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Structural Optimization of Large Structural Systems By Optimality Criteria Methods

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STRUCTURAL OPTIMIZATION OF LARGE STRUCTURAL SYSTEMS BY OPTIMALITY CRITERIA METHODS

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SUMMARY

The fundamental concepts of the optimality criteria method of structural optimization are presented. The effect of the separability properties of the objective and constraint functions on the optimality criteria expressions is emphasized. The single constraint case is treated first followed by the multiple constraint case with a more complex evaluation of the Lagrange multipliers. Examples illustrate the efficiency of the method.

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INTRODUCTION

The primary objective of this introductory discussion is to place the Optimality Criteria concepts in a proper framework relative to nonlinear mathematical programming and to dispel the often-stated erroneous characterization that these concepts lead to approximate optima. There are two major classes of approaches that have been vigorously developed for application in structural optimization. These are somewhat misleadingly referred to as Nonlinear Mathematical Programming (MP) and Optimality Criteria (OC) methods. The former seeks to improve the design at each iteration based on local information about the design space, thus directly optimizing the objective function. The latter seeks to satisfy the stated or derived optimality criteria, thus indirectly optimizing the objective function. Direct and indirect methods of nonlinear mathematical programming would be the correct and more informative designations. However, in this paper the abbreviations MP and OC are used, even though they are more traditional than correct designations.

The intent of the sections that follow is to introduce the fundamental concepts of what became known as Discretized Optimality Criteria (DOC) approaches to structural optimization. These methods consist of an intuitively stated or exact, mathematically derived optimality condition or criterion and an algorithm that satisfies it. The advantage of DOC methods is that they provide superior computational economy even when they are used with detailed mathematical models with thousands of finite elements. Such a capability makes the investigation of alternate structural concepts and layouts computationally affordable.

It is worth mentioning that, although in principle the derived optimality criteria methods have general validity, the most successful implementations have been those that take advantage of the favorable mathematical properties of certain classes of constraints and of favorable structural behavior characteristics. DOC methods have been successfully applied to practical problems with thousands of independent size variables. MP methods have general applicability, but the price of their generality is their limitation to a few hundred design variables because of the rapidly increasing computational burden as the number of design variables increases. Hybrid capabilities that utilize the advantages of both approaches have yet to be developed.

Perhaps the simplest way to point out the difference between MP and DOC approaches to optimization is to consider the two ways that structural analysis problems can be solved by the Principle of the

Minimum of the Total Potential. In this case a displacement state is sought for a structure that minimizes the total potential relative to all admissible displacement states in order to satisfy the equations of equilibrium.

The first and customary approach is to develop the optimality conditions, in this case the equations of equilibrium in terms of the displacement variables. The equations are obtained by variational methods if the structure is considered a continuum or by differentiation relative to discretized displacement variables in the case of discrete or discretized structures, such as finite element models. The total potential is then indirectly minimized by the solution of the resulting differential or algebraic equations, respectively. This approach is similar to DOC.

An alternate method is, at least in the discretized case, to apply any of the many available MP methods to directly minimize the total potential relative to the discrete deflection variables. As a minimum is reached by this rarely used direct minimization approach, the same vector of displacement variables is obtained as is by the more customary indirect (i.e., DOC) method. In this sense finite element methods, for example, can be viewed as DOC methods, even if the principal interest is in viewing the results as the solution to an analysis rather than as the solution to an equivalent optimization problem.

In the case of structural optimization, there are two classes of DOC statements, intuitive and derivable. The venerable and powerful Fully Stressed Design (FSD) approach is usually viewed as an intuitive statement, but under certain conditions that are met by most practical structures, it actually provides the exact optimum. Its success served as both the motivation and model to also develop DOC methods for frequently considered stiffness constraints. Because of the mathematical structure of the stiffness constraints, it was possible to derive exact DOC statements. Currently, the original DOC method—that is, the FSD for the stress constraint case—is considered approximate and the stiffness DOC methods exact. This mixed view has led to the often stated, mistaken characterization of DOC methods as approximate.

The early advancements in derivable optimality criteria methods by Prager and coworkers were directed toward special continuum problems (refs. 1 to 4). The results showed the elegance and the combined power of variational calculus and structural energy theorems in structural optimization problems. The resulting optimality conditions were differential equations, and their solutions defined the optimum structure.

The original development (refs. 5 and 6) of derivable DOC methods for structural optimization problems was consequently approached from a combination of three different directions: (1) engineering considerations of structural behavior, (2) motivation for similarity with the FSD approach in simplicity and computational economy, and (3) utilization of Lagrange multiplier methods of constrained optimization to discretize the continuum approach of the Prager school. In the discussion that follows, these motivations are considered with an engineering emphasis much along the lines of the historical development of DOC methods.

