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Conservative-Variable Average States for Equilibrium Gas Multi-Dimensional Fluxes

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CONSERVATION-VARIABLE AVERAGE STATES FOR EQUILIBRIUM GAS MULTI-DIMENSIONAL FLUXES

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Modern split component evaluations of the flux vector jacobians are thoroughly analyzed for equilibrium-gas average-state determinations. It is shown that all such derivations satisfy a fundamental eigenvalue consistency theorem. A conservation - variable average state is then developed for arbitrary equilibrium-gas equations of state and curvilinear-coordinate fluxes.

Original expressions for eigenvalues, sound speed, Mach number, and eigenvectors are then determined for a general average jacobian, and it is shown that the average eigenvalues, Mach number, and eigenvectors may not coincide with their classical pointwise counterparts.

A general equilibrium-gas equation of state is then discussed for conservation-variable CFD Euler formulations. The associated derivations lead to unique compatibility relations that constrain the pressure jacobian derivatives. Thereafter, alternative forms for the pressure variation and average sound speed are developed in terms of two average pressure jacobian derivatives. Significantly, no additional degree of freedom exists in the determination of these two average partial derivatives of pressure. Therefore, they are simultaneously computed exactly without any auxiliary relation, hence without any geometric solution projection or arbitrary scale factors.

Several alternative formulations are then compared and key differences highlighted with emphasis on the determination of the pressure variation and average sound speed. The relevant underlying assumptions are identified, including some subtle approximations that are inherently employed in published average-state procedures.

Finally, a representative test case is discussed for which an intrinsically exact average state is determined. This exact state is then compared with the predictions of recently published methods, and their inherent approximations are appropriately quantified.

Introduction

Since the notable paper by Roe [3], the classical average state and associated flux difference splitting algorithm have attracted the attention of several CFD researchers. Liou and van Leer [5] use this approach, and compare it to several flux vector splitting schemes for selected quasi 1-D problems, while Simpson [6] conducts a similar analysis for various 2-D airfoil flow field determinations. One often cited advantage of an average-state flux difference splitting algorithm lies in its reduced amount of intrinsic dissipation with respect to that inherent in the classical Steger-Warming and van Leer flux vector splittings. Furthermore, its capability to yield crisp shock resolutions makes it an attractive algorithm. These attributes have thus promoted continued research on real gas extensions, which ushered in the conspicuous works of Glaister [4], Liou, van Leer & Shuen [8], and more recently Vinokur & Montagne [9], among others.

The linking thread of these fundamental contributions is the utilization of the classical ideal-gas average state as a building block in seeking a kindred real gas average state. As detailed in Section 4, all such derivations separate the pressure from the kinetic/energy terms, which corresponds to a decomposition of the flux vector into convection and acoustic-wave components. The kinematic/energy relations are then exactly satisfied by the ideal-gas average state, whereas the pressure variation relation is evaluated at some specially devised average partial derivatives of pressure. The advantage of this procedure is that for an n-dimensional formulation, one eigenvalue of the associated average jacobian coincides with the ideal-gas average velocity, with algebraic and geometric multiplicity equal to n, while the other eigenvalues resemble the eigenvalues of the pointwise jacobian.

In the published procedures [4,8,9], the terminal average jacobian is obtained by restating the pressure jacobian in terms of two average thermodynamic pressure derivatives. The pivotal equation to be satisfied is then the pressure variation relation in terms of these two average derivatives, which are essentially regarded as two auxiliary independent variables. Hence, the pressure variation relation is viewed as a single admissible-state straight line equation in the two initially unknown partial derivatives of pressure. Therefore, an absence on mathematical closure is perceived, and additional constraints are then devised to complement the pressure variation equation. As detailed in Section 9, these constraints are inherently based upon an ideal-gas average-state evaluation of selected expressions within the jacobian pressure variation, a choice of convenient integration paths and associated auxiliary states, and a utilization of geometric trial solution projections and scale factors.

The specific avenue to develop the terminal average pressure derivatives then distinguishes each procedure. For example, in conjunction with an extra average mass-specific internal energy, Glaister [4] introduces additional intermediate states, whereas Liou et al. [8] utilize additional auxiliary partial derivatives of pressure.

These procedures actually correspond to an approximate integration of the pressure differential, upon selection of convenient integration paths, as discussed in Section 9. This notion is expressly utilized by Vinokur et al. [9] who provide an exact integral representation for these derivatives. They then integrate these expressions numerically for an alternative determination of another set of average partial derivatives. These constructions employ an independent internal energy variation, even though this variation directly depends upon the state variable, as detailed in Section 8. Futhermore, this approach generally yields approximate solutions that may not satisfy the pressure variation relation. Consequently, the terminal derivatives that do satisfy this relation are determined by projecting the approximate solutions onto the admissible-state straight line, according to a set of scale factors.

Putting aside the legitimate concern that additional intermediate states may not be permissible by the given real gas equations of state, the crucial issue is that these procedures require several assumptions and introduce different sets of average partial derivatives of pressure and sound speed expressions. And even though these relations satisfy a fundamental eigenvalue consistency theorem, they generally do not coincide with one another, or they may not exist in the domain of the analytical partial derivatives, or they may correspond to conservation-variable evaluations states outside the hyper-segment connecting the two given cell states, or they may yield indeterminate expressions. Thus, they may correspond to a state weakly related to the given cell states, notwithstanding the exact algebraic enforcement of the pressure variation expression.

In this connection, a representative test case in Section 10 illustrates the difference between an exact average state and the predictions of some of these methods. The average-state jacobian may thus become less representative of the local wave interaction patterns, thereby defeating the concept of an accurate approximate local Riemann solver for which the average-state notion was introduced in the first place. Moreover, citing Vinokur [9], the amount of intrinsic dissipation in these conditions may become a vaguely identified quantity.

In this report it is shown that the published equilibrium-gas average states [4,8,9] satisfy a fundamental eigenvalue consistency theorem. According to this theorem, if the flow is locally subsonic, the eigenvalue set of any associated average jacobian will contain mixed sign eigenvalues, whereas for a locally supersonic flow all average eigenvalue sets will contain uniform sign eigenvalues. The most recent average-state determination procedures are discussed, and a set of conservation-variable average states is then developed for arbitrary equilibrium-gas equations of state and curvilinear- coordinate fluxes. These states correspond to an internal average state belonging to the hyper-segment connecting the given computational cell left and right states.

A detailed characteristic analysis of the general curvilinear coordinate average jacobian is performed which leads to original expressions for eigenvalues, sound

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speed and Mach number in terms of the pressure jacobian. These relations, which revert to the classical thermodynamic expressions, show that the average eigenvalues, Mach number and eigenvectors may not coincide with the classical pointwise forms.

Since for an equilibrium-gas formulation, pressure is a function of only two thermodynamic variables, it is then shown that the pressure jacobian derivatives are not functionally independent. On the contrary, they satisfy a compatibility constraint that leads to alternative expressions for the pressure variation and average sound speed in terms of only two average pressure jacobian derivatives. These expressions do not explicitly depend upon the thermodynamic internal energy, since in a CFD formulation this variable is not independent, but is expressed in terms of the conservation state variable. Considering that within the mean value theorem resolution of the pressure variation, the pressure derivatives are intrinsic functions of a conservation state variable, it is shown that no additional degree of freedom exists in their computation. Therefore, these derivatives are simultaneously determined exactly without resorting to any geometric solution projection and arbitrary scale factors. The developed formulae, which bear a striking similarity with the classical thermodynamic relations, apply for totally arbitrary equilibrium-gas equations of state, and yield as approximate cases several expressions utilized in reported procedures. These alternative formulations are then compared and their intrinsic underlying assumptions are identified including some inherently employed subtle approximations.

To conclude, a representative test case is analyzed for which an intrinsically exact average state is determined without separating the kinematic/energy terms from the pressure relation in the flux difference resolution. Using this average state, the related exact average thermodynamic pressure derivatives and sound speed are computed. These results are then compared with the predictions of recent methods and their inherent approximations are appropriately quantified.

This report is organized in 10 Sections. Section 1 delineates the relevant form of the curvilinear-coordinate Euler equations, while Section 2 contains several theoretical average state considerations. Section 3 presents the eigenvalue consistency theorem, while Section 4 details the split component evaluation of the flux vector jacobian. Sections 5-6 are devoted to the development of the conservation-variable average state for one- and multi- dimensional curvilinear coordinate fluxes, and a discussion of the ideal gas average state. Section 7 details the characteristic analysis derivation of the eigenvalues and the sound speed expression in terms of the pressure jacobian, while the pivotal average partial derivatives of pressure are directly determined in Section 8. Finally, several comparisons with other average-state formulations are presented in Section 9, while representative results are discussed in Section 10.

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1 Governing Equations

The cartesian coordinate Euler equations in conservation law form are

$$\frac{\partial q}{\partial t} + \frac{\partial f_j}{\partial x_j} = 0$$

$$\text{on } \Re^+ \times \Omega, \ \ t \geq t_0, \ \ t, t_0 \in \Re^+, \ \ \Omega \subset \Re^n, \ \ 1 \leq n \leq 3 \tag{1}$$

where n denotes the spatial dimensionality. The state variable $q = q(\boldsymbol{x}, t)$, and the flux vector $f_j = f_j(q)$ are

$$q \equiv \left\{ \begin{array}{c} \rho \\ \boldsymbol{m} \\ E \end{array} \right\} \ , \ f_j \equiv \left\{ f_j^\ell \right\} = \left\{ \begin{array}{c} m_j \\ \frac{m_j}{\rho} \boldsymbol{m} + p \boldsymbol{\delta}_j \\ \frac{m_j}{\rho}(E + p) \end{array} \right\} \ , \ 1 \leq \ell \leq n + 2 \tag{2}$$

where ρ is the fluid density, $\boldsymbol{m} = \rho \boldsymbol{u}$ is the linear momentum vector, p is the static pressure, $\boldsymbol{\delta}_j \equiv \{\delta_j^i\}$, $1 \leq i \leq n$, is the Kronecker delta mixed tensor, and E is the volume specific total energy. System (1)-(2) is then closed with an equation of state for pressure. For a homogeneous gas in thermo-chemical equilibrium, this equation of state may be expressed as a function of q, differentiable almost everywhere [1] in Ω, of the form

$$p = p(q(\boldsymbol{x}, t)) \tag{3}$$

as detailed in Section 8.

With these specifications, the conservation state variable q constitutes the differetiation variable for determining the jacobian matrix $A_j(q)$ of $f_j(q)$. Consequently, any other variable introduced to compute $A_j(q)$ will depend upon q. Hence, the partial derivatives of $p(q)$ in $A_j(q)$ coincide with the jacobian of $p(q)$ with respect to q. This pressure jacobian can certainly be computed by way of the differentiation chain rule and the thermodynamic derivatives of pressure. Nonetheless, the required additional thermodynamic variables are themselves functions of q. These considerations have significant repercussions on the interpretation and evaluation of the pressure variation relation, as thoroughly documented in Section 8.

For arbitrary-geometry computations, the cartesian coordinates $\boldsymbol{x}$ are transformed into curvilinear coordinates $\boldsymbol{\eta}$ via the relation

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\eta}) \tag{4}$$

Function (4) must be single valued and have continuous first partial derivatives. The jacobian J of the coordinate transformation (4) is

$$J = \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\eta}} \right] \equiv \left[\frac{\partial x_j}{\partial \eta_k} \right] \tag{5}$$

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The relation between the x_i and η_j partial derivatives is then

$$\left\{\frac{\partial}{\partial \eta_k}\right\} = \left[\frac{\partial x_j}{\partial \eta_k}\right]\left\{\frac{\partial}{\partial x_j}\right\} = J\left\{\frac{\partial}{\partial x_j}\right\} \tag{6}$$

Therefore, the partial derivatives with respect to the cartesian coordinates x_j are expressed using the inverse of (6) as

$$\left\{\frac{\partial}{\partial x_j}\right\} = J^{-1}\left\{\frac{\partial}{\partial \eta_k}\right\} = \frac{1}{\det J}\left[\det J \cdot J^{-1}\right]\left\{\frac{\partial}{\partial \eta_k}\right\} \tag{7}$$

where $\det J$ does not vanish and keeps a uniform sign in Ω. With these specifications, the matrix $[\det J \cdot J^{-1}]$ is denoted as $[e_{jk}]$. The metric data e_{jk}, with continuous partial derivatives in their definition domain, then satisfy the fundamental invariance relation

$$\sum_{k=1}^{n} \frac{\partial e_{jk}}{\partial \eta_k} = 0 \quad , \quad 1 \le j \le n \tag{8}$$

Inserting (7)-(8) into (1) yields the contravariant conservation law system

$$\frac{\partial q}{\partial t} + \frac{1}{\det J}\frac{\partial \tilde{f}_k}{\partial \eta_k} = 0 \tag{9}$$

where the contravariant flux $\tilde{f}_k$ is defined as

$$\tilde{f}_k \equiv e_{jk}\left\{f_j^{\ell}\right\} = e_{jk}f_j = \left\{\begin{array}{c} e_{jk}m_j \\ e_{jk}\dfrac{m_j}{\rho}\boldsymbol{m} + e_{jk}p\delta_j \\ e_{jk}\dfrac{m_j}{\rho}(E+p) \end{array}\right\} \tag{10}$$

The curvilinear coordinate average states are then determined for the jacobian of this contravariant flux, as detailed in Section 4.

2 Theoretical Average States

A set of average states for expressing the flux variation exists by virtue of the mean value theorem. For an ideal gas, the original resolution [3] accrues from one average state and one average enthalpy. For a real gas, an analogous resolution also depends upon the partial derivatives of pressure.

The variation of each flux component f_j^{ℓ} in (2), (10) can be expressed by way of the multi-dimensional mean value theorem as

$$\begin{aligned} \Delta f_j^{\ell} &= \frac{\partial f_j^{\ell}}{\partial q}\left(q^L + \alpha^{\ell}\left(q^R - q^L\right)\right) \cdot \left(q^R - q^L\right) \\ &= \frac{\partial f_j^{\ell}}{\partial q}\left(\tilde{q}^{\ell}\right) \cdot \left(q^R - q^L\right) \end{aligned} \tag{11}$$

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where the partial derivatives in (11) are evaluated at the internal state

$$\widehat{q}^\ell \equiv q^L + \alpha^\ell \left(q^R - q^L \right) \tag{12}$$

In (11)-(12) superscripts L and R denote the available computational-cell left and right states respectively. Furthermore, each weight parameter α^ℓ satisfies the relation

$$0.0 \le \alpha^\ell \le 1.0 \quad , \quad 1 \le \ell \le n+2 \tag{13}$$

In general, these parameters may not coincide with one another, and hence the internal states $\widehat{q}^\ell$, $1 \le \ell \le n+2$, may differ from one another. Therefore, the variation of the flux vector (2), (10) is cast as

$$A_j \left(\widehat{q}^1, \widehat{q}^2, \cdots, \widehat{q}^{n+2} \right) \Delta q = \Delta f_j \tag{14}$$

where, following (11), the ℓth row of A_j is evaluated at $\widehat{q}^\ell$. Furthermore, more than one state may exist for a fixed ℓ. Thus, the average state is in general non-unique.

