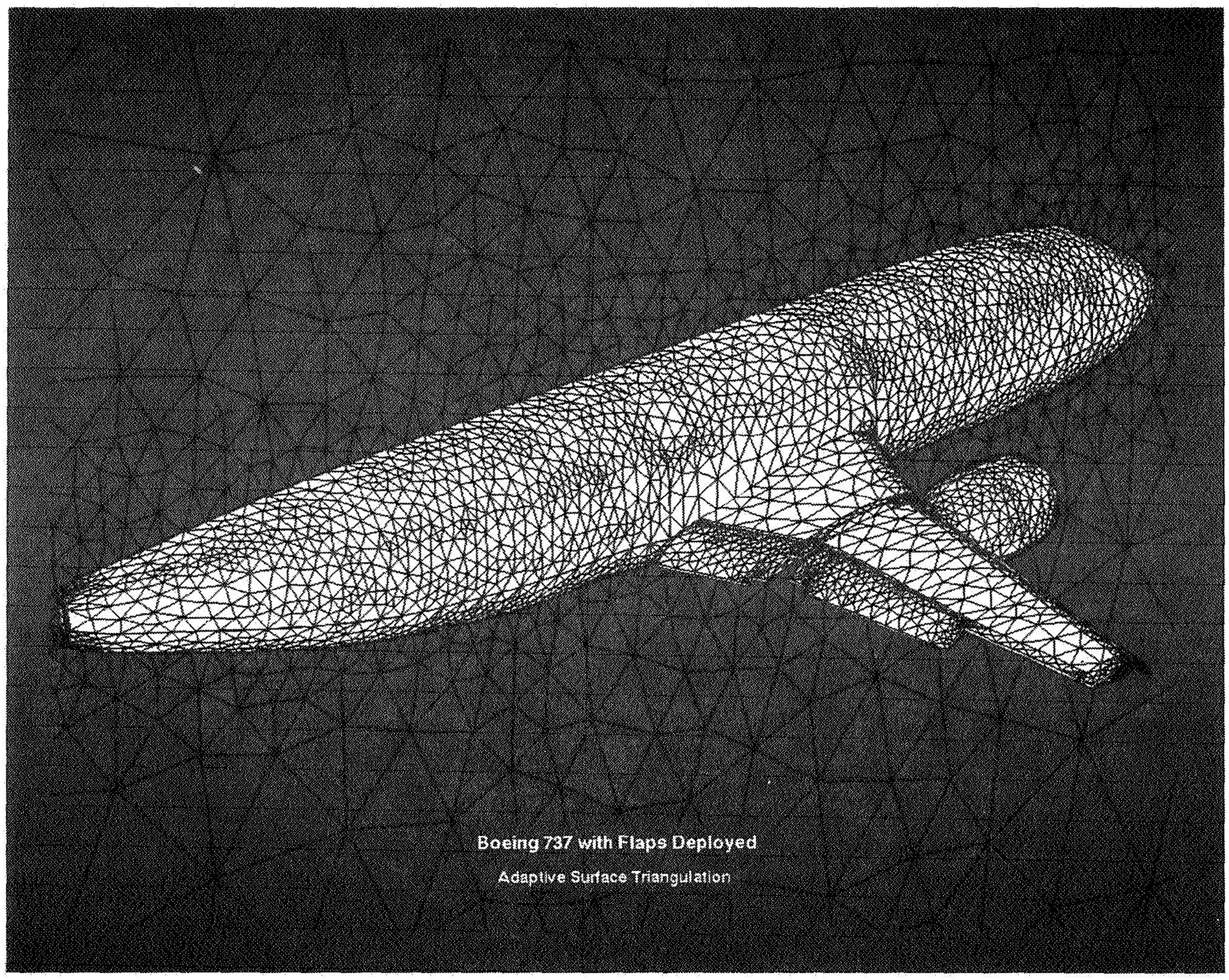
Adaptive Patch Triangulation

The triangulation takes place in the two dimensional (s, t) plane. The triangulation is adaptively refined using Steiner point insertion to minimize the maximum user specified absolute error and curvature tolerance on each patch. The absolute error is approximated by the perpendicular distance from the triangle centroid (projected back to 3-space) to the true surface. The user can further refine based on triangle aspect ratio in the (s, t) plane if desired.