FULLY STRESSED DESIGN AS MOTIVATION FOR DISCRETIZED OPTIMALITY CRITERIA METHODS

The intuition of a structural engineer is to strengthen a component if analysis shows it to be too weak and weaken it if it is shown to be too strong. If every member is finally at its proper strength for all load conditions after repeated resizing and analysis, the structure is accepted as optimum for the

purpose. As soon as computers became available, this fundamental attitude was formalized for computer implementation as the well-known FSD. As a brief definition we can say that in an FSD every structural member is either at its maximum allowable stress or is at a passive (minimum or maximum) size. In the case of statically determinate structures, only one FSD sizing iteration is needed, because the internal forces S_i do not change on the right side of a sizing formula, indicated in a general sense by

$$x_i \geq f_i(S_i, \sigma_{io}) \tag{1}$$

Here x_i is the needed cross-sectional property, such as area or section modulus, and σ_{io} is the stress allowable in the ith member. As is well known, if the structure is statically indeterminate, then the (generalized) internal member forces S_i are functions of the chosen cross-sectional properties, and equation (1) becomes nonlinear. The success of FSD in a given case depends on the sensitivity of S_i to changes in the cross-sectional properties and, of course, on whether the optimum is an FSD or not.

In most practical structures, such as a wing box (which is essentially a cantilever beam), this sensitivity is very low regardless of the theoretical number of indeterminacy of the finite element model. Equilibrium governs rather than compatibility. In such cases a few resizing iterations converge to the final design even when there are thousands of independent size variables. If the sensitivity is high, as for the popular 10-bar truss, more iterations are needed than for a large wing box model with its thousands of size variables. The important point is that in case of sizing formulas similar to equation (1) the needed number of iterations depends on structural behavior (a concept well understood by structural engineers), and not on the number of size variables. Before we discuss the development of exact DOC statements, a few remarks have to be made relative to FSD because of its importance in practice and because of its role as motivation for derivable DOC methods.

STRESS CONSTRAINTS AND THE FULLY STRESSED DESIGN APPROACH

As is now well known, FSD is not a valid general statement of a necessary or sufficient condition for optimality, because the optimum point in certain cases also can be a point where the constraint and objective functions are tangential. For that to happen, the number of active constraints has to be less than the number of size variables, in which case the design is not an FSD by definition.

When the FSD condition is satisfied exactly, the set of n active stress constraint equations completely defines the n member sizes. The objective function in that case plays no role at all. In terms of the design space, it is a vortex point of the n stress constraint surfaces. This point, as a matter of fact, usually is the theoretical optimum point for stress constraints for most practical large structures.

In some cases an FSD still can provide an acceptable practical approximation if the two kinds of points are sufficiently close together. Of the two kinds of points, FSD can provide an acceptable solution with excellent computational economy even for models with thousands of size variables. On the other hand, the exact optimum in such cases requires the use of direct search procedures of nonlinear mathematical programming methods that limit the number of size variables to a few hundred at best. To refer to FSD strictly as an intuitive and approximate method is a somewhat fuzzy proposition.

It is widely known that for most practical structures of closely uniform material properties FSD is the optimum. In addition, before judging FSD too harshly, one should remember that practically all major structures—from bridges and high-rise buildings to ships and space shuttle wings—have been designed essentially by manual or automated FSD procedures.

Under uniform material properties, FSD is equivalent to the exact optimum that can be derived for the special constraint of limiting the external work of the actual load system, also referred to as the compliance constraint. The theoretical optimum structure for such a constraint is uniformly strained and can be scaled to satisfy uniform stress constraints, resulting in an FSD design. Such a design is also equivalent to the uniform strain energy density statements. These concepts simplify the stress constraint case by converting the n stress constraints to a single external work constraint. One also can convert the n stress constraints to relative displacement constraints and obtain exact optimality criteria for stress constraints indirectly. For uniform structural properties, all these statements become equivalent. In case of deviations from uniform properties, care has to be exercised in using these statements.

The customary stress ratio resizing algorithm that is used to satisfy FSD also has a weakness. It has a tendency to eliminate the good members that have higher allowable stresses. FSD might be the optimum design in such cases, but the stress ratio algorithm diverges from it. For fundamentally dissimilar materials, such as composite material systems, the uncertainties associated with the case-by-case validity of an FSD or energy-density-based approach suggest caution.

The previous discussion was given mainly to clarify the exact or approximate nature of DOC methods. Because FSD is viewed, in general, simply as an intuitive DOC, often all DOC methods are mistakenly termed approximate, as opposed to MP methods that are termed exact methods. Such a view is obviously quite simplistic.

STRUCTURAL OPTIMIZATION AND SEPARABLE PROGRAMMING

Stress and stiffness constraints constitute two different classes of optimization problems. The most important mathematical difference is that stress constraints, unless expressed as constraints on member deformations, do not lead to so-called separable mathematical programming problems. Stiffness-related constraints, on the other hand, can be easily written in terms of equality of internal and external work quantities; and if stated that way, they will lead to separable programming problems and to exact, yet simple, DOC statements.

The important consequence of such formulation of DOC methods is that the resulting optimality criteria are both simple and exact, and not only for statically determinate problems but also for indeterminate problems. The fundamental concepts of DOC remain valid for nonseparable problems also; the difference is in the evaluation of the necessary quantities, which involves sensitivity calculations that are almost trivial for separable problems. Extensions to nonseparable problems are the subject of current research. In either case the number of reanalyses, similarly to that for FSD, is independent of the number of size variables: it depends on the structural behavior. This very property, the number of iterations being a function of structural behavior rather than of the number of size variables, is the fundamental advantage of DOC methods.