While the mean value theorem evaluation relies on at most $n+2$ states, the most recent developments only need fewer such states upon introduction of an additional average enthalpy. This reduction accrues from an independent application of the mean value theorem to specific components in the resolution of each flux vector component variation. For example, (11) can be cast as

$$\Delta f_j^\ell = \Delta f_j^{c^\ell} + \Delta f_j^{a^\ell} = \frac{\partial f_j^{c^\ell}}{\partial q} \left(q^L + \widehat{\alpha}^\ell \left(q^R - q^L \right) \right) \Delta q + \frac{\partial f_j^{a^\ell}}{\partial q} \left(q^L + \widetilde{\alpha}^\ell \left(q^R - q^L \right) \right) \Delta q \tag{15}$$

whereupon, the mean value theorem independently applied to $\Delta f_j^{c^\ell}$ and $\Delta f_j^{a^\ell}$ yields

$$\Delta f_j^{c^\ell} = \frac{\partial f_j^{c^\ell}}{\partial q} \left(q^L + \widehat{\alpha}^\ell \left(q^R - q^L \right) \right) \Delta q \tag{16}$$

$$\Delta f_j^{a^\ell} = \frac{\partial f_j^{a^\ell}}{\partial q} \left(q^L + \widetilde{\alpha}^\ell \left(q^R - q^L \right) \right) \Delta q \tag{17}$$

As shown in Section 4, this procedure corresponds to a convection/acoustic-wave flux vector decomposition that is intrinsically utilized in [4,8,9].

Even though several CFD papers [3, 5, 8, 9] employ the appealing notation

$$A_j \left(\widehat{q} \right) \Delta q = \Delta f_j \tag{18}$$

which relies on a free interpretation of the multi-dimensional mean value theorem and presumes the existence of a single average state $\widehat{q}$ valid for each and every flux vector component, the original ideal-gas average state [3] violates the mean value theorem (11) in the flux vector energy component variation, as detailed in Section 6.

Nevertheless, the original average state successfully yields a closed form resolution for (11) using an additional average enthalpy $\widehat{H}$. Thus (18) can be more accurately cast as

$$A_j\left(\hat{q},\widehat{H}\right)\Delta q = \Delta f_j \qquad (19)$$

and is at variance with the developments in [3]. As proved in Section 5, expression (19) for an ideal gas can be alternatively cast as

$$A_j\left(\hat{q},\hat{p}\right)\Delta q = \Delta f_j \qquad (20)$$

upon identification of an internal-average total energy with associated average pressure $\hat{p}$. For a general equilibrium gas equation of state, expression (20) is expanded as

$$A_j\left(\hat{q},\hat{p},\left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{m,E},\left(\widetilde{\frac{\partial p}{\partial m}}\right)_{\rho,E},\left(\widetilde{\frac{\partial p}{\partial E}}\right)_{\rho,m}\right)\Delta q = \Delta f_j \qquad (21)$$

where a tilde denotes average partial derivatives. In this form, relation (21) indicates the crucial functional dependence of the flux vector jacobian upon the partial derivatives of pressure.

3 Eigenvalue Invariance Relations

Considering the non-uniqueness of the average states, several dissimilar average jacobians can be developed for a single state-variable pair. Consequently, the spectra of these jacobians will not coincide with one another. While this situation may be unavoidable, physical consistency demands that, for a local flow of a specific character (subsonic or supersonic), all associated average jacobians for any average state must correspond to a uniform set of eigenvalues. Therefore, if the flow is locally subsonic, the eigenvalue set of any associated average jacobian must contain positive and negative eigenvalues, whereas for a locally supersonic flow, the eigenvalue set of all associated average jacobians can only contain uniform sign eigenvalues. If this property were not always met, then for some subsonic flows, a flux difference splitting implementation may lead to a fully upwind scheme for all flux vector components, which may in turn engender numerical solution instability. As proven by the following n-dimensional-formulation theorem, those average jacobians that satisfy (21) and share the ideal-gas average velocity eigenvalue, with algebraic and geomentric multiplicity equal to n, display consistent eigenvalue sets with mixed- or uniform-sign eigenvalues for a subsonic or supersonic flow state respectively.

Definition: two non-singular matrices are defined eigenvalue consistent when their spectra have the same number of positive and negative eigenvalues.

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Theorem 1 *Let two non-singular and diagonalizable average jacobians correspond to any single set of state variable and flux variations Δq and Δf. These jacobians are then always eigenvalue consistent for supersonic flows. For subsonic flows they are eigenvalue consistent if for an n dimensional formulation they share one eigenvalue with algebraic and geometric multiplicity equal to n.*

<u>Proof</u>. For any state variable variation Δq, the resolution of the associated flux variation Δf for any two generally different average states can be expressed as

$$A\left(\hat{q}_1,\hat{p}_1,\left(\widetilde{\frac{\partial p}{\partial q}}\right)_1\right)\Delta q = \Delta f \quad , \quad A\left(\hat{q}_2,\hat{p}_2,\left(\widetilde{\frac{\partial p}{\partial q}}\right)_2\right)\Delta q = \Delta f \quad (22)$$

Given the purely algebraic nature of the following derivations, (22) is abridged as

$$A_1\Delta q = \Delta f \quad , \quad A_2\Delta q = \Delta f \tag{23}$$

Since each matrix in (23) is diagonalizable, each possesses a complete set of eigenvectors. Hence Δq can be expressed as

$$\Delta q = X_1 b \quad , \quad \Delta q = X_2 c \tag{24}$$

In (24), X_1 and X_2 are the eigenvector matrices for A_1 and A_2 respectively, while b and c are the associated eigenvector coefficient set. From (24) it follows

$$X_1 b = X_2 c \tag{25}$$

Furthermore, expressing the matrices A_1 and A_2 by way of a similarity transformation yields

$$X_1\Lambda_1 X_1^{-1}\Delta q = \Delta f \quad , \quad X_2\Lambda_2 X_2^{-1}\Delta q = \Delta f \tag{26}$$

where Λ_1 and Λ_2 are the diagonal matrices containing the eigenvalues of A_1 and A_2 respectively. Inserting (24) into (26) leads to

$$X_1\Lambda_1 b = \Delta f \quad , \quad X_2\Lambda_2 c = \Delta f \tag{27}$$

whence

$$X_1\Lambda_1 b = X_2\Lambda_2 c \tag{28}$$

Thereafter, (25) and (28) lead to the following expressions for b

$$b = \left(X_1^{-1}X_2\right)c \tag{29}$$

$$b = \Lambda_1^{-1}\left(X_1^{-1}X_2\right)\Lambda_2 c \tag{30}$$

It must be shown that when Λ_1 displays a certain number of positive and negative eigenvalues, Λ_2 has the same number of positive and negative eigenvalues. Therefore, for any Δq, hence for any b, consider the positive bilinear form

$$0 < b^T \mathcal{I}_1 \Lambda_1 b \tag{31}$$

In (31), $\mathcal{I}_1$ is a diagonal matrix with entries σ_1^j, $1 \leq j \leq n + 2$, equal to $+1$ or -1 only, depending on the algebraic sign of the eigenvalues in Λ_1, such that the matrix $\mathcal{I}_1 \Lambda_1$ is positive definite. Expressing b^T and b in (31) by way of (29) and (30) respectively yields

$$0 < c^T \left[\left(X_1^{-1} X_2 \right)^T \mathcal{I}_1 \left(X_1^{-1} X_2 \right) \Lambda_2 \right] c \tag{32}$$

According to (31), the matrix in (32) is positive definite. Consequently, all of its principal minors

$$\det \left[\mathcal{M} \right]_i \equiv \det \left[\left(X_1^{-1} X_2 \right)^T \mathcal{I}_1 \left(X_1^{-1} X_2 \right) \Lambda_2 \right]_i \tag{33}$$

must be positive [2]. Considering the diagonality of Λ_2, this matrix minor positivity property is expressed as

$$\begin{aligned} 0 \;&<\; \det \left[\left(X_1^{-1} X_2 \right)^T \mathcal{I}_1 \left(X_1^{-1} X_2 \right) \Lambda_2 \right]_i \\ &=\; \det \left[\left(X_1^{-1} X_2 \right)^T \mathcal{I}_1 \left(X_1^{-1} X_2 \right) \right]_i \cdot \prod_{j=1}^{i} \lambda_2^j \;,\quad 1 \leq i \leq n + 2 \end{aligned} \tag{34}$$

where λ_2^j, $1 \leq j \leq n + 2$, denote the eigenvalues in Λ_2. For a supersonic flow, all the eigenvalues in Λ_1 have the same algebraic sign. Hence, all of the entries σ_1^j in $\mathcal{I}_1$ are equal to one another, and (34) thus becomes

$$0 < \det \left[\left(X_1^{-1} X_2 \right)^T \left(X_1^{-1} X_2 \right) \right]_i \cdot \prod_{j=1}^{i} \left(\sigma_1^j \lambda_2^j \right) \;,\quad 1 \leq i \leq n + 2 \tag{35}$$

Furthermore, the matrix

$$Y = \left(X_1^{-1} X_2 \right)^T \left(X_1^{-1} X_2 \right) \tag{36}$$

is certainly positive definite, as well as symmetric, since it corresponds to the matrix product of a non-singular matrix and its transpose. Consequently, the matrix minors

$$\det \left[Y \right]_i = \det \left[\left(X_1^{-1} X_2 \right)^T \left(X_1^{-1} X_2 \right) \right]_i \;,\quad 1 \leq i \leq n + 2 \tag{37}$$

are also positive. Therefore, it follows from (35) that

$$\sigma_1^j \lambda_2^j > 0 \ , \quad 1 \leq j \leq n+2 \qquad (38)$$

Hence, if $\lambda_1^j > 0$, $1 \leq j \leq n+2$, then $\sigma_1^j = +1$, $1 \leq j \leq n+2$, and $\lambda_2^j > 0$, $1 \leq j \leq n+2$, whereas if $\lambda_1^j < 0$, $1 \leq j \leq n+2$, then $\sigma_1^j = -1$, $1 \leq j \leq n+2$, and $\lambda_2^j < 0$, $1 \leq j \leq n+2$, which proves eigenvalue consistency.

For subsonic flows, Λ_1 has mixed sign eigenvalues. It thus has two eigenvalues of opposite sign in addition to one eigenvalue with multiplicity n. Therefore, $\mathcal{I}_1$ has entries with different algebraic sign and consequently its determinant can be either positive or negative. For $i = n+2$, expression (34) yields

$$0 < \det[Y] \cdot \det[\mathcal{I}_1] \cdot \prod_{j=1}^{n+2} \lambda_2^j \qquad (39)$$

Consequently,

$$0 < \det[\mathcal{I}_1] \cdot \prod_{j=1}^{n+2} \lambda_2^j \qquad (40)$$

The eigenvalues λ_2^j, $1 \leq j \leq n+2$, cannot all have the same algebraic sign, otherwise, according to the previous conclusion, the same would hold true for λ_1^j, $1 \leq j \leq n+2$, and the flow would be supersonic. Therefore, Λ_2 will display at least two eigenvalues of opposite sign. By hypothesis, Λ_1 and Λ_2 already share one eigenvalue with multiplicity n, and for one- and three- dimensional formulations, n is an odd number. In this case, if the common eigenvalue is positive, respectively negative, $\det[\mathcal{I}_1]$ is negative, respectively positive. Thus, from (40), $\det[\Lambda_2]$ is negative, respectively positive, but in either case Λ_2 has n eigenvalues with the same sign and two eigenvalues with opposite sign. For a two-dimensional formulation, n is an even number and $\det[\mathcal{I}_1]$ is always negative. Therefore, from (40) $\det[\Lambda_2]$ will always be negative, and thus Λ_2 has again n eigenvalues with the same sign and two eigenvalues with opposite sign. Hence, also for subsonic flows Λ_1 and Λ_2 have the same number of positive and negative eigenvalues.

Finally, since this proof is valid for any two average jacobians that satisfy the stated hypotheses, it remains valid for all such jacobians $\Box$.

By virtue of this theorem, a flux-difference splitting procedure will always yield the appropriate upwind scheme, for both subsonic and supersonic flows, for any average state that satisfies (21), provided that the associated average jacobian has

one eigenvalue with algebraic and geometric multiplicity equal to n. Therefore, it is not strictly indispensable to upwind scheme appropriateness that the eigenvalues of all average jacobians, associated with a single local flow character, coincide with one another. However, the closer the average eigenvalue sets are in an appropriate norm to the pointwise set, the more accurate the average-state flux vector difference resolution becomes. The following theorem shows that this eigenvalue convergence property directly depends upon the eigenvector matrices, as can be easily imagined.

Theorem 2 *If the matrices A_1 and A_2 in Theorem 1 share the same eigenvector matrix X, then they also share the same eigenvalues.*

Proof. Theorem 1 yields the relations

$$b = \left(X_1^{-1} X_2\right) c \ , \quad b = \Lambda_1^{-1} \left(X_1^{-1} X_2\right) \Lambda_2 c \tag{41}$$

Since $X_1 = X_2 = X$, (41) yields

$$c = \Lambda_1^{-1} \Lambda_2 c \tag{42}$$

Therefore, A_1 and A_2 have the same eigenvalues $\square$.

Consequently, it becomes convenient for the average eigenvector matrix to remain close in an appropriate norm to the pointwise eigenvector matrix. This is especially important for transonic or stagnation-point flows where one or more average eigenvalues may vanish, hence some hypotheses of the eigenvalue consistency theorem may not be satisfied.

4 Reference Jacobian and Current Evaluation Procedures

Specific average states are determined such that the relations corresponding to the flux difference resolution

$$A_j \left(\widehat{q}, \widehat{p}, \left(\widetilde{\frac{\partial p}{\partial \rho}} \right)_{m,E}, \left(\widetilde{\frac{\partial p}{\partial m}} \right)_{\rho,E}, \left(\widetilde{\frac{\partial p}{\partial E}} \right)_{\rho,m} \right) \Delta q = \Delta f_j \tag{43}$$

are formally satisfied. For a reference one-dimensional formulation, the average jacobian in (43) is

$$A_1 \left(\widehat{q}, \widehat{p}, \left(\widetilde{\frac{\partial p}{\partial \rho}} \right)_{m,E}, \left(\widetilde{\frac{\partial p}{\partial m}} \right)_{\rho,E}, \left(\widetilde{\frac{\partial p}{\partial E}} \right)_{\rho,m} \right) =$$

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$$\left(\begin{array}{ccc} 0 & , & 1 & , & 0 \\ -\dfrac{\widehat{m}^2}{\widehat{\rho}^2}+\left(\dfrac{\widetilde{\partial p}}{\partial \rho}\right)_{m,E} & , & 2\dfrac{\widehat{m}}{\widehat{\rho}}+\left(\dfrac{\widetilde{\partial p}}{\partial m}\right)_{\rho,E} & , & \left(\dfrac{\widetilde{\partial p}}{\partial E}\right)_{\rho,m} \\ -\dfrac{\widehat{m}}{\widehat{\rho}^2}(\widehat{E}+\widehat{p})+\dfrac{\widehat{m}}{\widehat{\rho}}\left(\dfrac{\widetilde{\partial p}}{\partial \rho}\right)_{m,E} & , & \dfrac{\widehat{E}+\widehat{p}}{\widehat{\rho}}+\dfrac{\widehat{m}}{\widehat{\rho}}\left(\dfrac{\widetilde{\partial p}}{\partial m}\right)_{\rho,E} & , & \dfrac{\widehat{m}}{\widehat{\rho}}\left(1+\left(\dfrac{\widetilde{\partial p}}{\partial E}\right)_{\rho,m}\right) \end{array}\right)$$

(44)

Therefore, (43) yields the identity

$$m^R - m^L = m^R - m^L \tag{45}$$

for the continuity equation,

$$-\frac{\widehat{m}^2}{\widehat{\rho}^2}\left(\rho^R - \rho^L\right) + 2\frac{\widehat{m}}{\widehat{\rho}}\left(m^R - m^L\right) + \quad ,$$

$$\left(\frac{\widetilde{\partial p}}{\partial \rho}\right)_{m,E}\left(\rho^R - \rho^L\right) + \left(\frac{\widetilde{\partial p}}{\partial m}\right)_{\rho,E}\left(m^R - m^L\right) + \left(\frac{\widetilde{\partial p}}{\partial E}\right)_{\rho,m}\left(E^R - E^L\right) = \quad \cdot$$

$$\frac{\left(m^R\right)^2}{\rho^R} + p^R - \frac{\left(m^L\right)^2}{\rho^L} - p^L \tag{46}$$

for the momentum equation, and

$$-\frac{\widehat{m}}{\widehat{\rho}^2}\left(\widehat{E}+\widehat{p}\right)\left(\rho^R - \rho^L\right) + \frac{\widehat{E}+\widehat{p}}{\widehat{\rho}}\left(m^R - m^L\right) + \frac{\widehat{m}}{\widehat{\rho}}\left(E^R - E^L\right) +$$

$$\frac{\widehat{m}}{\widehat{\rho}}\left(\left(\frac{\widetilde{\partial p}}{\partial \rho}\right)_{m,E}\left(\rho^R - \rho^L\right) + \left(\frac{\widetilde{\partial p}}{\partial m}\right)_{\rho,E}\left(m^R - m^L\right) + \left(\frac{\widetilde{\partial p}}{\partial E}\right)_{\rho,m}\left(E^R - E^L\right)\right) =$$

$$\frac{m^R}{\rho^R}\left(E^R + p^R\right) - \frac{m^L}{\rho^L}\left(E^L + p^L\right) \tag{47}$$

for the energy equation.