- Adaptive Delaunay triangulation in (s, t) plane.
    - Adaptive insertion based on absolute error and maximum curvature tolerance.



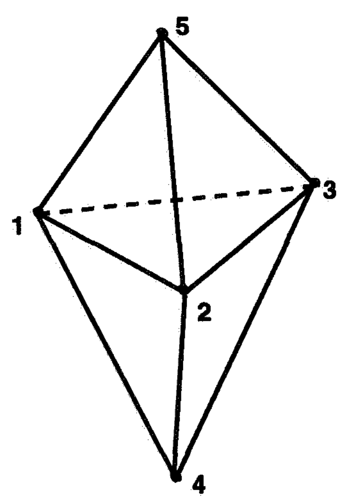
- Calculation of triangulation absolute error by measurement of distance from face centroid to true surface.
- Optional refinement based on triangle aspect ratio.


Boeing 737 with Flaps Deployed
Adaptive Surface Triangulation

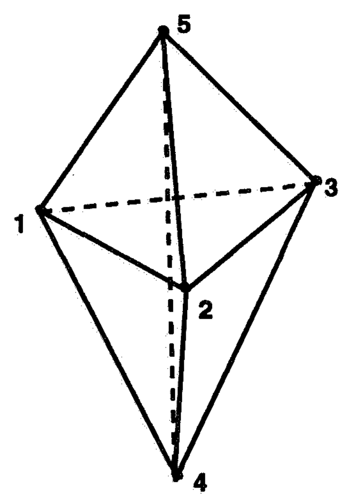
3-D Incremental Insertion and Optimization

The 3-D edge swapping methodology is based on Lawson's result [5] that there exists at most two ways to triangulate $n+1$ points in R^n. It can be shown that if a configuration of 5 points in 3-D allows only one triangulation, it will satisfy the Delaunay circumsphere test. If a configuration of 5 points in 3-D allows two triangulations, then only one will satisfy the Delaunay circumsphere test. The edge swapping algorithm is based on locally swapping convex non-Delaunay triangulations of 5 points into its Delaunay counterpart. It has been shown that during the optimization process for a general mesh the procedure may get stuck in a local optimum. However, Rajan [1] proved that inserting a site into an existing Delaunay triangulation followed by edge swapping is guaranteed to recover the new Delaunay triangulation. This forms the basis of our 3-D Incremental Insertion and Optimization algorithm for 3-D Volume Mesh Generation.

- Combinatorial theorem (Lawson): There exist at most two ways to triangulate $n+1$ points in $\mathbf{R}^n$ (subject to convexity).



2 Tetrahedron Configuration



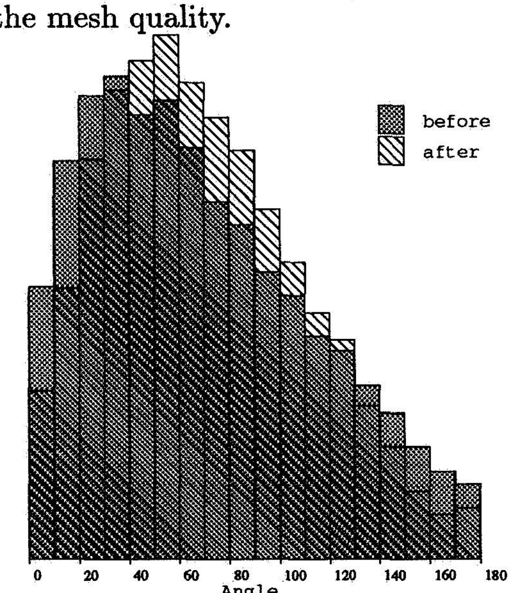
3 Tetrahedron Configuration

- Edge swapping an arbitrary 3-D triangulation (subject to circumsphere optimization) is not sufficient to guarantee that a Delaunay triangulation will be produced.
- Site insertion into an existing Delaunay triangulation followed by edge swapping will produce a new Delaunay triangulation (Rajan, 1991).