For the programming problem to be separable, both the objective function $W(X)$ and the constraint function $C(X)$, where $X = (x_1, x_2, \ldots, x_n)$, have to satisfy the following two separability conditions:

$$W(X) = \sum_{i=1}^{n} w_i(x_i) \qquad C(X) = \sum_{i=1}^{n} c_i(x_i) \tag{2}$$

and

$$\frac{\partial^2 w_i(x_i)}{\partial x_i \, \partial x_j} = \frac{\partial^2 c_i(x_i)}{\partial x_i \, \partial x_j} = 0 \tag{3}$$

that is,

$$\frac{\partial C(X)}{\partial x_i} = \frac{dc_i(x_i)}{dx_i}$$

The first condition states that the functions $W(X)$ and $C(X)$ can be written as a sum of terms, each dependent explicitly only on a single variable. For all $c_i(x_i)$ containing implicit dependence on all x_j, as for indeterminate structures, the second condition also must hold. These conditions, when satisfied both by the objective and constraint functions, result in a separable programming problem and in simple uncoupled and exact DOC statements that involve only the ith size variable. In the following sections, methods are shown as to how to utilize such DOC expressions to derive recursion relationships, the hallmark of DOC methods.

DISCRETIZED OPTIMALITY CRITERIA FOR A SINGLE STIFFNESS CONSTRAINT

The case of a single constraint is not only convenient to use in discussing the fundamental ideas of DOC methods, but it is also an important class of problems. Many times in practical cases there is a single troublesome response behavior, such as a strength design obtained by FSD or other methods, which is deficient in some stiffness-related behavior. As shown next, simple and powerful DOC approaches can be derived to deal with such constraints.

The standard statement for our constrained optimization problem is to minimize

$$W(X) \tag{4}$$

subject to

$$G(X) = C(X) - C^* = 0 \tag{5}$$

where

$$X = (x_1,\ldots,x_n) \qquad x_i > 0 \tag{6}$$

$W(X)$ and $C(X)$ are required to satisfy the separability conditions. The functions $c_i(x_i)$ represent the internal energy contribution of the ith variable to satisfy the constraint C^*, which is expressed as external work, such as the external virtual work of a unit virtual force along the single displacement to be constrained. This is a very important point because it is this formulation that leads to not just $W(X)$ but also $G(X)$ to satisfy the separability conditions.

The first step is to form the Lagrangian as

$$L(X,\lambda) = \sum_{i=1}^{n} w_i(x_i) - \lambda \left[\sum_{i=1}^{n} c_i(x_i) - C^* \right] \tag{7}$$

Invoking the separability conditions, the optimality criteria become

$$\frac{\partial L(X,\lambda)}{\partial x_i} = \frac{dw_i(x_i)}{dx_i} - \lambda \frac{dc_i(x_i)}{dx_i} = 0 \qquad (i = 1,\ldots,n) \tag{8}$$

or

$$\frac{\dfrac{dc_i(x_i)}{dx_i}}{\dfrac{dw_i(x_i)}{dx_i}} = \frac{1}{\lambda} = \text{constant} = \frac{\text{change in performance}}{\text{change in cost}} \tag{9}$$

This optimality condition states that at optimum the "return on investment" is the same for all variables. As mentioned earlier, this statement is valid also for the general case where $W(X)$ and $G(X)$ do not satisfy the separability conditions. The difference is that potentially costly sensitivity analyses may have to be performed in the general case, whereas the special case leads to equation (9), which involves only the uncoupled ith terms.

At this point it should be mentioned that usually not all variables are free to change, and they get separated into active and passive constraints in order to satisfy minimum or maximum size variables, or because some structural members are to remain a certain size for practical reasons. For these discussions it is assumed that the contribution of the passive variables is migrated into the value of C^*, modifying its original value.

For specific cases of specialization of $w_i(x_i)$ and $c_i(x_i)$, equation (8) or (9) can be solved for x_i, resulting in explicit formulas for the design variables that are similar to equation (1), our model formula.

Consider the specializations

$$w_i(x_i) = w_i x_i \qquad c_i(x_i) = \frac{c_i}{x_i} \tag{10}$$

which, for example, for truss members can be further specialized as

$$w_i(x_i) = L_i \rho_i A_i \qquad w_i = L_i \rho_i \qquad x_i = A_i \tag{11}$$

and

$$c_i(x_i) = \frac{S_i^P \, S_i^V \, L_i}{E_i A_i} \qquad c_i = \frac{S_i^P \, S_i^V \, L_i}{E_i} \tag{12}$$

In these expressions, A_i, L_i, and ρ_i are the cross-sectional area, the length, and the specific weight of the ith bar, respectively. S_i^P is the axial member force due to the actual load system P. If the constrained quantity C^* is the virtual work of a virtual load system V along actual displacements, then $c_i(x_i)$ is the contribution of the ith member to the internal virtual work to equal C^*, the constrained external virtual work. In that case S_i^V is the axial member force due to the virtual load system V. Examples of V are the unit dummy load at and along the constrained deflection, a unit couple to control twist, or P = V when the work of the actual load system P is being limited as the compliance constraint mentioned earlier.