These three relations would constitute a very appealing formulation if a single average state $\widehat{q}$ were efficiently identifiable for the evaluation of the jacobian state vector terms as well as pressure and its partial derivatives. Certainly (46)-(47) form a non-linear system of two equations for the three unknown components of $\widehat{q}$. Hence, the average state $\widehat{q}$ could coincide with the solution of this system. However, no complete information exists to date regarding solution existence, let alone uniqueness.

Therefore, the current formulations seek an average state by separating the pressure from the kinematic/energy terms in the momentum and energy equations, following (15)-(17). This procedure corresponds to a decomposition of the flux

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vector $f_j(q)$ into the physically distinct convection and acoustic-wave components $f_j^c(q)$ and $f_j^a(q)$

$$f_j(q) = f_j^c(q) + f_j^a(q) \tag{48}$$

with definitions

$$f_j^c(q) \equiv \frac{m_j}{\rho} \left\{ \begin{array}{c} \rho \\ \boldsymbol{m} \\ E+p \end{array} \right\} \quad , \quad f_j^a(q) \equiv \left\{ \begin{array}{c} 0 \\ p\boldsymbol{\delta}_j \\ 0 \end{array} \right\} \tag{49}$$

The jacobian $A_j(q)$ of $f_j(q)$ is then

$$A_j(q) \equiv \frac{\partial f_j}{\partial q} = \frac{\partial f_j^c}{\partial q} + \frac{\partial f_j^a}{\partial q} \tag{50}$$

Consequently, the resolution of the flux vector $f_j(q)$ can be expressed as in (15)-(17). The associated convection and acoustic-wave average states may then be distinct from one another considering the difference between these two processes.

Thereafter, the variations in the kinematic/energy terms in the convection flux are expressed in terms of the ideal gas average state, whereas the pressure variation in the acoustic-wave flux is evaluated by way of some specially devised average partial derivatives of pressure. The developed average density and velocity (or momentum) that satisfy the momentum equation are then inserted into the energy equation. This technique yields an average enthalpy (or energy with associated pressure as shown in Section 5) in closed form which however does not satisfy the mean value theorem for the energy equation, as proven in Section 6. Nonetheless, a distinctive advantage of this modus operandi is that one eigenvalue of the associated average jacobian is always the ideal-gas average velocity, with algebraic and geometric multiplicity equal to n, which satisfies the hypotheses of the eigenvalue consistency theorem. Furthermore, the other eigenvalues resemble those of the pointwise jacobians, as detailed in Section 7.

For the convection flux resolution, the separate evaluation method yields the identity

$$m^R - m^L = m^R - m^L \tag{51}$$

for the continuity equation,

$$-\frac{\widehat{m}^2}{\widehat{\rho}^2}\left(\rho^R - \rho^L\right) + 2\frac{\widehat{m}}{\widehat{\rho}}\left(m^R - m^L\right) = \frac{\left(m^R\right)^2}{\rho^R} - \frac{\left(m^L\right)^2}{\rho^L} \tag{52}$$

for the momentum equation, and

$$-\frac{\widehat{m}}{\widehat{\rho}^2}\widehat{E}\left(\rho^R - \rho^L\right) + \frac{\widehat{E}}{\widehat{\rho}}\left(m^R - m^L\right) + \frac{\widehat{m}}{\widehat{\rho}}\left(E^R - E^L\right) = \frac{m^R}{\rho^R}E^R - \frac{m^L}{\rho^L}E^L \tag{53}$$

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$$-\frac{\widehat{m}}{\widehat{\rho}^2}\widehat{p}\left(\rho^R - \rho^L\right) + \frac{\widehat{p}}{\widehat{\rho}}\left(m^R - m^L\right) + \frac{\widehat{m}}{\widehat{\rho}}\left(p^R - p^L\right) = \frac{m^R}{\rho^R}p^R - \frac{m^L}{\rho^L}p^L \qquad (54)$$

for the energy equation.

The terminal form of (54) is obtained by virtue of the pressure variation expression associated with the acoustic flux. From (49)-(50) and for an n dimensional formulation, this expression is

$$\left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\boldsymbol{m},E}\left(\rho^R - \rho^L\right) + \left(\widetilde{\frac{\partial p}{\partial \boldsymbol{m}}}\right)_{\rho,E}\cdot\left(\boldsymbol{m}^R - \boldsymbol{m}^L\right) + \left(\widetilde{\frac{\partial p}{\partial E}}\right)_{\rho,\boldsymbol{m}}\left(E^R - E^L\right) = p^R - p^L$$

$$(55)$$

which thus becomes the crucial relation to be satisfied. As fully detailed in Section 9, the various procedures published thus far [4,8,9] differ in the specific avenue to evaluate this pressure relation, even though they share the common feature of algebraically constructing sets of average partial derivatives of pressure assuming several geometric solution projection rules and associated scale factors. Consequently, the corresponding solutions remain inextricably tied up with these assumptions. On the contrary, the developments in Section 8 dispense with these assumptions by exactly determining the intrinsic acoustic-wave average state that satisfies (55).

5   Conservation Variable Average States

It is often reported that an average density $\widehat{\rho}$ can be chosen freely [8], and that there is no physical basis for introducing specific expressions for $\widehat{\rho}$ [9]. However, the classical average density $\widehat{\rho}$, as well as the average momentum $\widehat{m}$, and volume specific total energy $\widehat{E}$ can all be analytically determined in a rigorous manner invoking the mean value theorem for the flux difference resolution (44). The resulting expressions

$$\widehat{\rho} = \rho^L + \alpha\left(\rho^R - \rho^L\right) \qquad (56)$$

$$\widehat{m} = m^L + \alpha\left(m^R - m^L\right) \qquad (57)$$

$$\widehat{E} = E^L + \alpha\left(E^R - E^L\right) \qquad (58)$$

$$\widehat{p} = p^L + \alpha\left(p^R - p^L\right) \qquad (59)$$

employ a single weight parameter α, $0 \leq \alpha \leq 1$, and apply for both one-dimensional and curvilinear-coordinate multi-dimensional formulations.

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5.1 One-Dimensional Average States

Expression (52) constitutes a quadratic equation for $\left(\dfrac{\widehat{m}}{\widehat{\rho}}\right)$

$$\left(\frac{\widehat{m}}{\widehat{\rho}}\right)^2 \left(\rho^R - \rho^L\right) - 2\left(\frac{\widehat{m}}{\widehat{\rho}}\right)\left(m^R - m^L\right) + \frac{\left(m^R\right)^2}{\rho^R} - \frac{\left(m^L\right)^2}{\rho^L} = 0 \qquad (60)$$

with internal-state solution

$$\frac{\widehat{m}}{\widehat{\rho}} = \frac{\dfrac{m^L}{\rho^L}\sqrt{\rho^L} + \dfrac{m^R}{\rho^R}\sqrt{\rho^R}}{\sqrt{\rho^L} + \sqrt{\rho^R}} \qquad (61)$$

Furthermore, the insertion of expressions (56)-(57) into (61) yields a single equation for α with solution

$$0 \le \alpha = \frac{\sqrt{\rho^L}}{\sqrt{\rho^L} + \sqrt{\rho^R}} \le 1 \qquad (62)$$

With this solution, (56) becomes

$$\widehat{\rho} = \sqrt{\rho^L \rho^R} \qquad (63)$$

which coincides with the classical form.

Inserting the average density and momentum into the two leading terms in the lhs of (53) yields the equality

$$-\frac{\widehat{m}}{\widehat{\rho}^2}\widehat{E}\left(\rho^R - \rho^L\right) + \frac{\widehat{E}}{\widehat{\rho}}\left(m^R - m^L\right) = \widehat{E}\left(\frac{m^R}{\rho^R} - \frac{m^L}{\rho^L}\right) \qquad (64)$$

Hence, (53) becomes

$$\Delta\left(\frac{m}{\rho}E\right) = \frac{m^R}{\rho^R}E^R - \frac{m^L}{\rho^L}E^L = \widehat{E}\left(\frac{m^R}{\rho^R} - \frac{m^L}{\rho^L}\right) + \frac{\widehat{m}}{\widehat{\rho}}\left(E^R - E^L\right) \qquad (65)$$

which constitutes the average-state resolution of the flux of E. A similar operation sequence on (54) yields

$$\Delta\left(\frac{m}{\rho}p\right) = \frac{m^R}{\rho^R}p^R - \frac{m^L}{\rho^L}p^L = \widehat{p}\left(\frac{m^R}{\rho^R} - \frac{m^L}{\rho^L}\right) + \frac{\widehat{m}}{\widehat{\rho}}\left(p^R - p^L\right) \qquad (66)$$

which constitutes the average-state resolution of the flux of p. Whereupon, inserting (61) into (65) yields

$$\begin{aligned}
\widehat{E} &= E^L \frac{\sqrt{\rho^R}}{\sqrt{\rho^R} + \sqrt{\rho^L}} + E^R \frac{\sqrt{\rho^L}}{\sqrt{\rho^R} + \sqrt{\rho^L}} \\
&= E^L + \alpha\left(E^R - E^L\right)
\end{aligned} \qquad (67)$$

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where α is defined by (62). Hence, the average total energy $\widehat{E}$ is also determined with exactly the same weight as the average density and momentum. Finally, the average pressure $\widehat{p}$ is analogously expressed as

$$\widehat{p} = p^L + \alpha \left(p^R - p^L \right) \tag{68}$$

owing to the identical structure of (65) and (66).

5.2   Multi-Dimensional Average States

The derivations of the average state for the jacobian of the multi-dimensional flux (10) with embedded metric data e_{jk} are similar to the previous developments, and show that the average one-dimensional conservation variables also apply for multi-dimensional curvilinear coordinate formulations. The variation of the contravariant flux $\widetilde{f}_k$ is expressed as

$$A_k \left(\widehat{q}, \widehat{p}, \left(\widetilde{\frac{\partial p}{\partial \rho}} \right)_{\boldsymbol{m},E}, \left(\widetilde{\frac{\partial p}{\partial m}} \right)_{\rho,E}, \left(\widetilde{\frac{\partial p}{\partial E}} \right)_{\rho,\boldsymbol{m}}, e_{\underline{jk}} \right) \Delta q = \Delta \left(e_{jk} f_j \right) \tag{69}$$

In order to simplify the notation, the partial derivatives of pressure in (69) are denoted as

$$p_\rho = \left(\frac{\partial p}{\partial \rho} \right)_{\boldsymbol{m},E} \quad , \quad p_{m_i} = \left(\frac{\partial p}{\partial m_i} \right)_{\rho,m_j,E,i\neq j} \quad , \quad p_E = \left(\frac{\partial p}{\partial E} \right)_{\rho,\boldsymbol{m}} \tag{70}$$

Therefore, the average state jacobian in (69) becomes

$$\widehat{A}_k \equiv A_k \left(\widehat{q}, \widehat{p}, \widetilde{p_\rho}, \widetilde{pm}, \widetilde{p_E}, e_{\underline{jk}} \right) = \left(\frac{\partial (e_{jk} f_j)}{\partial q} \right) =$$

$$\begin{pmatrix} 0 & , & e_{\ell k} & , & 0 \\ -e_{jk}\dfrac{\widehat{m_j}}{\widehat{\rho}^2}m_i + e_{ik}\widetilde{p_\rho} & , & e_{\ell k}\dfrac{\widehat{m_i}}{\widehat{\rho}} + e_{ik}\widetilde{p_{m_\ell}} + e_{jk}\dfrac{\widehat{m_j}}{\widehat{\rho}}\delta_{i\ell} & , & e_{ik}\widetilde{p_E} \\ -e_{jk}\dfrac{\widehat{m_j}}{\widehat{\rho}^2}\left(\widehat{E} + \widehat{p} \right) + e_{jk}\dfrac{\widehat{m_j}}{\widehat{\rho}}\widetilde{p_\rho} & , & \dfrac{e_{\ell k}}{\widehat{\rho}}\left(\widehat{E} + \widehat{p} \right) + e_{jk}\dfrac{\widehat{m_j}}{\widehat{\rho}}\widetilde{p_{m_\ell}} & , & e_{jk}\dfrac{\widehat{m_j}}{\widehat{\rho}}\left(1 + \widetilde{p_E} \right) \end{pmatrix} \tag{71}$$

where, $1 \leq (i,\ell) \leq n$ are the row and column indices, for the n-dimensional resolution of momentum. In the sequel, the various formulae are exemplified and demonstrated for a two-dimensional formulation. In this case, the average jacobian $\widehat{A}_k$ becomes a 4×4 matrix, where the middle two rows and columns respectively derive from the middle row and column in (71) for $i, \ell = 1, 2$. For constant metric data e_{jk} in each and every computational cell, the contravariant flux variation in (69) becomes

$$\Delta \left(e_{jk} f_j \right) = e_{jk} \Delta \left(f_j \right) \tag{72}$$

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Therefore for the convection and acoustic-wave flux vector decomposition, expression (69) yields the identity

$$e_{1k}\left(m_1^R - m_1^L\right) + e_{2k}\left(m_2^R - m_2^L\right) = e_{jk}m_j^R - e_{jk}m_j^L \tag{73}$$

for continuity equation,

$$-e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}^2}\widehat{m}_1\left(\rho^R - \rho^L\right) + \left(e_{1k}\frac{\widehat{m}_1}{\widehat{\rho}} + e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}}\right)\left(m_1^R - m_1^L\right) + e_{2k}\frac{\widehat{m}_1}{\widehat{\rho}}\left(m_2^R - m_2^L\right) =$$

$$e_{jk}\frac{m_j^R}{\rho^R}m_1^R - e_{jk}\frac{m_j^L}{\rho^L}m_1^L \tag{74}$$

$$e_{1k}\left(\left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\boldsymbol{m},E}\left(\rho^R - \rho^L\right) + \left(\widetilde{\frac{\partial p}{\partial \boldsymbol{m}}}\right)_{\rho,E}\cdot\left(\boldsymbol{m}^R - \boldsymbol{m}^L\right) + \left(\widetilde{\frac{\partial p}{\partial E}}\right)_{\rho,\boldsymbol{m}}\left(E^R - E^L\right)\right) =$$