Local and Global Optimization

As long as a metric can be defined for the two triangulations of a configuration of 5 points in three dimensions, edge swapping locally to optimize that metric can be performed. In this way, edge swapping provides a general framework which allows for different criteria of judging mesh quality. This permits criteria other than the Delaunay circumsphere test. For instance, we can maximize the minimum face (or solid) angle, or minimize the maximum face (or solid) angle, or reduce the number of edges and volumes in the mesh. Local optimization is not guaranteed to yield the globally optimal mesh, but typically it produces significantly improved meshes.

- 3-D edge swapping permits local optimization based on many possible criterion such as:
  - Circumsphere
  - Maximize the minimum face (or solid) angle
  - Minimize the maximum face (or solid) angle
  - Minimize the total number of edges and volume
- Local optimization usually does not produce a globally optimal triangulation but can significantly improve the mesh quality.

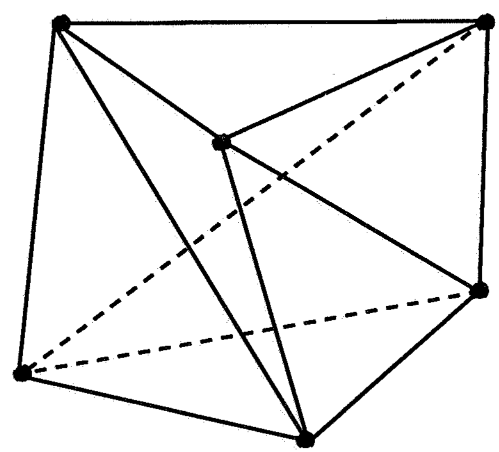


Histogram showing improvement in maximum face angle using edge swapping procedure on triangulation of 500 random sites in unit cube.