With the frequently valid specializations of equation (10), we can form the Lagrangian as

$$L(X,\lambda) = \sum_{i=1}^{n} w_i x_i + \lambda \left(\sum_{i=1}^{n} \frac{c_i}{x_i} - C^* \right) \tag{13}$$

and obtain the optimality criteria as

$$\frac{\partial L(X,\lambda)}{\partial x_i} = w_i - \lambda \frac{c_i}{x_i^2} = 0 \qquad (i = 1,\ldots,n) \tag{14}$$

that also can be written in the more useful forms,

$$x_i^{new} = \sqrt{\lambda \frac{c_i}{w_i}}^{1/2 \text{ old}} \tag{15}$$

or

$$x_i^{new} = \left(\lambda \frac{c_i}{w_i x_i} \right)^{old} \tag{16}$$

Note that equation (15), in a way similar to FSD, provides the optimum design in a single iteration as a direct formula. Equations (15) and (16) also can be written in a form that will be used in a more general fashion to develop the recursion relations to satisfy these equations iteratively:

By multiplying equation (17) on both sides by x_i^q and taking the qth root of both sides, one obtains the

$$1 = \lambda \frac{c_i}{w_i x_i^2} = D_i \tag{17}$$

fundamental DOC recursion relation:

$$x_i^{new} = \left(x_i D_i^{1/q}\right)^{old} \tag{18}$$

For $q = 2$ and $q = 1$, equation (18) reduces to equations (16) and (17), respectively. The exponent q is a step-size parameter that governs the modifying power of D_i, a quantity that approaches unity as the optimality criteria are becoming satisfied during iterations. The larger the exponent, the larger the modification initially, but it may result in numerical instability. A small exponent may result in a large number of iterations. It is often advantageous to provide an additional flexibility by multiplying $1/q$ at each iteration by a number $(1 + e)$ where e is a small positive or negative number depending on which strategy is used (a larger or smaller initial exponent).

Equation (18) can be linearized by starting out with the following modification:

$$x_i = x_i D_i^{1/q} \rightarrow x_i = x_i (1 + D_i - 1)^{1/q} \tag{19}$$

Considering that close to the optimum $1 >> (D_i - 1)$ leads to the linearized expression

$$x_i^{new} = \left\{ x_i \left[1 + \frac{1}{q} (D_i - 1) \right] \right\}^{old} \tag{20}$$

If equation (18) is written in terms of reciprocal variables and then similarly linearized and finally written again in terms of the original variables, one obtains the expression

$$x_i^{new} = \left\{ \frac{x_i}{\left[1 - \frac{1}{q} (D_i - 1) \right]} \right\}^{old} \tag{21}$$

Equations (18), (20), and (21) are our three basic recursion relations, also referred to as the exponential, the linearized, and the linearized reciprocal recursion relationships.

Lagrange Multiplier for a Single Constraint

When the optimality criteria are satisfied for a preselected choice of the single multiplier, one obtains an optimum relative distribution of the design variables, but one that yields a constraint value $C(X)$ that is most likely different from C^*, the required value. For this case of a single constraint, the

design can be simply scaled to satisfy the constraint equation. Equation (15) also can be substituted into the constraint equation and solved for the multiplier, yielding a formula for it. There have been many approaches proposed to update the multiplier during iterations to satisfy the constraint, but scaling is the simplest in this case. The role of the multipliers becomes more important and more complex in the case of multiple constraints.

In preparation for the multiple constraint case, two obvious ways to update the multiplier are presented here. They utilize the diminishing difference between the calculated value of $C(X)$ and the required value C^* to update the multiplier. The first method adds a modification to the current multiplier:

$$\lambda^{new} = [\lambda(1 + pG)]^{old} \qquad G \to O \quad \text{as} \quad C \to C^* \tag{22}$$

As the constraint is satisfied, G (the value of the constraint function) vanishes, and the modifications stop. It is interesting to note that, if an extended Lagrangian formulation would have been used by adding the square of $G(X)$ as a penalty term, equation (22) would have been obtained.

In the second method, the ratio of C and C^*, the current and the required values of the constraint, is utilized to modify the multiplier:

$$\lambda^{new} = \left[\lambda\left(\frac{C}{C^*}\right)^p\right]^{old} \tag{23}$$

This ratio approaches unity as C approaches C^*. In both cases p is a step-size parameter to control the rate of the modifications, with $p = 2$ being a good starting value and $p = 1$ being equivalent to scaling. Again the modifying power is diminishing as the constraint is nearly satisfied, and the multiplication of p by a factor $(1 + e)$ at each iteration (as discussed earlier) can be advantageous.