$$e_{1k}\left(p^R - p^L\right) \tag{75}$$

for the longitudinal-momentum equation,

$$-e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}^2}\widehat{m}_2\left(\rho^R - \rho^L\right) + e_{1k}\frac{\widehat{m}_2}{\widehat{\rho}}\left(m_1^R - m_1^L\right) + \left(e_{2k}\frac{\widehat{m}_2}{\widehat{\rho}} + e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}}\right)\left(m_2^R - m_2^L\right) =$$

$$e_{jk}\frac{m_j^R}{\rho^R}m_2^R - e_{jk}\frac{m_j^L}{\rho^L}m_2^L \tag{76}$$

$$e_{2k}\left(\left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\boldsymbol{m},E}\left(\rho^R - \rho^L\right) + \left(\widetilde{\frac{\partial p}{\partial \boldsymbol{m}}}\right)_{\rho,E}\cdot\left(\boldsymbol{m}^R - \boldsymbol{m}^L\right) + \left(\widetilde{\frac{\partial p}{\partial E}}\right)_{\rho,\boldsymbol{m}}\left(E^R - E^L\right)\right) =$$

$$e_{2k}\left(p^R - p^L\right) \tag{77}$$

for the transverse-momentum equation, and

$$-e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}^2}\widehat{E}\left(\rho^R - \rho^L\right) + e_{jk}\frac{\widehat{E}}{\widehat{\rho}}\left(m_j^R - m_j^L\right) + e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}}\left(E^R - E^L\right) =$$

$$e_{jk}\frac{m_j^R}{\rho^R}E^R - e_{jk}\frac{m_j^L}{\rho^L}E^L \tag{78}$$

$$-e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}^2}\widehat{p}\left(\rho^R - \rho^L\right) + e_{jk}\frac{\widehat{p}}{\widehat{\rho}}\left(m_j^R - m_j^L\right) + e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}}\left(p^R - p^L\right)$$

$$e_{jk}\frac{m_j^R}{\rho^R}p^R - e_{jk}\frac{m_j^L}{\rho^L}p^L \tag{79}$$

for the energy equation, where (79) implies (75), and (77).

Expressing the principal-momentum flux variation in homogeneous form yields

$$e_{1k}\left[-\frac{\widehat{m}_1}{\widehat{\rho}^2}\widehat{m}_1\left(\rho^R-\rho^L\right)+2\frac{\widehat{m}_1}{\widehat{\rho}}\left(m_1^R-m_1^L\right)-\left(\frac{m_1^R}{\rho^R}m_1^R-\frac{m_1^L}{\rho^L}m_1^L\right)\right]+$$

$$e_{2k}\left[-\frac{\widehat{m}_2}{\widehat{\rho}^2}\widehat{m}_1\left(\rho^R-\rho^L\right)+\frac{\widehat{m}_2}{\widehat{\rho}}\left(m_1^R-m_1^L\right)+\right.$$

$$\left.\frac{\widehat{m}_1}{\widehat{\rho}}\left(m_2^R-m_2^L\right)-\left(\frac{m_2^R}{\rho^R}m_1^R-\frac{m_2^L}{\rho^L}m_1^L\right)\right]=0 \tag{80}$$

and since the metric data are linearly independent, the two expressions between brackets will simultaneously vanish. Whereupon, the first expression reverts to (60), hence the average density coincides with (56), while the average principal momentum is

$$\widehat{m}_1 = m_1^L + \alpha\left(m_1^R-m_1^L\right) \tag{81}$$

where α is defined by (62). Thereafter, inserting the average density and principal momentum into the second expression yields

$$\widehat{m}_2\left(-m_1^L\rho^R+m_1^R\rho^L\right)+\widehat{\rho}\left(\widehat{m}_1\left(m_2^R-m_2^L\right)\right)=\rho^L m_2^R m_1^R-\rho^R m_2^L m_1^L \tag{82}$$

which defines the average transverse momentum as

$$\widehat{m}_2 = m_2^L + \alpha\left(m_2^R-m_2^L\right) \tag{83}$$

where α is again defined by (62). The homogeneous form of the transverse - momentum flux variation is then

$$e_{1k}\left[-\frac{\widehat{m}_1}{\widehat{\rho}^2}\widehat{m}_2\left(\rho^R-\rho^L\right)+\frac{\widehat{m}_2}{\widehat{\rho}}\left(m_1^R-m_1^L\right)+\right.$$

$$\left.\frac{\widehat{m}_1}{\widehat{\rho}}\left(m_2^R-m_2^L\right)-\left(\frac{m_1^R}{\rho^R}m_2^R-\frac{m_1^L}{\rho^L}m_2^L\right)\right]+$$

$$e_{2k}\left[-\frac{\widehat{m}_2}{\widehat{\rho}^2}\widehat{m}_2\left(\rho^R-\rho^L\right)+2\frac{\widehat{m}_2}{\widehat{\rho}}\left(m_2^R-m_2^L\right)-\left(\frac{m_2^R}{\rho^R}m_2^R-\frac{m_2^L}{\rho^L}m_2^L\right)\right]=0 \tag{84}$$

After interchanging the momentum subscripts, this expression reverts to the previous principal-momentum flux variation, and consequently it is identically satisfied by the developed average density and momentum components. The average partial derivatives in expressions (75), (77) are then exactly determined in Section 8.

The insertion of the average density and momentum components into the two leading terms in the lhs of (78) yields

$$-e_{jk}\frac{\widehat{m}_j}{\widehat{\rho}^2}\widehat{E}\left(\rho^R-\rho^L\right)+e_{jk}\frac{\widehat{E}}{\widehat{\rho}}\left(m_j^R-m_j^L\right)=$$

$$\widehat{E}\left(e_{jk}\frac{m_j^R}{\rho^R}-e_{jk}\frac{m_j^L}{\rho^L}\right) \tag{85}$$

Hence, (78) becomes

$$\Delta\left(e_{jk}\frac{m_j}{\rho}E\right) = e_{jk}\frac{m_j^R}{\rho^R}E^R - e_{jk}\frac{m_j^L}{\rho^L}E^L = \widehat{E}\left(e_{jk}\frac{m_j^R}{\rho^R} - e_{jk}\frac{m_j^L}{\rho^L}\right) + e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}}\left(E^R - E^L\right)$$

(86)

which constitutes the average-state resolution for the variation of the contravariant flux of E. A similar operation sequence on (79) yields

$$\Delta\left(e_{jk}\frac{m_j}{\rho}p\right) = e_{jk}\frac{m_j^R}{\rho^R}p^R - e_{jk}\frac{m_j^L}{\rho^L}p^L = \widehat{p}\left(e_{jk}\frac{m_j^R}{\rho^R} - e_{jk}\frac{m_j^L}{\rho^L}\right) + e_{jk}\frac{\widehat{m_j}}{\widehat{\rho}}\left(p^R - p^L\right)$$

(87)

which constitutes the average-state resolution for the variation of the contravariant flux of p. Whereupon, inserting the average density and momentum components into (86) and considering the linear independence of the metric data yields the average volume specific total energy

$$\widehat{E} = E^L + \alpha\left(E^R - E^L\right)$$

(88)

exactly as before. The average pressure is then again expressed as

$$\widehat{p} = p^L + \alpha\left(p^R - p^L\right)$$

(89)

owing to the identical structure of (86) and (87).

6   Reference Average State

The original convection average state [3] relies on an average velocity and enthalpy, and can be expressed as

$$\widehat{\rho} = \sqrt{\rho^L\rho^R}$$

(90)

$$\widehat{u} = u^L\frac{\sqrt{\rho^L}}{\sqrt{\rho^L} + \sqrt{\rho^R}} + u^R\frac{\sqrt{\rho^R}}{\sqrt{\rho^L} + \sqrt{\rho^R}}$$

(91)

$$\widehat{H} = \frac{H^L\sqrt{\rho^L} + H^R\sqrt{\rho^R}}{\sqrt{\rho^L} + \sqrt{\rho^R}}$$

(92)

No average pressure $\widehat{p}$ or volume specific total energy $\widehat{E}$ is explicitly defined in the original derivations. However, the developed conservation variable average state (56)-(59) identically reverts to (90)-(92). This is proven by inserting the average density, momenta, energy and pressure states (56)-(59) into the definition of each

velocity component and enthalpy which yields

$$\hat{u} = \frac{\widehat{m}}{\widehat{\rho}} = \frac{u^L \sqrt{\rho^L} + u^R \sqrt{\rho^R}}{\sqrt{\rho^L} + \sqrt{\rho^R}} \tag{93}$$

$$\widehat{H} = \frac{\widehat{E} + \widehat{p}}{\widehat{\rho}} = \frac{H^L \sqrt{\rho^L} + H^R \sqrt{\rho^R}}{\sqrt{\rho^L} + \sqrt{\rho^R}} \tag{94}$$

Therefore, the developed convection average states (56)-(59) are totally equivalent to (90)-(92). It is occasionally reported that the average pressure should be obtained using its own equation of state evaluated at (90)-(92). However, if this course of action is selected, then the corresponding average total energy cannot coincide with the internal average (58). In either case, only in the enthalpy function do pressure and total energy emerge in the conservation variable jacobian. Therefore, any set of $\widehat{p}$ and $\widehat{E}$ that does not alter (94), will yield the same convection average state.

As stated in Sections 2-3, the average enthalpy does not satisfy the mean value theorem for the energy equation. This is proved observing that the legitimate mean value theorem variables are the jacobian differentiation variables which coincide with the entries in q. Hence the total enthalpy

$$H(q) = \frac{1}{\rho} \left(E + p(\rho, m, E) \right) \tag{95}$$

is itself a function of the dependent variable q. Consequently, its mean value theorem average is

$$\widehat{H}^{MVT} = \frac{1}{\widehat{\rho}} \left(\widehat{E} + p\left(\widehat{\rho}, \widehat{m}, \widehat{E} \right) \right) \tag{96}$$

where superscript MVT denotes such an average. However, the average enthalpy (94) may not coincide with this expression. In fact, for the ideal gas equation of state (144), the difference between (96) and (95) is

$$\widehat{H}^{MVT} - \widehat{H} = \gamma \left(\frac{\widehat{E}}{\widehat{\rho}} - \left(\widehat{\frac{E}{\rho}} \right) \right) - \frac{\gamma - 1}{2} \left((\hat{u})^2 - \widehat{(u)^2} \right) \tag{97}$$

$$= -\frac{\gamma - 1}{2} \left((\hat{u})^2 - \widehat{(u)^2} \right) \tag{98}$$

where the terminal minimal variation form accrues from the internal average total energy (58).

As detailed in Section 7, the average sound speed squared is exactly expressed as

$$\widehat{c}^2 = \left(\widetilde{\frac{\partial p}{\partial \rho}} \right)_{m,E} + \left(\widetilde{\frac{\partial p}{\partial E}} \right)_{\rho,m} \cdot \left(\widehat{H} - \frac{\widetilde{m}^T}{\widetilde{\rho}} \cdot \frac{\widehat{m}}{\widehat{\rho}} \right) \tag{99}$$

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and depends upon the average enthalpy (94). This dependence, which results from the insertion of the average density and momentum into the energy flux resolution (78), constitutes the fundamental reason why (99) may supply an external average sound speed, for some left and right states. For example, using (144), the ideal gas specification of the average sound speed squared becomes

$$\widehat{c^2} = (\gamma - 1)\frac{\widehat{m}^T \cdot \widehat{m}}{2\widehat{\rho}^2} + (\gamma - 1)\left(\widehat{H} - \frac{\widehat{m}^T}{\widehat{\rho}} \cdot \frac{\widehat{m}}{\widehat{\rho}}\right) \tag{100}$$

The insertion of the average states (91) and (92) into (100) then yields

$$\widehat{c^2} = \frac{\alpha(\alpha - 1)(\gamma - 1)}{2}\left(\frac{m^R}{\rho^R} - \frac{m^L}{\rho^L}\right)^T \cdot \left(\frac{m^R}{\rho^R} - \frac{m^L}{\rho^L}\right) + \alpha\left(c^L\right)^2 + (1 - \alpha)\left(c^R\right)^2 \tag{101}$$

which, as also determined in [7], adds an extra term to an internal average for c^2. Conversely, if $\widehat{H}$ in (99) were replaced by (96), this extra term would not emerge. However, (99) is the only correct expression compatible with the flux difference resolutions (43), (69). This result should not be considered detrimental, since the fundamental flux difference relations (43), (69) are rigorously satisfied for arbitrary left and right states. On the contrary, (99), hence (101), may be used to determine whether some external average eigenvalues occur in the computed flow field, and, consequently, whether a local enhancement of the basic flux difference scheme is warranted.

7   Average Characteristic Relations

A completely general eigenvalue analysis performed on the multi-dimensional curvilinear - coordinate jacobian (71) yields an original expression for the speed of sound which is shown to revert identically to the classical thermodynamic relation. The same analysis yields the average eigenvalues, Mach number, and eigenvectors as functions of both the average state (56)-(59) and the partial derivatives of pressure. Importantly, these results apply for any specific form of these derivatives.

7.1   Average Eigenvalues

The trace of matrix (71), which coincides with the sum of its eigenvalues, is

$$\text{tr}\left(\widehat{A}_k\right) = (n + 2)e_{jk}\frac{\widehat{m}_j}{\widehat{\rho}} + e_{jk}\left(\frac{\widetilde{\partial p}}{\partial m_j}\right)_{\rho,m_i,E,i \neq j} + e_{jk}\frac{\widehat{m}_j}{\widehat{\rho}}\left(\frac{\widetilde{\partial p}}{\partial E}\right)_{\rho,m} \tag{102}$$

Considering that in general the acoustic-wave average state corresponding to the average partial derivatives of pressure may differ from $\widehat{q}$, the trailing two-term expression in (102) may not vanish since it may not satisfy the compatibility relation

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(127) derived in Section 8. Hence, expression (102) may not reduce to the usual result $\mathrm{tr}\left(\widehat{A}_k\right) = (n+2)e_{jk}\dfrac{\widehat{m}_j}{\widehat{\rho}}$. Consequently, the eigenvalues of (71) may not coincide with the classical expressions, which for a three-dimensional cartesian-coordinate reference frame are

$$\lambda_{k_{1,3}} = \frac{\widehat{m}_k}{\widehat{\rho}} \quad , \quad \lambda_{k_{4,5}} = \frac{\widehat{m}_k}{\widehat{\rho}} \pm c \tag{103}$$

where c denotes the speed of sound. In fact, the analytical solution of the characteristic equation

$$\det\left[A_k\left(\widehat{q},\widehat{p},\left(\frac{\widetilde{\partial p}}{\partial \rho}\right)_{\boldsymbol{m},E},\left(\frac{\widetilde{\partial p}}{\partial \boldsymbol{m}}\right)_{\rho,E},\left(\frac{\widetilde{\partial p}}{\partial E}\right)_{\rho,\boldsymbol{m}} e_{jk} \, \underline{}\,\right) - \lambda_k I\right] = 0 \tag{104}$$

yields

$$\lambda_{k_{1,3}} = e_{jk}\frac{\widehat{m}_j}{\widehat{\rho}} \tag{105}$$

$$\lambda_{k_{4,5}} = e_{jk}\frac{\widehat{m}_j}{\widehat{\rho}} + e_{jk}\widehat{u}_{d_j} \pm \left(\left(e_{jk}\widehat{u}_{d_j}\right)^2 + e_{j\underline{k}}e_{jk}\widehat{a^2}\right)^{1/2} \tag{106}$$

where $\widehat{\boldsymbol{u}}_d \equiv \{\widehat{u}_{d_j}\}^T$, $1 \le j \le n$, and $\widehat{a^2}$ are directly obtained as part of the eigenvalue solution itself as

$$\widehat{\boldsymbol{u}}_d\left(\widehat{q},\left(\frac{\widetilde{\partial p}}{\partial \boldsymbol{m}}\right)_{\rho,E},\left(\frac{\widetilde{\partial p}}{\partial E}\right)_{\rho,\boldsymbol{m}}\right) \equiv \frac{1}{2}\left(\left(\frac{\widetilde{\partial p}}{\partial \boldsymbol{m}}\right)_{\rho,E}^T + \frac{\widehat{m}}{\widehat{\rho}}\left(\frac{\widetilde{\partial p}}{\partial E}\right)_{\rho,\boldsymbol{m}}\right) \tag{107}$$

$$\widehat{a^2}\left(\widehat{q},\widehat{p},\left(\frac{\widetilde{\partial p}}{\partial q}\right)\right) \equiv \left(\frac{\widetilde{\partial p}}{\partial \boldsymbol{m}}\right)_{\rho,E} \cdot \frac{\widehat{m}}{\widehat{\rho}} + \left(\frac{\widetilde{\partial p}}{\partial \rho}\right)_{\boldsymbol{m},E} + \frac{\widehat{E}+\widehat{p}}{\widehat{\rho}}\left(\frac{\widetilde{\partial p}}{\partial E}\right)_{\rho,\boldsymbol{m}} \tag{108}$$

With these expressions the sum of eigenvalues (105)-(106) then coincides with (102).