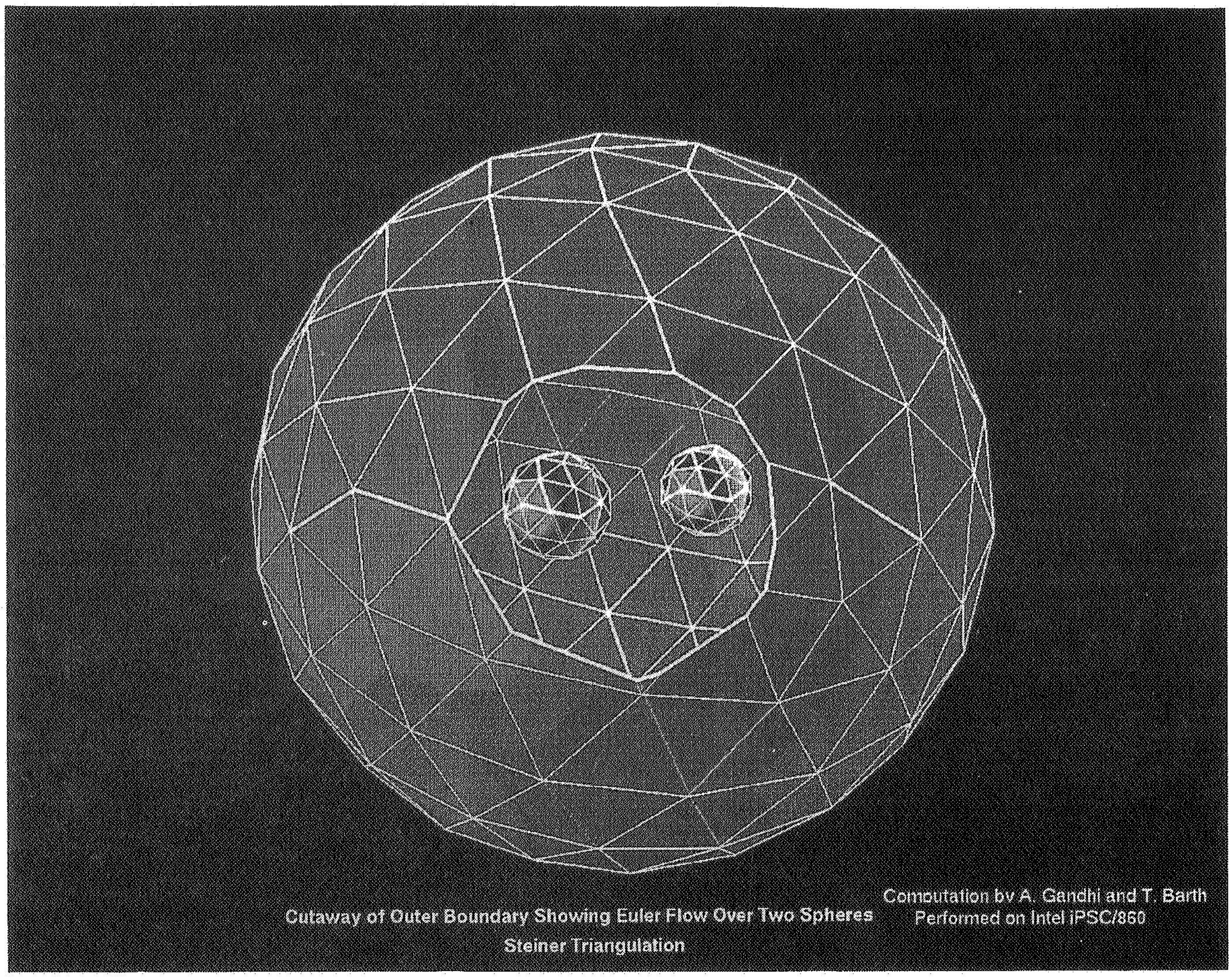
3-D Volume Triangulation

In order to perform our 3-D Incremental Insertion and Optimization algorithm for 3-D Volume Mesh Generation, we need a constrained Delaunay triangulation of the boundary in which we can incrementally add new sites. This Constrained Delaunay triangulation of the boundary sites is constructed using the Tanamura algorithm. Steiner points are introduced into this triangulation in a way that optimizes some criterion of mesh quality. We have been working with trying to minimize the maximum cell (tetrahedron) aspect ratio by introducing a site at the circumcenter of the tetrahedron. It should be noted that construction of constrained Delaunay triangulation is not always possible and we may need to add Steiner points to the interior of the untetrahedralizable volume to make it tetrahedralizable.

- Constrained Delaunay triangulation of boundary sites is constructed using Tanamura algorithm implemented by Marshal Merriam at NASA Ames
- Steiner points are inserted into this triangulation which minimize the maximum tetrahedron aspect ratio (or any other user specified criterion).
- Caveat Emptor: Construction of constrained Delaunay triangulation of boundary sites is not always possible unless Steiner points are added to the interior of the volume. Example:



Nontetrahedralizable twisted prism with reflex exterior faces


Cutaway of Outer Boundary Showing Euler Flow Over Two Spheres
Steiner Triangulation
Computation by A. Gandhi and T. Barth
Performed on Intel iPSC/860

CONCLUSIONS

We have demonstrated the feasibility of adaptive Steiner point insertion and edge swapping to generate surface meshes about complex geometries. Edge swapping algorithms have the advantage over certain other standard triangulation techniques because of the ability to improve mesh quality through local optimization. We have also shown how to extend the edge swapping idea to three dimensions using the combinatorial result of Lawson. The resulting algorithm permits adaptive Steiner triangulations about complex geometries.

- Demonstrated the feasibility of adaptive Steiner point insertion and edge swapping to generate surface meshes about complex geometries.
- Implemented an incremental insertion and edge swapping algorithm to generate 3-D volume meshes.
- Demonstrated the unique capability of edge swapping to improve mesh quality using various optimization criteria.

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5. Lawson, C. L., "Properties of n-dimensional Triangulations" CAGD, Vol. 3, April 1986, pp. 231–246.