Finite Element Formulations

Displacement constraints.—There are three cases of stiffness constraints that have been formulated during the early developments of DOC (ref. 7). These approaches also pioneered the development of what became known as sensitivity analysis. First, the approach is given in some detail for the case of generalized displacement constraints given in terms of external virtual work. Consider the following definitions:

C^* $\quad V_j r_j^P$ (sum on j)

K system stiffness matrix

P actual loads

r^P system displacements due to actual loads P

r^V system displacements due to virtual loads V

V virtual loads

Because of using values from solutions during the evaluations of D_i, the following relations hold:

$$Kr^P = P \qquad Kr^V = V \qquad K_i r_i^P = S_i^P \qquad K_i r_i^V = S_i^V \tag{24}$$

The Lagrangian takes the form

$$L(X,\lambda) = \sum_{i=1}^{n} w_i x_i + \lambda\left(r^P K r^V - C^*\right) \tag{25}$$

and the optimality condition becomes

$$\frac{\partial L(X,\lambda)}{\partial x_i} = w_i + \lambda\left(\frac{\partial r^P}{\partial x_i} K r^V + r^P \frac{\partial K}{\partial x_i} r^V + r^P K \frac{\partial r^V}{\partial x_i}\right) = 0 \tag{26}$$

The three terms in the parenthesis appear complicated, but they actually contain a very simple relationship. Utilizing the solution for the load system P, one can write in transposed form the following relationships:

$$r^P K = P \qquad \frac{\partial r^P}{\partial x_i} K + r^P \frac{\partial K}{\partial x_i} = 0 \qquad \frac{\partial r^P}{\partial x_i} = -r^P \frac{\partial K}{\partial x_i} K^{-1} \tag{27}$$

Similarly for the virtual load system V, one obtains

$$Kr^V = V \qquad \frac{\partial K}{\partial x_i} r^V + K \frac{\partial r^V}{\partial x_i} = 0 \qquad \frac{\partial r^V}{\partial x_i} = -K^{-1} \frac{\partial K}{\partial x_i} r^V \tag{28}$$

Substitution of equations (27) and (28) into equation (26) and performing the simplifications finally yields

$$w_i - \lambda\left(r^P \frac{\partial K}{\partial x_i} r^V\right) = 0 \tag{29}$$

Note that separability was not assumed, but proven, because the first and last terms that contain the implicit derivative terms have vanished. Furthermore, if we have linear dependence of the system stiffness matrix on all x_i, the following relation holds:

$$\frac{\partial K}{\partial x_i} = \frac{1}{x_i} K_i \tag{30}$$

yielding the simple expression for D_i:

$$1 = \lambda \frac{r^P K_i r^V}{w_i x_i} = D_i \tag{31}$$

Buckling and frequency constraints.—The derivations for D_i for the case of buckling constraints and vibration frequency constraints are very similar to the derivation discussed earlier for the displacement constraint. The detailed derivations are given in reference 7. Here only the final expressions are repeated:

$$1 = \lambda \frac{y^T K_i y}{w_i x_i} = D_i \tag{32}$$

for the case of buckling constraints, and

$$1 = \lambda \frac{y^T K_i y - \omega^{*2} y^T m_i y}{w_i x_i} = D_i \tag{33}$$

for the case of frequency constraints, where y is the associated eigenfunction and ω^* is the required frequency. Equations (32) and (33) satisfy the optimality criteria. For the buckling case, they also satisfy the constraint by scaling. For the case of vibration frequency control, the presence of nonstructural mass is usually required to effect a change in frequency, and constraint satisfaction in this case requires special caution. Reference 8 points out possible difficulties and is suggested reading in this respect.

Examples

Example 1: Forward-swept wing.—As stated earlier, the simple constraint case is often important in practice when one particular response is not satisfied, for example, by a strength-based design. Figure 1 shows the configuration of the X-29A forward-swept wing airplane, and figure 2 shows the model used in the design studies with over 4000 finite elements. The FSD design of the composite wing was deficient in divergence velocity, and a DOC method (ref. 9) was used to correct the deficiency. The solutions converged typically in around half a dozen iterations for the few thousand design variables, indicating the efficiency of the optimality criteria approach.

Example 2: Effect of the choice of step-size parameters.—Figure 3 shows a version of the classical three-bar truss used here for illustration mostly because it produces a small data set that can be presented in its entirety (table I). The constraint is a 1-in. extension of bar 1, which is also equivalent to a stress constraint and is always satisfied by scaling as indicated in table I by the constant value for stress.

The merit function was artificially chosen to be $W = 100A_1 + 10A_2$, resulting in the "design space" given in figure 4.