7.2 Speed of Sound

Relation (108) delivers an original alternative expression for the square of the speed of sound that identically reverts to the classical thermodynamic expression

$$c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{\mathrm{isen}} = \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon + \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \frac{p}{\rho^2} \tag{109}$$

as directly obtained using the equation of state, the first principle of thermodynamics, and the definition of speed of sound. This coincidence accrues from a differentiation chain rule restatement of the partial derivatives of pressure in (108)

using expressions (124)-(126), detailed in Section 8. Insertion of these expressions into the generic form of (108) leads to

$$
\begin{aligned}
a^2\left(q, p, \left(\frac{\partial p}{\partial q}\right)\right) &= \left(\frac{\partial p}{\partial \boldsymbol{m}}\right)_{\rho, E} \cdot \frac{\boldsymbol{m}}{\rho} + \left(\frac{\partial p}{\partial \rho}\right)_{\boldsymbol{m}, E} + \frac{E + p}{\rho}\left(\frac{\partial p}{\partial E}\right)_{\rho, \boldsymbol{m}} \\
&= \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \cdot \left(-\frac{\boldsymbol{m}^T \cdot \boldsymbol{m}}{\rho^3}\right) + \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon + \\
&\quad \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \cdot \left(-\frac{E}{\rho^2} + \frac{\boldsymbol{m}^T \cdot \boldsymbol{m}}{\rho^3}\right) + \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \cdot \left(\frac{E}{\rho^2} + \frac{p}{\rho^2}\right) \\
&= \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon + \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \frac{p}{\rho^2} = c^2 \qquad (110)
\end{aligned}
$$

which coincides with (109). Consequently, a^2 is indeed an alternative relation for c^2. Whereupon, insertion into (108) of the pressure derivative compatibility relations (127), derived in Section 8, yields the average conservation-variable speed of sound squared

$$
\widehat{c^2} = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\boldsymbol{m}, E} + \left(\widetilde{\frac{\partial p}{\partial E}}\right)_{\rho, \boldsymbol{m}} \cdot \left(\frac{(\widehat{E} + \widehat{p})}{\widehat{\rho}} - \frac{\widetilde{\boldsymbol{m}}^T}{\widetilde{\rho}} \cdot \frac{\widehat{\boldsymbol{m}}}{\widehat{\rho}}\right) \qquad (111)
$$

which depends on only two partial derivatives, hence it closely resembles the classical thermodynamic relation (109). Expression (111) does not depend on an average internal energy, hence is not affected by any assumption connected with such an additional variable.

7.3   Average Mach Number

The eigenvalues of the one-dimensional average jacobian (44) are obtained from (105)-(106) as

$$
\lambda_1 = \frac{\widehat{m}}{\widehat{\rho}} \qquad (112)
$$

$$
\lambda_{2,3} = \frac{\widehat{m}}{\widehat{\rho}} + \widehat{u}_d \pm \left(\widehat{u}_d^2 + \widehat{a^2}\right)^{1/2} \qquad (113)
$$

Expression (113) leads to an average-Mach number definition as follows. For a positive velocity, the eigenvalues (112)-(113) are all non-negative if

$$
\frac{\widehat{m}}{\widehat{\rho}} + \widehat{u}_d \geq \left(\widehat{u}_d^2 + \widehat{a^2}\right)^{1/2} \qquad (114)
$$

which yields

$$
\left(\frac{\widehat{m}}{\widehat{\rho}}\right)^2 + 2\frac{\widehat{m}}{\widehat{\rho}}\widehat{u}_d \geq \widehat{a^2} \qquad (115)
$$

Consequently, the average eigenvalues have all uniform sign if the expression

$$\widehat{M} \equiv \frac{\left(\left(\frac{\widehat{m}}{\widehat{\rho}}\right)^2 + \frac{\widehat{m}}{\widehat{\rho}}\left(\left(\widetilde{\frac{\partial p}{\partial m}}\right)_{\rho,E} + \frac{\widehat{m}}{\widehat{\rho}}\left(\widetilde{\frac{\partial p}{\partial E}}\right)_{\rho,m}\right)\right)^{1/2}}{\left(\widehat{a^2}\right)^{1/2}} \tag{116}$$

is greater than one. Therefore, $\widehat{M}$ defines the average Mach number. The same result is achieved by enforcing non positivity of λ_2, for a negative velocity. Expression (116) depends on both $\widehat{q}$ and the partial derivatives of pressure, and in general differs from the classical expression evaluated at the ideal-gas average state, unless the average partial derivatives correspond to $\widehat{q}$ itself, see Section 8.

7.4 Average Eigenvectors

The three right eigenvectors for the one-dimensional similarity transformation

$$A\left(\widehat{q},\widehat{p},\left(\widetilde{\frac{\partial p}{\partial q}}\right)\right) = X\Lambda X^{-1} \tag{117}$$

are

$$\begin{pmatrix} 1 & , & 1 \\ \frac{\widehat{m}}{\widehat{\rho}} & , & \frac{\widehat{m}}{\widehat{\rho}} \pm \widehat{u}_d + \left(\widehat{u}_d^2 + \widehat{a^2}\right)^{1/2} \\ -\frac{1}{\widetilde{p_E}}\left(\frac{\widehat{m}}{\widehat{\rho}}\widetilde{p_m} + \widetilde{p_\rho}\right) & , & \frac{\widehat{E}+\widehat{p}}{\widehat{\rho}} \pm \frac{\widehat{m}}{\widehat{\rho}}\left(\widehat{u}_d^2 + \widehat{a^2}\right)^{1/2} + \frac{\widehat{m}}{\widehat{\rho}}\widehat{u}_d \end{pmatrix} \tag{118}$$

where the second and third column for X respectively correspond to the positive and negative sign of the ensemble $\pm$ in the second column of (118). Therefore, the average conservation-variable eigenvalues, Mach number, and eigenvectors (105)-(106), (116), and (118) differ from the classical forms due to their dependence upon the average deviation velocity $\widehat{u}_d$, in addition to the average sound speed squared $\widehat{a^2}$. Consequently, the sign of (106), (113) may not coincide with the sign of the pointwise eigenvalues in transonic flow regions, while X may not revert to the pointwise eigenvectors. Hence, in equilibrium real gas situations, any approximation to these exact internal conservation variable average eigenvalues and eigenvectors may yield for some thermodynamic states a terminal flux splitting scheme with locally inadequate intrinsic dissipation. For an ideal-gas specification the average partial derivatives correspond to $\widehat{q}$ itself, see Section 8. Hence, the deviation speed vanishes and consequently (105)-(106), (116), and (118) revert to the classical forms.

8 Partial Derivatives of Pressure

In the published procedures [4,8,9] the average partial derivatives of pressure are essentially regarded as additional independent variables. Hence, they are algebraically determined using additional geometric projections and scale factors. In distinction, these derivatives are herein simultaneously computed by determining their intrinsic acoustic-wave average state, for the mean value theorem resolution of the pressure variation jacobian.

8.1 Pressure Jacobian and Compatibility Relations

For a homogeneous gas in thermo-chemical equilibrium, the pressure equation of state may be expressed as the function differentiable almost everywhere [1] in Ω

$$p = p(\rho, \varepsilon) \tag{119}$$

where ε denotes the mass-specific internal energy. Therefore, for a given thermodynamic state (ρ, ε), both p and its partial derivatives are theoretically determined. For a CFD simulation, hence for a given q, the mass-specific internal energy is cast as

$$\varepsilon(q) = \frac{1}{\rho} \left(E - \frac{m^T \cdot m}{2\rho} \right) \tag{120}$$

Therefore, ε is no longer an independent variable, but intrinsically depends on the conservation state variable q. Consequently, within an equilibrium-gas CFD algorithm, p is expressed as

$$p = p(\rho, \varepsilon(\rho, m, E)) = p(q(x, t)) \tag{121}$$

Hence, the pressure jacobian is also a function of q

$$\frac{\partial p}{\partial q} \equiv \frac{\partial p}{\partial q}(q) \tag{122}$$

Consequently, the average pressure jacobian corresponds to

$$\widetilde{\frac{\partial p}{\partial q}} \equiv \frac{\partial p}{\partial q}(\tilde{q}) \tag{123}$$

where $\tilde{q}$ denotes the associated intrinsic evaluation state.

The partial derivatives of pressure with respect to the state variable q are expressed by way of the differentiation chain rule and the thermodynamic derivatives as

$$\left(\frac{\partial p}{\partial \rho} \right)_{m,E} = \left(\frac{\partial p}{\partial \rho} \right)_\varepsilon + \left(\frac{\partial p}{\partial \varepsilon} \right)_\rho \left(\frac{\partial \varepsilon}{\partial \rho} \right)_{m,E} = \left(\frac{\partial p}{\partial \rho} \right)_\varepsilon + \left(\frac{\partial p}{\partial \varepsilon} \right)_\rho \cdot \frac{1}{\rho^2} \left(\frac{m^T \cdot m}{\rho} - E \right) \tag{124}$$

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$$\left(\frac{\partial p}{\partial m}\right)_{\rho,E} = \left(\frac{\partial p}{\partial \varepsilon}\right)_{\rho} \left(\frac{\partial \varepsilon}{\partial m}\right)_{\rho,E} = \left(\frac{\partial p}{\partial \varepsilon}\right)_{\rho} \cdot \left(-\frac{m^T}{\rho^2}\right) \tag{125}$$

$$\left(\frac{\partial p}{\partial E}\right)_{\rho,m} = \left(\frac{\partial p}{\partial \varepsilon}\right)_{\rho} \left(\frac{\partial \varepsilon}{\partial E}\right)_{\rho,m} = \left(\frac{\partial p}{\partial \varepsilon}\right)_{\rho} \cdot \left(\frac{1}{\rho}\right) \tag{126}$$

Whereupon these formulae lead to the fundamental compatibility relation

$$\left(\frac{\partial p}{\partial m}\right)_{\rho,E} + \frac{m^T}{\rho} \left(\frac{\partial p}{\partial E}\right)_{\rho,m} = 0 \tag{127}$$

The developments of Sections 4-5 yield the jacobian pressure variation expression

$$p^R - p^L = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{m,E} \left(\rho^R - \rho^L\right) + \left(\widetilde{\frac{\partial p}{\partial m}}\right)_{\rho,E} \cdot \left(m^R - m^L\right) + \left(\widetilde{\frac{\partial p}{\partial E}}\right)_{\rho,m} \left(E^R - E^L\right) \tag{128}$$

Whereupon, insertion of the compatibility relation (127) into (128) and noting the dependence on the intrinsic evaluation state $\tilde{q}$ yield the reduced form

$$p^R - p^L = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{m,E} \left(\rho^R - \rho^L\right) + \left(\widetilde{\frac{\partial p}{\partial E}}\right)_{\rho,m} \cdot \left(\left(E^R - E^L\right) - \frac{\widetilde{m}^T}{\tilde{\rho}} \cdot \left(m^R - m^L\right)\right) \tag{129}$$

which is universally valid for both ideal and real gas equations of state, it only depends upon two partial derivatives, and it thereby closely resembles the classical thermodynamic expression

$$p^R - p^L = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\varepsilon} \Delta\rho + \left(\widetilde{\frac{\partial p}{\partial \varepsilon}}\right)_{\rho} \Delta\varepsilon \tag{130}$$

Expression (129) coincides with a special form of (130). This result is obtained by inserting (124) and (126) into (129), which yields

$$p^R - p^L = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\varepsilon} \Delta\rho + \left(\widetilde{\frac{\partial p}{\partial \varepsilon}}\right)_{\rho} \left[\frac{1}{\tilde{\rho}^2} \left(\frac{\widetilde{m}^T \cdot \widetilde{m}}{\tilde{\rho}} - \tilde{E}\right) \Delta\rho - \frac{\widetilde{m}^T}{\tilde{\rho}^2} \cdot \Delta m + \frac{\Delta E}{\tilde{\rho}}\right] \tag{131}$$

This relation reverts to (130) provided that

$$\Delta\varepsilon(\tilde{q}, \Delta q) \equiv \frac{1}{\tilde{\rho}^2} \left(\frac{\widetilde{m}^T \cdot \widetilde{m}}{\tilde{\rho}} - \tilde{E}\right) \Delta\rho - \frac{\widetilde{m}}{\tilde{\rho}^2} \cdot \Delta m + \frac{\Delta E}{\tilde{\rho}} \tag{132}$$

This expression corresponds to the variation of (120) in terms of its first differential, and in general does not coincide with the simple independent variation

$$\Delta\varepsilon \equiv \varepsilon\left(q^R\right) - \varepsilon\left(q^L\right) \tag{133}$$

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even though the difference between (132) and (133) is of order $\mathcal{O}\left(\Delta q^2\right)$. When the acoustic-wave average state $\widetilde{q}$ coincides with the convection average state $\widehat{q}$, then (132) coincides with (133)

$$\Delta\varepsilon\left(\widehat{q},\Delta q\right) \equiv \frac{1}{\widehat{\rho}^2}\left(\frac{\widehat{m}^T\cdot\widehat{m}}{\widehat{\rho}} - \widehat{E}\right)\Delta\rho - \frac{\widehat{m}}{\widehat{\rho}^2}\cdot\Delta m + \frac{\Delta E}{\widehat{\rho}} = \varepsilon\left(q^R\right) - \varepsilon\left(q^L\right) \quad (134)$$

However, the convection average state $\widehat{q}$ may not always satisfy (129) or (130). Therefore, within a CFD algorithm, the pressure variation (130) is in general expressed as

$$p^R - p^L = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\widetilde{\frac{\partial p}{\partial\varepsilon}}\right)_\rho \Delta\varepsilon\left(\widetilde{q},\Delta q\right) \quad (135)$$

For a non-vanishing simple variation $\Delta\varepsilon$, expression (135) is cast as

$$\begin{aligned}
p^R - p^L &= \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\left(\widetilde{\frac{\partial p}{\partial\varepsilon}}\right)_\rho \frac{\Delta\varepsilon\left(\widetilde{q},\Delta q\right)}{\Delta\varepsilon}\right)\Delta\varepsilon \\
&= \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\widetilde{\frac{\partial p}{\partial\varepsilon}}\right)_\rho \Delta\varepsilon \quad (136)
\end{aligned}$$

which formally reverts to (130), and corresponds to the actual pressure variation relation utilized in several reported procedures, as detailed in Section 9. Following (122)-(123), relations (129) and (135) are properly expressed as

$$\begin{aligned}
p^R - p^L &= \left(\frac{\partial p}{\partial \rho}(\widetilde{q})\right)_{m,E} \Delta\rho + \left(\frac{\partial p}{\partial E}(\widetilde{q})\right)_{\rho,m}\cdot\left(\Delta E - \frac{\widetilde{m}^T}{\widetilde{\rho}}\cdot\Delta m\right) \quad (137) \\
p^R - p^L &= \left(\frac{\partial p}{\partial \rho}(\widetilde{q})\right)_\varepsilon \Delta\rho + \left(\frac{\partial p}{\partial\varepsilon}(\widetilde{q})\right)_\rho \Delta\varepsilon\left(\widetilde{q},\Delta q\right) \quad (138)
\end{aligned}$$

Therefore, all of these derivatives can be simultaneously determined by identifying the single state $\widetilde{q}$, as detailed in the next section.