Many experimental runs were made for this small problem to study convergence behavior for various combinations of equations (18), (20), and (21) (sizing options 1, 2, and 3, respectively) and of the two update formulas for the multiplier, equations (22) and (23) (λ options 1 and 2). All possible combinations give a very similar performance of seven to eight iterations to four-figure convergence with default values of $Q = 1/q = 0.5$ and $p = 2$. Convergence in four iterations was also obtained by starting with slightly larger exponents ($Q = 0.7$; $p = 2.2$) and a moderate value of $\alpha = 1.02$. The smoothest "tuned" performance to four-digit accuracy was obtained in only two iterations by starting with relatively large exponents ($Q = 1.0$; $p = 2.5$) and reducing them with $\alpha = 0.8$ as the optimum was being approached. Table I shows the corresponding iterations. The point is that, like with any other method, the parameters are problem dependent, and tuned values can cause dramatic improvement. The comparisons given by many researchers showing the number of iterations for their method versus other methods is usually simplistic. How many iterations does DOC need for this problem, 8 or 4 or 2?

In a single constraint case, the relation $\lambda = W/C^*$ holds, and with $C^* = 1$ this results in $W = \lambda$ for this problem. The optimality criteria are satisfied whenever $D_1 = D_2$, and they are equal to unity when the proper value of the multiplier is reached. Because scaling corrects for discrepancies in the value of λ, the correct optimum is obtained even without the correct value of the multiplier or of D_1 and D_2, provided they are equal.

Example 3: Buckling constraint for a truss column.—A particularly simple case is used to illustrate the use of equation (32) to optimize a structure for a buckling constraint. Figure 5 shows a 50-member truss column and the normalized relative chord areas of the optimized structure. The continuous curve is from reference 10 and represents the continuous change of the face sheet thickness of a sandwich column obtained by variational methods. A good approximation of the chord areas can even be calculated by hand, if one assumes, for example, a sinusoidal eigenfunction. The data presented were obtained by a few iterations involving eigenfunction calculations. It is a good exercise to perform the first iteration by hand. The relative distribution of the chord areas could be scaled to satisfy the prescribed buckling requirement.

DISCRETIZED OPTIMALITY CRITERIA FOR MULTIPLE CONSTRAINTS

Multiple constraints represent only a minor change to the recursion relations, but they introduce a major problem for the identification of the constraint set that is active at the optimum. The multiple constraints are usually required to be satisfied as inequality constraints or, in some cases, as equality constraints to be enforced within physical feasibility. In the latter case the constraints in question are declared active, but in the former case the constraints are allowed to separate into active and inactive constraint sets, with some constraints shifting between the two sets during iterations. A thorough treatment of active set strategies is beyond the scope of this paper. The problem is discussed only within the context of the approaches that are presented for the evaluation of multiple Lagrange multipliers that no longer can be viewed as simple scalers.

The Lagrangian in this case involves a sum of constraints, each weighted with a Lagrange multiplier. The number of constraints is m, and the multipliers and the $c_i(x_i)$ terms acquire the additional index j representing the jth constraint:

$$L(X,\lambda) = \sum_{i=1}^{n} w_i x_i + \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} \frac{c_{ij}}{x_i} - C_j^* \right) \tag{34}$$

The optimality criteria become

$$\frac{\partial L(X,\lambda)}{\partial x_i} = w_i - \sum_{j=1}^{m} \lambda_j \frac{c_{ij}}{x_i^2} = 0 \qquad (k = 1,\ldots,n) \tag{35}$$

or

$$1 = \sum_{j=1}^{m} \lambda_j \frac{c_{ij}}{w_i x_i^2} = D_i \tag{36}$$

The three recursion relations, equations (18), (20), and (21), remain the same, with equation (36) replacing equation (17).

For the evaluation of the Lagrange multipliers, equation (36) is rewritten in the following form:

$$\lambda_1 \begin{bmatrix} \frac{c_{11}}{x_1} \\ \frac{c_{21}}{x_2} \\ \vdots \\ \frac{c_{n1}}{x_n} \end{bmatrix} + \lambda_2 \begin{bmatrix} \frac{c_{12}}{x_1} \\ \frac{c_{22}}{x_2} \\ \vdots \\ \frac{c_{n2}}{x_n} \end{bmatrix} + \ldots + \lambda_m \begin{bmatrix} \frac{c_{1m}}{x_i} \\ \frac{c_{2m}}{x_2} \\ \vdots \\ \frac{c_{nm}}{x_n} \end{bmatrix} = \begin{bmatrix} w_1 x_1 \\ w_2 x_2 \\ \vdots \\ w_n x_n \end{bmatrix} \tag{37}$$

$$\lambda_1 C_1^* + \lambda_2 C_2^* + \ldots + \lambda_m C_m^* = W_{min}$$

It can be noted that equation (37) is to be satisfied both "horizontally" as the optimality criteria and also "vertically" as the constraint equations. It is coupled in both directions and requires alternating iterations between the two sets of equations. The columns are m design vectors with $w_i x_i$ as the weighted design variable. Each vector is to add up to the required constraint, and if it falls short, it is an obvious thought to enhance its participation by the update formulas of its multiplier presented earlier as equations (22) and (23). In many cases either one of these simple approaches are satisfactory.

Four observations can be made at this point. One is that the vectors associated with each constraint are, in general, not the optimum designs for that constraint, and one cannot use the simple procedure of obtaining the m optimum vectors and then using them to find their optimum weighted

sum as the optimum for the multiple constraint case. This would not be equivalent to the correct dual-programming approach. The second observation is that the simple equations (22) and (23) are uncoupled, and as they correct for one constraint, they modify the previous degree of satisfaction of the other constraints. The third observation represents the good news that equations (22) and (23) automatically tend to eliminate the consistently oversatisfied constraints. The fourth observation is the bad news that one needs a starting value for each multiplier. Relative starting values can be estimated by considering the constraints one at a time in the expressions at the bottom of equations (37) by using the relation

$$\lambda_j = \frac{W}{C_j^*} \tag{38}$$

There are a number of approaches with essentially equivalent final algorithms that have been proposed in the literature to develop coupled methods to evaluate the multipliers. In the dual-programming concept the optimality criteria expressions are used to eliminate the design variables from the Lagrangian in terms of the multipliers. The stationary conditions of the dual Lagrangian relative to the multipliers then provide the conditions with the multipliers as unknowns for the constraints to be also satisfied. The optimality criteria expressions also can be used to eliminate the design variables from the constraint equations and to obtain a set of nonlinear equations to be solved for the multipliers. A linearization results in the same set of linear equations as the stationary conditions in the dual-problem formulation.

The approach presented in reference 11 is the simplest way to derive the same set of equations, which would be valid at the optimum. One can simply combine the constraint and optimality criteria equations by replacing the operation of the summation in the constraint equations with multiplication with a unit vector. The unit vector is then written as a vector of the D_i expressions that are equal to unity at the optimum:

$$\left[\frac{c_{ij}}{x_i}\right]_{m\times n}\left[\frac{c_{ik}}{w_i x_i^2}\right]_{n\times m}\left[\lambda_j\right]_{m\times 1} = \left[C_j^*\right]_{m\times 1} \tag{39}$$

If one carries out the indicated operations, a set of m-coupled linear equations are obtained for the m constraints:

$$\left[E_{jk}\right]\left[\lambda_j\right] = \left[C_j^*\right]$$

where

$$E_{jk} = \frac{c_{ij}c_{ik}}{w_i x_i^3} \tag{40}$$

The formula for E_{ij} is typical of the terms obtained by other approaches in deriving a set of equations for the multipliers. Equation (40) also can be written in an iterative form as

$$\left[\lambda_j\right]^{new} = \left[E_{jk}\right]^{-1}\left[(p+1)C_j - pC_j^*\right]^{old} \tag{41}$$

with the step-size parameter p appearing again. Equation (41) is suggested to be used instead of equation (40).

Example: Truss-Slab Structural Tailoring Problem

The fundamental usefulness of DOC methods derives from their ability to arrive at an optimum within a small number of iterations, each iteration requiring only a single analysis with no need to use any additional analyses or any approximation concepts to generate sensitivities. Foremost, the number of iterations is independent of the number of design variables when the same structure is modeled with an increasing number of variables toward higher fidelity needed at final design stages. As a simple illustration of these points, a model generator was created to conveniently prepare the input data for truss-slabs with an increasing number of subdivisions to result in an increasing number of independent size variables. Figure 6 shows a set of truss-slabs with increasing numbers of members and the associated convergence curves. The two end corners of the slab are loaded, with one corner receiving twice the load. This would lead to twisting of the free end, but twisting was prevented by equal prescribed displacements of the two corners to create a simple structural tailoring problem. As can be seen from the convergence curves, the number of iterations to convergence is independent of the number of size variables, in this case, the bar areas. The convergence in about 25 iterations probably could be "tuned" to something much less, as was illustrated for the three-bar truss.

Generalized Optimality Criteria

The discussions up to this point have relied heavily on the concept of separability. With separability satisfied, the optimality criteria equations uncouple and become diagonal. In that case, one can solve for the design variables and arrive at a formula similar to equation (1). However, it is advantageous to make the formula more flexible by introducing the quantity D_i, and to allow for flexibility in the various step-size parameters. The use of the quantity D_i also circumvents the need for solving explicitly for the design variables and allows the possible treatment of cases where that is not possible or not convenient. If D_i can be computed in its general form,

$$1 = \sum_{j=1}^{m} \lambda_j \frac{\partial w(X) / \partial x_i}{\partial G(X) / \partial x_i} = D_i \tag{42}$$

then one can attempt to use, in principle, the optimality criteria iterative concepts for nonseparable cases also. Equation (42) requires sensitivity analyses, which are now available for most constraints. A number of researchers are now exploring these possibilities, and reference 12 is a suggested reading. It also discusses advanced scaling procedures to augment the Lagrange multipliers toward satisfying the constraints. More research and experience is needed to explore the power of DOC methods under totally general conditions and produce reliable software packages. Stress constraints are also discussed in reference 12, which uses the concepts of Prager's compliance constraint as the special case of displacement constraints with the work of the external load system being constrained. The equivalence of such compliance constraint concepts and the case of general arbitrary stress constraints have to be treated with

caution. In principle one always can choose an arbitrary distribution of stress allowables that has nothing to do with the design obtained by satisfying a compliance constraint, referred to in reference 12 as a generalized stiffness constraint and discussed earlier in references 6 and 11.