8.2 Intrinsic Evaluation State Determination

The existence of an internal state $\widetilde{q}$ for the evaluation of the partial derivatives in (137) is ensured by the multi-dimensional mean value theorem (11), since p is a differentiable function with continuous partial derivatives almost everywhere in Ω [1]. The determination of the intrinsic state $\widetilde{q} \equiv q^L + \widetilde{\alpha}\left(q^R - q^L\right)$ state is then achieved by the equivalent direct computation of the single weight coefficient $\widetilde{\alpha}$. To this end, (137) is re-expressed as

$$F(\tilde{\alpha}) = \left(\frac{\partial p}{\partial \rho}\left(q^L + \tilde{\alpha}\left(q^R - q^L\right)\right)\right)_{m,E} \Delta\rho +$$

$$\left(\frac{\partial p}{\partial E}\left(q^L + \tilde{\alpha}\left(q^R - q^L\right)\right)\right)_{\rho,m} \left(\Delta E - \frac{\left(m^L + \tilde{\alpha}\left(m^R - m^L\right)\right)^T}{\rho^L + \tilde{\alpha}\left(\rho^R - \rho^L\right)} \cdot \Delta m\right) - \Delta p = 0 \tag{139}$$

This relation constitutes a single equation for the unknown $\tilde{\alpha}$ which can be solved to an arbitrary degree of accuracy by way of a Newton-Raphson iteration strategy

$$\alpha^{s+1} - \alpha^s = -\frac{F(\alpha^s)}{\left(\frac{\Delta F}{\Delta \alpha}\right)_{\alpha^s}} \tag{140}$$

where superscript "s" denotes the iteration index. Owing to the multi-dimensional mean value theorem, the solution $\tilde{\alpha}$ satisfies the inequality

$$0.0 \leq \tilde{\alpha} \leq 1.0 \tag{141}$$

This relation circumscribes the root of (139), and an initial estimate α^0 for a first mesh-node computation may then coincide with (62), while $\Delta\alpha \ll 1.0$. Whereupon, α^0 at subsequent mesh nodes may coincide with the value of $\tilde{\alpha}$ at a neighboring point where (139) is already solved. This choice leads to an efficient computational strategy that only requires two or three iterations owing to the good initial estimate. Therefore, evaluating (140) is relatively inexpensive, it leads to a super linear convergent process, and directly yields $\tilde{\alpha}$. Hence, the average state $\tilde{q}$ as well as the average partial derivatives that satisfy (137)-(138) are simultaneously determined. This is the procedure utilized for the benchmark test in Section 10.

A viable expression for ΔF in the denominator of (140) is

$$(\Delta F)_{\alpha^s} \equiv (\Delta p_\rho)_{\alpha^s} \cdot \Delta\rho + (\Delta p_E)_{\alpha^s} \cdot \left(\Delta E - \frac{\widetilde{m}^T}{\tilde{\rho}} \cdot \Delta m\right)_{\alpha^s} +$$

$$\left(\frac{\widetilde{p_E}}{\tilde{\rho}}\right)_{\alpha^s} \left(\Delta\rho \frac{\widetilde{m}}{\tilde{\rho}} - \Delta m\right)^T_{\alpha^s} \cdot \Delta m \Delta\alpha \tag{142}$$

where the last term corresponds to the differential of the coefficient of Δp_E with respect to α. Furthermore, $(\Delta p_\rho)_{\alpha^s}$ and $(\Delta p_E)_{\alpha^s}$ are

$$(\Delta p_\rho)_{\alpha^s} \equiv p_\rho\left(q^L + (\alpha^s + \Delta\alpha)\left(q^R - q^L\right)\right) - p_\rho\left(q^L + \alpha^s\left(q^R - q^L\right)\right)$$

$$(\Delta p_E)_{\alpha^s} \equiv p_E\left(q^L + (\alpha^s + \Delta\alpha)\left(q^R - q^L\right)\right) - p_E\left(q^L + \alpha^s\left(q^R - q^L\right)\right) \tag{143}$$

Note that at covergence, the solution $\tilde{\alpha}$ no longer depends upon (142)-(143).

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8.3 Ideal Gas Partial Derivatives

For an ideal gas, (139) is identically satisfied for $\tilde{\alpha}$ coinciding with (62). This conclusion follows from the state equation

$$p = (\gamma - 1) \left(E - \frac{\boldsymbol{m}^T \cdot \boldsymbol{m}}{2\rho} \right) \tag{144}$$

and associated average partial derivatives

$$\left(\widehat{\frac{\partial p}{\partial \rho}} \right)_{\boldsymbol{m},E} = (\gamma - 1) \frac{\widehat{\boldsymbol{m}}^T \cdot \widehat{\boldsymbol{m}}}{2\widehat{\rho}^2} \quad , \quad \left(\widehat{\frac{\partial p}{\partial \boldsymbol{m}}} \right)_{\rho,E} = -(\gamma - 1) \frac{\widehat{\boldsymbol{m}}^T}{\widehat{\rho}} \quad , \quad \left(\widehat{\frac{\partial p}{\partial E}} \right)_{\rho,\boldsymbol{m}} = (\gamma - 1) \tag{145}$$

The insertion of (144)-(145) into (139) yields

$$\Delta E - \frac{1}{2} \Delta \left(\frac{\boldsymbol{m}^T \cdot \boldsymbol{m}}{\rho} \right) = \frac{\widehat{\boldsymbol{m}}^T \cdot \widehat{\boldsymbol{m}}}{2\widehat{\rho}^2} \Delta\rho - \frac{\widehat{\boldsymbol{m}}^T}{\widehat{\rho}} \cdot \Delta\boldsymbol{m} + \Delta E \tag{146}$$

hence

$$-\frac{\widehat{\boldsymbol{m}}^T \cdot \widehat{\boldsymbol{m}}}{\widehat{\rho}^2} \Delta\rho + 2\frac{\widehat{\boldsymbol{m}}^T}{\widehat{\rho}} \cdot \Delta\boldsymbol{m} - \Delta \left(\frac{\boldsymbol{m}^T \cdot \boldsymbol{m}}{\rho} \right) = 0 \tag{147}$$

This expression is identically satisfied by the average density and momentum components (56)-(57), since (147), for a two-dimensional formulation, constitutes the sum of the first relation in (80) and the second relation in (84), which are themselves identically satisfied by the convection average state. The same conclusion is achieved for a three-dimensional formulation.

9 Comparisons with Other Formulations

The pressure variation and sound speed expressions reported in [4,8,9] are analyzed and their underlying hypotheses identified. These derivations algebraically determine the partial derivatives in (138) without reference to the intrinsic state $\tilde{q}$. The various solutions for each partial derivative set generally do not coincide with one another. Furthermore, it is a-priori unknown whether some evaluation state $\tilde{q}_a$ exists in the domain of the analytical partial derivatives such that

$$\left(\widetilde{\frac{\partial p}{\partial q}} \right)_a = \left(\frac{\partial p}{\partial q}(\tilde{q}_a) \right) \tag{148}$$

where subscript a denotes algebraic determination. And even if such states existed, they may correspond to external conservation-variable states.

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9.1 Pressure and Sound Speed Expressions

The pressure and sound speed expressions associated with the current formulations are listed in Tables 1 and 2 along with the the thermodynamic relation for the sound speed.

Table 1: Pressure Variation Expressions

Author		Δp
Glaister	[4]	$\left(\widetilde{\dfrac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\widetilde{\dfrac{\partial p}{\partial \varepsilon}}\right)_\rho \Delta\varepsilon$
Liou et al.	[8]	$\left(\widetilde{\dfrac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\widetilde{\dfrac{\partial p}{\partial \varepsilon}}\right)_\rho \Delta\varepsilon$
Vinokur et al.	[9]	$\left(\widetilde{\dfrac{\partial p}{\partial \rho}}\right)_{\rho\varepsilon} \Delta\rho + \left(\widetilde{\dfrac{\partial p}{\partial \rho\varepsilon}}\right)_\rho \Delta(\rho\varepsilon)$
Present		$\left(\widetilde{\dfrac{\partial p}{\partial \rho}}\right)_{m,E} \Delta\rho + \left(\widetilde{\dfrac{\partial p}{\partial E}}\right)_{\rho,m} \cdot \left(\Delta E - \dfrac{\widetilde{m}^T}{\widetilde{\rho}} \cdot \Delta m\right)$

Table 2: Sound Speed Expressions

Author	$\widehat{c^2}$
(Thermodynamics)	$\left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon + \left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho \dfrac{p}{\rho^2}$
Glaister	$\left(\widetilde{\dfrac{\partial p}{\partial \rho}}\right)_\varepsilon + \left(\widetilde{\dfrac{\partial p}{\partial \varepsilon}}\right)_\rho \cdot \left[\dfrac{\widehat{H} - \widehat{\varepsilon}}{\widehat{\rho}} - \dfrac{\widehat{m}^T \cdot \widehat{m}}{2\widehat{\rho}^3}\right]$
Liou et al.	$\left(\widetilde{\dfrac{\partial p}{\partial \rho}}\right)_\varepsilon + \left(\widetilde{\dfrac{\partial p}{\partial \varepsilon}}\right)_\rho \cdot \left[\dfrac{\widehat{H} - \widehat{\varepsilon}}{\widehat{\rho}} - \dfrac{\widehat{m}^T \cdot \widehat{m}}{2\widehat{\rho}^3}\right]$
Vinokur et al.	$\left(\widetilde{\dfrac{\partial p}{\partial \rho}}\right)_{\rho\varepsilon} + \left(\widetilde{\dfrac{\partial p}{\partial \rho\varepsilon}}\right)_\rho \cdot \left[\widehat{H} - \dfrac{\widehat{m}^T \cdot \widehat{m}}{2\widehat{\rho}^2}\right]$
Present	$\left(\widetilde{\dfrac{\partial p}{\partial \rho}}\right)_{m,E} + \left(\widetilde{\dfrac{\partial p}{\partial E}}\right)_{\rho,m} \cdot \left[\widehat{H} - \dfrac{\widetilde{m}^T \cdot \widetilde{m}}{\widetilde{\rho}\widehat{\rho}}\right]$

The reported formulae [4,8,9] use the thermodynamic derivatives of pressure and some combinations of flow variables in the respective sound speed expressions. Several derivations also introduce an extra average mass specific internal energy. The average derivatives in these constructions are synthesized using the assumptions that the general jacobian pressure variation and sound speed squared can be reliably satisfied using the following constraints

1. ideal-gas average state evaluation of selected expressions within the pressure variation

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2. selection of simple integration paths and associated auxiliary state

3. geometric solution projection with arbitrary scale factors

Significantly, the developed exact expressions (111)-(129), which are independent of these assumptions, lead to the reported formulae when these constraints are explicitly used.

The equation of state used by Glaister [4] and Liou et al. [8] is

$$p = p(\rho, \varepsilon) \tag{149}$$

Thus, the partial derivatives of pressure in (44) are cast as

$$\left(\frac{\partial p}{\partial \rho}\right)_{m,E} = \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon + \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \cdot \frac{1}{\rho}\left(\frac{m^T \cdot m}{2\rho^2} - \varepsilon\right) \tag{150}$$

$$\left(\frac{\partial p}{\partial m}\right)_{\rho,E} = \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \cdot \left(-\frac{m^T}{\rho^2}\right) \tag{151}$$

$$\left(\frac{\partial p}{\partial E}\right)_{\rho,m} = \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho \cdot \left(\frac{1}{\rho}\right) \tag{152}$$

The insertion of (150)-(152) into (129) yields

$$\Delta p = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\widetilde{\widetilde{\frac{\partial p}{\partial \varepsilon}}}\right)_\rho \left[\frac{\Delta\varepsilon}{\Delta\varepsilon(\widetilde{q}, \Delta q)} \cdot \left[\left(\frac{\widetilde{m}^T \cdot \widetilde{m}}{2\widetilde{\rho}^3} - \frac{\widetilde{\varepsilon}}{\widetilde{\rho}}\right) \Delta\rho - \frac{\widetilde{m}^T}{\widetilde{\rho}^2} \cdot \Delta m + \frac{\Delta E}{\widetilde{\rho}}\right]\right] \tag{153}$$

Whereupon, the intrinsic assumption in the formulations of Glaister and Liou et al. is that (153) can be reliably satisfied by both evaluating the terms between brackets at the ideal-gas average state, and introducing the average mass specific internal energy

$$\widehat{\varepsilon} = \frac{\varepsilon^L\sqrt{\rho^L} + \varepsilon^R\sqrt{\rho^R}}{\sqrt{\rho^L} + \sqrt{\rho^R}} \tag{154}$$

With this assumption, the components of the deviation velocity $\widehat{u}_d$ vanish, while (153) becomes

$$\begin{aligned}
\Delta p &= \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\widetilde{\widetilde{\frac{\partial p}{\partial \varepsilon}}}\right)_\rho \cdot \left\{\frac{1}{\widetilde{\rho}}\left[\Delta\left(E - \frac{m^T \cdot m}{2\rho}\right) - \widehat{\varepsilon}\Delta\rho\right]\right\} \\
&= \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\widetilde{\widetilde{\frac{\partial p}{\partial \varepsilon}}}\right)_\rho \cdot \left\{\frac{1}{\widetilde{\rho}}\left[\Delta(\rho\varepsilon) - \widehat{\varepsilon}\Delta\rho\right]\right\} \\
&= \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon \Delta\rho + \left(\widetilde{\widetilde{\frac{\partial p}{\partial \varepsilon}}}\right)_\rho \Delta\varepsilon \tag{155}
\end{aligned}$$

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which coincides with (136) and the expression used in [4,8]. The average partial derivatives in (155) are then determined by projection, as highlighted in Section 9.2. Note that even if these partial derivatives were exactly determined, using the procedure of Section 8.2 for example, there would be no a-priori assurance that the evaluation state $(\tilde{\rho}, \tilde{\varepsilon})$ corresponds to an internal conservation-variable state. The expression for the speed of sound in terms of ρ and ε is then obtained by inserting (150)-(152) into (111) as

$$\hat{c}^2 = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon + \left(\widetilde{\frac{\partial p}{\partial \varepsilon}}\right)_\rho \frac{\Delta \varepsilon}{\Delta \varepsilon(\tilde{q}, \Delta q)} \cdot \left\{ \frac{1}{\tilde{\rho}} \left[\widehat{H} - \tilde{\varepsilon} - \frac{\widetilde{m}^T}{\tilde{\rho}} \cdot \left(\frac{\widehat{m}}{\widehat{\rho}} - \frac{\widetilde{m}}{2\tilde{\rho}} \right) \right] \right\} \tag{156}$$

With the stated assumptions, (156) becomes

$$\hat{c}^2 = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon + \left(\widetilde{\frac{\partial p}{\partial \varepsilon}}\right)_\rho \cdot \left[\frac{\widehat{H} - \tilde{\varepsilon}}{\tilde{\rho}} - \frac{\widehat{m}^T \cdot \widehat{m}}{2\tilde{\rho}^3} \right] \tag{157}$$

which coincides with the corresponding expression in [4,8]. Considering that (154) is not evaluated at an internal conservation-variable state, but results from a direct average, expressions (155)-(157) do no revert to the standard ideal-gas expressions. Finally, the internal-state average jacobian (44) is also modified by restating the partial derivatives of pressure therein, with expressions (150)-(152) with coefficients evaluated at the ideal gas average. This operation then yields the matrix utilized in [4,8]. Therefore, it is not a-priori guaranteed that this matrix is evaluated at internal conservation-variable states, with unpredictable consequences on the eigenvector matrix, hence on the accuracy of the associated flux-difference-splitting results.

Similar conclusions carry over to the developments of Vinokur et al. [9]. Their reference equation of state is

$$p = p(\rho, \rho\varepsilon) \tag{158}$$

Consequently, by using (120) the partial derivatives of pressure in (129) are cast as

$$\left(\frac{\partial p}{\partial \rho}\right)_{m,E} = \left(\frac{\partial p}{\partial \rho}\right)_{\rho\varepsilon} + \left(\frac{\partial p}{\partial \rho\varepsilon}\right)_\rho \cdot \left(\frac{m^T \cdot m}{2\rho^2}\right) \tag{159}$$

$$\left(\frac{\partial p}{\partial m}\right)_{\rho,E} = \left(\frac{\partial p}{\partial \rho\varepsilon}\right)_\rho \cdot \left(-\frac{m^T}{\rho}\right) \tag{160}$$

$$\left(\frac{\partial p}{\partial E}\right)_{\rho,m} = \left(\frac{\partial p}{\partial \rho\varepsilon}\right)_\rho \tag{161}$$

The insertion of (159)-(161) into (129) yields

$$\Delta p = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\rho\varepsilon} \Delta\rho + \left(\widetilde{\frac{\partial p}{\partial \rho\varepsilon}}\right)_\rho \cdot \left[\frac{\widetilde{m}^T \cdot \widetilde{m}}{2\tilde{\rho}^2} \Delta\rho - \frac{\widetilde{m}^T}{\tilde{\rho}} \cdot \Delta m + \Delta E \right] \tag{162}$$

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Whereupon, the intrinsic assumption in the formulation of Vinokur et al. is that (162) can be reliably satisfied by evaluating the terms between brackets at the ideal-gas average state. With this assumption, the components of the deviation velocity $\hat{u}_d$ vanish, while (162) becomes

$$
\begin{aligned}
\Delta p &= \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\rho\varepsilon} \Delta\rho + \left(\widetilde{\widetilde{\frac{\partial p}{\partial \rho\varepsilon}}}\right)_{\rho} \cdot \left[\Delta E - \Delta\left(\frac{\boldsymbol{m}^T \cdot \boldsymbol{m}}{2\rho}\right)\right] \\
&= \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\rho\varepsilon} \Delta\rho + \left(\widetilde{\widetilde{\frac{\partial p}{\partial \rho\varepsilon}}}\right)_{\rho} \Delta(\rho\varepsilon)
\end{aligned}
\tag{163}
$$

which coincides with the corresponding expression used in [9]. The average partial derivatives in (163) are then determined by projection, as highlighted in Section 9.2. The same conclusions stated after (155) also apply to (163). The expression for the speed of sound in terms of ρ and $\rho\varepsilon$ is then obtained by inserting (159)-(161) into (111) as

$$
\hat{c}^2 = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\rho\varepsilon} + \left(\widetilde{\frac{\partial p}{\partial \rho\varepsilon}}\right)_{\rho} \cdot \left[\widehat{H} - \frac{\widetilde{\boldsymbol{m}}^T}{\widetilde{\rho}} \cdot \left(\frac{\widehat{\boldsymbol{m}}}{\widehat{\rho}} - \frac{\widetilde{\boldsymbol{m}}}{2\widetilde{\rho}}\right)\right]
\tag{164}
$$

With the stated assumption, (164) becomes

$$
\hat{c}^2 = \left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_{\rho\varepsilon} + \left(\widetilde{\widetilde{\frac{\partial p}{\partial \rho\varepsilon}}}\right)_{\rho} \cdot \left[\widehat{H} - \frac{\widehat{\boldsymbol{m}}^T \cdot \widehat{\boldsymbol{m}}}{2\widehat{\rho}^2}\right]
\tag{165}
$$

which coincides with the expression in [9]. Expressions (163)-(165) then revert to the standard ideal-gas expressions, since no additional average mass specific internal energy is utilized. Finally, the internal state average jacobian (44) is also modified by restating the partial derivatives of pressure therein, with expressions (159)-(161) with coefficients evaluated at the ideal gas average. This operation then yields the matrix utilized in [9], which again is not guaranteed to depend consistently on internal conservation-variable states.

9.2 Algebraic Partial Derivatives

The average partial derivatives in (155), (157), (163), and (165) are algebraically determined without specific reference to their intrinsic evaluation state. This algebraic determination accrues from a projection of a set of trial partial derivatives onto the admissible-state line. These trial derivatives are obtained through an approximate evaluation of the line integrals in the exact pressure variation expression

$$
\Delta p = \int_\Gamma dp = \int_\Gamma \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon d\rho + \int_\Gamma \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho d\varepsilon
$$

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$$= \overline{\left(\frac{\partial p}{\partial \rho}\right)_\varepsilon} \Delta\rho + \overline{\overline{\left(\frac{\partial p}{\partial \varepsilon}\right)_\rho}} \Delta\varepsilon \qquad (166)$$

Therefore, the trial partial derivatives are defined as

$$\overline{\left(\frac{\partial p}{\partial \rho}\right)_\varepsilon} \simeq \frac{1}{\Delta\rho} \int_\Gamma \left(\frac{\partial p}{\partial \rho}\right)_\varepsilon d\rho \quad , \quad \overline{\left(\frac{\partial p}{\partial \varepsilon}\right)_\rho} \simeq \frac{1}{\Delta\varepsilon} \int_\Gamma \left(\frac{\partial p}{\partial \varepsilon}\right)_\rho d\varepsilon \qquad (167)$$

while (166) is one form of the admissible-state line equation. Note that the developments of Vinokur et al. are obtained from (166)-(167) by replacing ε with $\rho\varepsilon$. At this point, a multitude of integration paths can be selected for the exact integration of dp since this is an exact differential and consequently Δp is independent of the integration path. Nevertheless, the individual partial derivatives in (166) do depend upon the integration path, even though Δp does not. Hence, different sets of partial derivatives (167) and associated speeds of sound are in general obtained with different paths, which in turn will depend upon selected auxiliary states. The specific auxiliary states employed in [4,8,9] are

$$A \equiv \left(\rho^R, \varepsilon^L\right) \quad , \quad B \equiv \left(\rho^L, \varepsilon^R\right) \quad , \quad C \equiv (\hat{\rho}, \hat{\varepsilon}) \qquad (168)$$

while the selected integration paths are

Table 3: Integration Paths

Author		Γ
Glaister	[4]	$LABR, LBAR$
Liou et al.	[8]	LCR
Vinokur et al.	[9]	LR

With these paths, the chosen forms of (167) are

Table 4a-b: Trial Partial Derivatives

Author	$\overline{\left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon}$
Glaister	$\dfrac{p^A - p^L}{\Delta\rho}$
Liou et al.	$\left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon (\hat{\rho}, \hat{\varepsilon})$
Vinokur et al.	Approximate integration of (167)

Author	$\overline{\left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho}$
Glaister	$\dfrac{p^B - p^L}{\Delta\varepsilon}$
Liou et al.	$\left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho (\hat{\rho}, \hat{\varepsilon})$
Vinokur et al.	Approximate integration of (167)

The projections of these approximations onto the admissible-state straight line, corresponding to (166), are then the coordinates of the point shared by this line and the approximation line. The latter line contains the trial derivative approximation point, with coordinates based on (167), and intersects the admissible-state line at a suitable angle, as determined by selected scale factors. Representative closed form solutions are

Table 5a-b: Average Partial Derivatives

Author	$\left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon$
Glaister [4]	$\dfrac{1}{2}\left[\dfrac{\Delta p}{\Delta \rho} + \dfrac{p^A - p^B}{\Delta \rho}\right]$
Liou et al. [8]	$\dfrac{1}{2}\left[\dfrac{\Delta p}{\Delta \rho} - \left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho \dfrac{\Delta \varepsilon}{\Delta \rho} + \left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon\right]$

Author	$\left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho$
Glaister	$\dfrac{1}{2}\left[\dfrac{\Delta p}{\Delta \varepsilon} + \dfrac{p^B - p^A}{\Delta \varepsilon}\right]$
Liou et al.	$\dfrac{1}{2}\left[\dfrac{\Delta p}{\Delta \varepsilon} - \left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon \dfrac{\Delta \rho}{\Delta \varepsilon} + \left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho\right]$

where Glaister's trial derivatives from Table 4 are used to obtain his solution from Liou's determination in this table.

Analytically, these average partial derivatives correspond to the solution (x, y) of the linear system

$$\begin{cases} \kappa x + \beta y = \gamma \\ \overline{\beta} x - \kappa y = \overline{\beta}\overline{x} - \kappa\overline{y} \end{cases} \qquad (169)$$

where the first equation is associated with (166), while the second represents the approximation line. The solution of (169) is

$$x = \frac{\kappa\gamma + \beta\left(\overline{\beta}\overline{x} - \kappa\overline{y}\right)}{\kappa^2 + \overline{\beta}\beta}$$

$$y = \frac{\overline{\beta}\gamma - \kappa\left(\overline{\beta}\overline{x} - \kappa\overline{y}\right)}{\kappa^2 + \overline{\beta}\beta} \qquad (170)$$

By author, the variables in (169)-(170) are defined as

Table 6a-c: Variables in (169)-(170).

Author		x	y
Glaister	[4]	$\dfrac{1}{\mathcal{R}}\left(\dfrac{\widetilde{\widetilde{\partial p}}}{\partial \varepsilon}\right)_\rho$	$\dfrac{1}{\mathcal{E}}\left(\dfrac{\widetilde{\widetilde{\partial p}}}{\partial \rho}\right)_\varepsilon$
Liou et al.	[8]	$\dfrac{1}{\mathcal{R}}\left(\dfrac{\widetilde{\partial p}}{\partial \varepsilon}\right)_\rho$	$\dfrac{1}{\mathcal{E}}\left(\dfrac{\widetilde{\partial p}}{\partial \rho}\right)_\varepsilon$
Vinokur et al.	[9]	$\left(\dfrac{\widetilde{\widetilde{\partial p}}}{\partial \rho \varepsilon}\right)_\rho^{-1}$	$\left(\dfrac{\widetilde{\partial p}}{\partial \rho}\right)_{\rho\varepsilon}\left(\dfrac{\widetilde{\widetilde{\partial p}}}{\partial \rho \varepsilon}\right)_\rho^{-1}$

Author	$\overline{x}$	$\overline{y}$
Glaister	$\dfrac{1}{\mathcal{R}}\left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho$	$\dfrac{1}{\mathcal{E}}\left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon$
Liou et al.	$\dfrac{1}{\mathcal{R}}\left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho$	$\dfrac{1}{\mathcal{E}}\left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon$
Vinokur et al.	$\left(\dfrac{\partial p}{\partial \rho \varepsilon}\right)_\rho^{-1}$	$\left(\dfrac{\partial p}{\partial \rho}\right)_{\rho\varepsilon}\cdot\left(\dfrac{\partial p}{\partial \rho \varepsilon}\right)_\rho^{-1}$

Author	κ	β	$\overline{\beta}$	γ
Glaister	$\mathcal{R}\dfrac{\Delta \varepsilon}{\Delta p}$	$\mathcal{E}\dfrac{\Delta \rho}{\Delta p}$	$\mathcal{E}\dfrac{\Delta \rho}{\Delta p}$	1
Liou et al.	$\mathcal{R}\dfrac{\Delta \varepsilon}{\Delta p}$	$\mathcal{E}\dfrac{\Delta \rho}{\Delta p}$	$\mathcal{E}\dfrac{\Delta \rho}{\Delta p}$	1
Vinokur et al.	Δp	$-\Delta \rho$	$-\mathcal{S}^2\Delta \rho$	$\Delta(\rho \varepsilon)$

Note that since Glaister and Liou et al. select $\beta = \overline{\beta}$, the admissible-state line is orthogonal to the approximation line in their implementation, whereas this feature is not present in the formulation of Vinokur and Montagne. The parameters κ, β, and $\overline{\beta}$ depend upon $\mathcal{R}$, $\mathcal{E}$, and $\mathcal{S}$ which are free are non-dimensionalizing scale factors. Therefore, even after specifying an integration path, a formulational non-uniqueness still persists upon variation of these scale factors.

With the specific selection

$$\mathcal{R} = \frac{\Delta p}{\Delta \varepsilon} \quad , \quad \mathcal{E} = \frac{\Delta p}{\Delta \rho} \tag{171}$$

and using the trial derivatives in Table 4, expressions (170) yield the solutions of Glaister and Liou et al. listed in Table 5. All of these average partial derivatives algebraically satisfy the pressure relation (166), even though they may become indeterminate for vanishing $\Delta \rho$ and $\Delta \varepsilon$, and generally do not coincide with one another.

10 Representative Results

As indicated in Section 2, the solution of (14) and (139) is in general non-unique. This feature is now demonstrated for a shock-tube test case [4] for which intrinsically exact average states are determined without separating the kinematic/energy terms from the pressure relations. With these solutions, the associated exact thermodynamic pressure derivatives and average sound speed squared are determined as indicated in Section 8.2. These results are then compared with the predictions of Liou's and Glaister's methods for an appropriate quantification of their inherent approximations, as induced by the algebraic synthesis of the average partial derivatives of pressure.

The test left and right states are

Table 7: Initial Shock Tube States

	Left	Right
p	100 atm	1 atm
T	9014 K	300 K
u	0 m/sec	0 m/sec

Using the left-state pressure and temperature as reference values, the associated non-dimensional states are

Table 8: Non-Dimensional States

	Left	Right	A	B
ρ	1.000	0.447	0.447	1.000
m	0.000	0.000	0.000	0.000
E	5.491	2.498E-02	2.454	5.580E-02
ϵ	5.491	5.580E-02	5.491	5.580E-02
p	1.000	1.000E-02	0.403	2.231E-02
T	1.000	3.327E-02	0.948	3.326E-02
$\left(\dfrac{\partial p}{\partial \rho}\right)_\epsilon$	1.053	2.231E-02	1.008	2.2311E-02
$\left(\dfrac{\partial p}{\partial \epsilon}\right)_\rho$	0.140	0.178	5.880E-02	0.399

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Figure 1 shows the variation of (139) for $0.0 \leq \alpha \leq 1.0$, for both ideal and real neutral air, corresponding to the left and right states in Table 8.
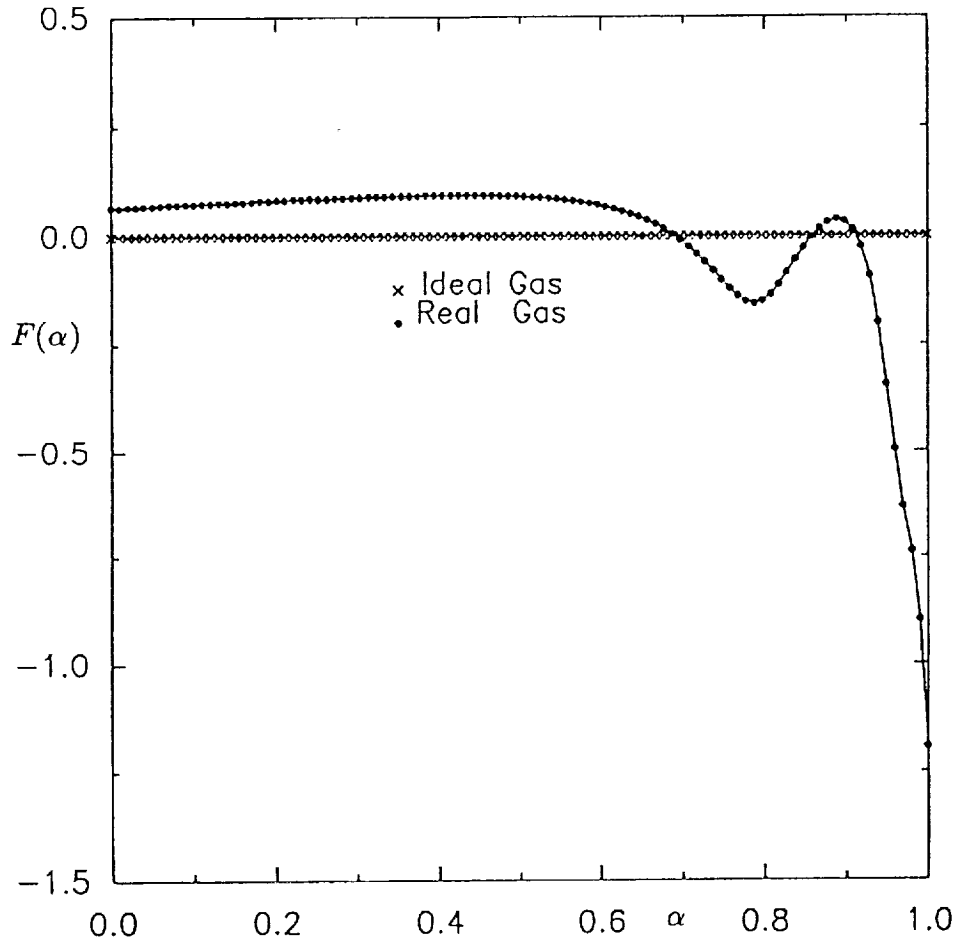

Figure 1: Variation of Function (139)

Evidently, the ideal-gas version of (139) is identically satisfied for any value of α, following (146)-(147), since $m = 0$. Conversely, for real air, not only does a solution exist, but as many as three are possible in this case. These solutions, as generated by (140) with $\Delta\alpha = 1.0E - 04$, are listed in the following table, along with the ideal-gas average state

Table 9: Ideal-Gas Average State and Exact Solutions.

	Average	Sol. I	Sol. II	Sol. III
α	0.599	0.689	0.860	0.908
ρ	0.669	0.618	0.524	0.498
m	0.000	0.000	0.000	0.000
E	2.216	1.719	0.789	0.527
ε	3.312	2.782	1.505	1.058
$\Delta\varepsilon(\widetilde{q}, \Delta q)$	-5.435	-6.356	-8.842	-9.800
$\Delta\varepsilon$	-5.435	-5.435	-5.435	-5.435
$\dfrac{\Delta\varepsilon(\widetilde{q}, \Delta q)}{\Delta\varepsilon}$	1.000	1.169	1.627	1.803
$\left(\dfrac{\partial p}{\partial \rho}\right)_\varepsilon$	0.709	0.618	0.369	0.290
$\left(\dfrac{\partial p}{\partial \varepsilon}\right)_\rho$	9.712E-02	0.102	8.895E-02	8.310E-02
$\widetilde{\left(\dfrac{\partial p}{\partial \varepsilon}\right)}_\rho$	9.712E-02	0.119	0.145	0.150
$\widehat{c}^2$	0.797	0.806	0.778	0.767

According to these results, the variation of mass specific internal energy $\Delta\varepsilon(\widetilde{q}, \Delta q)$ significantly depends upon the intrinsic state $\widetilde{q}$. Conversely, the dependence of the sound speed squared on $\widetilde{q}$ is less pronounced, while the eigenvalues corresponding to these three solutions all display the same algebraic sign structure, in agreement with the eigenvalue consistency theorem.

For this test case, the trial and projected average pressure derivatives for Glaister's and Liou's methods are

Table 10: Trial and Average Partial Derivatives

Author	$\overline{\left(\dfrac{\partial p}{\partial \rho}\right)}_\varepsilon$	$\widetilde{\left(\dfrac{\partial p}{\partial \rho}\right)}_\varepsilon$	$\overline{\left(\dfrac{\partial p}{\partial \varepsilon}\right)}_\rho$	$\widetilde{\left(\dfrac{\partial p}{\partial \varepsilon}\right)}_\rho$
Glaister [4]	1.079	0.551	0.180	0.126
Liou et al. [8]	0.709	0.772	9.712E-02	0.103

Furthermore, the admissible-state straight-line equation associated with (166) and the first equation in (169) is

$$\left(\widetilde{\frac{\partial p}{\partial \rho}}\right)_\varepsilon = -0.102\left(\widetilde{\widetilde{\frac{\partial p}{\partial \varepsilon}}}\right)_\rho + 0.182 \tag{172}$$

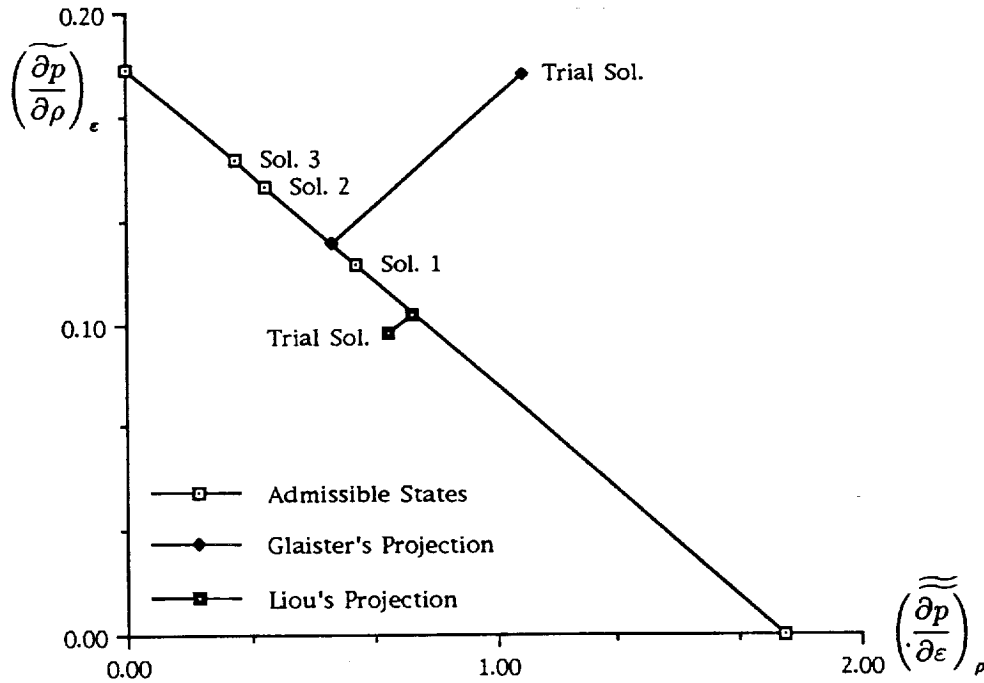Figure 2 graphs the admissible- state straight line along with Glaister's and Liou's solution projections.

Figure 2: Admissible-State Straight Line

The trial solution of Glaister's method is farther than Liou's from the admissible state line, even though its associated projected solution is closer to all three exact solutions than Liou's. Clearly, a different choice of the scale factors $\mathcal{R}$ and $\mathcal{E}$ may yield closer terminal projections, even though such factors would remain solution dependent, and there seems to exist no practical way to determine them accurately.

The solution closest to the ideal-gas average state is then isolated, since with this selection the exact average eigenvalues, Mach number, and eigenvectors (105), (106), (116), and (118) converge to the pointwise forms. Table 11 presents the exact solution I and the ideal-gas averages, as compared with the predictions of selected expressions in Tables 2 and 5.

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Table 11: Exact and Approximate Solutions

	Exact	Average	Glaister	Liou et. al
$\left(\dfrac{\widetilde{\partial p}}{\partial \rho}\right)_\epsilon$	0.618	0.709	0.551	0.772
$\epsilon\%$	0.00 %	-14.56%	-10.84 %	+24.91 %
$\left(\dfrac{\widetilde{\partial p}}{\partial \epsilon}\right)_\rho$	0.119	9.712E-02	0.126	0.103
ϵ %	0.00 %	18.38%	-5.88 %	+13.44 %
$\widetilde{c^2}$	0.806	0.797	0.666	0.867
ϵ %	0.00 %	1.11%	-17.36 %	+7.56 %

For the thermodynamic partial derivatives of pressure, the percent error $\epsilon\%$ reaches 24.91%, while that incurred in the speed of sound squared is as high as 17.36 %. Furthermore, in this case, these errors are not significantly different than those generated by simply evaluating all thermodynamic partial derivatives at the ideal-gas average state, as indicated in Table 11. In general, these errors will propagate to the eigenvector matrix with unpredictable consequences on the accurate flux-difference splitting computations of subtle flow features such as shock waves and contact discontinuities.

Concluding Remarks

The theoretical developments in this report yield unambiguous internal-average conservation variables for curvilinear- coordinate multi- dimensional flux difference resolution, including the speed of sound and the partial derivatives of pressure. These derivatives are determined for arbitrary equations of state without any extra assumption. Hence, no average internal energy is introduced, and no approximations and projections are employed.

The jacobian matrices (44) and (71) attain a tidy structure, and for equilibrium real-gas equations of state, their eigenvalues and associated Mach number are shown to differ from the classical forms. Only after introducing subtle assumptions, documented in Section 9, can the average eigenvalues formally coincide in these conditions with the classical forms. Of course, the unavoidable theoretical price to pay is the loss of an a-priori assurance that the jacobian partial derivatives of pressure and speed of sound either exist in the domain of the analytical partial derivatives, or correspond to an evaluation at internal-average conservation-variable states, with unpredictable consequences on the eigenvector matrix, and hence on highly accurate flux-difference-splitting computations of subtle flow features like shock waves and contact discontinuities. Conversely, the formulae presented in this research afford this a-priori assurance. Furthermore, their very existence, uniqueness, and numerical demonstration lead to the fundamental conclusion that the generalization of

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the average-state notion to an equilibrium real-gas equation of state introduces no additional degree of freedom in the determination of the average partial derivatives of pressure, unlike what is stated in [9].

Certainly, the reported average jacobians will converge to their respective classical pointwise forms when the given computational-cell left and right states approach each other, which for smooth flows occurs under appropriate mesh refinements. Therefore, although these average jacobians may correspond to external average conservation variables, with thermodynamic partial derivatives and speed of sound possibly affected by non negligible errors at shock locations, their utilization in significantly refined grid computations, e.g. [8], has led to satisfactory results. One however wonders whether these procedures depend upon uniform refined grids for their reliable operation.

The developed average jacobians and associated eigenvalues and eigenvectors, consistently depend on internal-average conservation variables and without extra assumptions they exactly satisfy the flux difference resolution even for arbitrary separation of the computational-cell left and right states. Furthermore, they are valid for arbitrary multi-dimensional curvilinear-coordinate meshes.

Acknowledgements

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References

[1] V. Hutson and J. S. Pym, *"Applications of Functional Analysis and Operator Theory"*, Academic Press (1980).

[2] R. A. Horn and C. R. Johnson, *"Matrix Analysis"*, Cambridge University Press (1985).

[3] P. L. Roe, "Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes", Journal of Computational Physics, 43, 357-372 (1981).

[4] P. Glaister, "An Approximate Linearised Riemann Solver for the Euler Equations for Real Gases", Journal of Computational Physics, 74, 382-408 (1988).

[5] M. S. Liou and B. van Leer, "Choice of Implicit and Explicit Operators for the Upwind Differencing Method", Tech. Paper AIAA 88-0624, 26th Aerospace Sciences Meeting, Reno NV (1988).

[6] B. Simpson, "Unsteady Three-Dimensional Thin Layer Navier- Stokes Solutions on Dynamic Blocked Grids", Tech. Report AFATL-TR-89-19, ABAB, AF Armament LAB, Eglin AFB, FL (1989).

[7] M. Vinokur, "An analysis of Finite Difference and Finite-Volume Formulations of Conservation Laws", Journal of Computational Physics, 81, 1-52 (1989).

[8] M. S. Liou, B. van Leer, and J. S. Shuen, "Splitting of Inviscid Fluxes for Real Gases", Journal of Computational Physics, 87, 1-24 (1990).

[9] M. Vinokur, and J. L. Montagne', " Generalized Flux-Vector Splitting and Roe Average for an Equilibrium Real Gas", Journal of Computational Physics, 89, 276-300 (1990).

[10] P. Cinnella and C. F. Cox, "An Efficient 'Black Box' Solver for The Equilibrium Composition of Real Gases", Tech. Paper AIAA 91-3322-cp (1991).

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Modern split component evaluations of the flux vector jacobians are thoroughly analyzed for equilibrium-gas average-state determinations. It is shown that all such derivations satisfy a fundamental eigenvalue consistency theorem. A conservative-variable average state is then developed for arbitrary equilibrium-gas equations of state and curvilinear-coordinate fluxes. Original expressions for eigenvalues, sound speed, Mach number, and eigenvectors are then determined for a general average jacobian, and it is shown that the average eigenvalues, Mach number, and eigenvectors may not coincide with their classical pointwise counterparts. A general equilibrium-gas equation of state is then discussed for conservative-variable CFD Euler formulations. The associated derivations lead to unique compatibility relations that constrain the pressure jacobian derivatives. Thereafter, alternative forms for the pressure variation and average sound speed are developed in terms of two average pressure jacobian derivatives. Significantly, no additional degree of freedom exists in the determination of these two average partial derivatives of pressure. Therefore, they are simultaneously computed exactly without any auxiliary relation, hence without any geometric solution projection or arbitrary scale factors. Several alternative formulations are then compared and key differences highlighted with emphasis on the determination of the pressure variation and average sound speed. The relevant underlying assumptions are identified, including some subtle approximations that are inherently employed in published average-state procedures. Finally, a representative test case is discussed for which an intrinsically exact average state is determined. This exact state is then compared with the predictions of recent methods, and their inherent approximations are appropriately quantified.

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