Another interesting application of the displacement constraints capability is to assign a preselected equilibrating internal load system in a statically indeterminate structure and require the resulting relative displacements to vanish at the "cuts" as somewhat of a reversed force method. Reference 13 discusses such applications, and reference 14 applies the concepts to optimization for stress constraints by using the force method formulation. Reference 15 presents a brief history of the evolution of DOC methods and also contains an extensive list of references. With the increasing need for efficient optimization capabilities, hybrid approaches exploiting the best features of both the general purpose MP and special purpose DOC methods should be the next area for development.

APPENDIX—SYMBOLS

A_i	cross-sectional area of the ith bar
$C(X)$	constraint function
C^*	external work; required value of the constraint function
$c_i(x_i)$	internal energy contribution of the ith variable to satisfy the constraint C^*
c_i, c_{ij}	specific internal energy contribution of the ith variable to satisfy the constraint C^*
D_i	quantity that approaches unity as the optimality criteria are becoming satisfied during iterations
e	small positive or negative number to modify conversions
j	index representing the jth constraint
K	system stiffness matrix
$L(X,\lambda)$	Lagrangian
L_i	length of the ith bar
m	number of constraints
m_i	specific mass of the ith member
n	number of design variables
P	actual load system
p	step-size parameter to control the rate of the modifications
q	step-size parameter that governs the modifying power of D_i

r^P	system displacements due to actual loads P
r^V	system displacements due to virtual loads V
S_i^P	axial member force due to the actual load system P
S_i^V	axial member force due to the virtual load system V
V	virtual load system
W(X)	objective function
$w_i(x_i)$	contribution of the ith member to the objective function
ω^*	value of frequency constraint
x_i	cross-sectional property
y	eigenfunction
y^T	transpose of the eigenfunction
α	convergence modifier
λ	Lagrange multiplier
ρ_i	specific weight of the ith bar
σ_{io}	stress allowable in the ith member

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TABLE I.—EFFECT OF STEP-SIZE PARAMETERS ON CONVERGENCE TO FOUR SIGNIFICANT FIGURES FOR THE THREE-BAR TRUSS

[Sizing option 2. Initial conditions: $A_1 = 1.00$; $A_2 = 1.00$; $\lambda = 100.0$.]

Iterations	Cross-sectional area of bar 1, A_1	Cross-sectional area of bar 2, A_2	Stress in bar 1	Objective function, W	λ	D_1	D_2
Step-size parameters Q = 0.50, p = 2.00, and $\alpha = 1.00$; λ option 1							
0	1.000000	1.000000	*********	110.000000	100.000000	1.171573	2.426407
1	1.309462	2.066130	70.710678	151.607531	265.934614	1.699445	2.100485
2	1.280669	2.320900	70.710678	151.275889	139.628364	0.918398	0.948426
3	1.277544	2.351477	70.710678	151.269153	151.019925	0.996577	1.008001
4	1.276401	2.362817	70.710678	151.268292	151.267328	0.999409	1.003150
5	1.276028	2.366544	70.710678	151.268199	151.268107	0.999808	1.001031
6	1.275906	2.367764	70.710678	151.268189	151.268180	0.999937	1.000337
7	1.275866	2.368163	70.710678	151.268188	151.268187	0.999980	1.000110
8	1.275853	2.368293	70.710678	151.268188	151.268188	0.999993	1.000036
9	1.275848	2.368336	70.710678	151.268188	151.268188	0.999998	1.000012
10	1.275847	2.368350	70.710678	151.268188	151.268188	0.999999	1.000004
11	1.275846	2.368354	70.710678	151.268188	151.268188	1.000000	1.000001
12	1.275846	2.368356	70.710678	151.268188	151.268188	1.000000	1.000000
13	1.275846	2.368356	70.710678	151.268188	151.268188	1.000000	1.000000
14	1.275846	2.368357	70.710678	151.268188	151.268188	1.000000	1.000000
15	1.275846	2.368357	70.710678	151.268188	151.268188	1.000000	1.000000
Step-size parameters Q = 0.70, p = 2.20, and $\alpha = 1.02$; λ option 2							
0	1.000000	1.000000	*********	110.000000	100.000000	1.171573	2.426407
1	1.282247	2.305701	70.710678	151.281676	284.003540	1.864957	1.946046
2	1.275093	2.375911	70.710678	151.268380	105.712326	0.699397	0.695849
3	1.275771	2.369110	70.710678	151.268190	175.421554	1.159765	1.159176
4	1.275850	2.368315	70.710678	151.268188	139.599526	0.922857	0.922883
5	1.275846	2.368358	70.710678	151.268188	157.860942	1.043583	1.043583
6	1.275846	2.368357	70.710678	151.268188	147.548082	0.975407	0.975407
7	1.275846	2.368357	70.710678	151.268188	153.561249	1.015159	1.015159
8	1.275846	2.368357	70.710678	151.268188	149.812918	0.990380	0.990380
9	1.275846	2.368357	70.710678	151.268188	152.253776	1.006516	1.006516
10	1.275846	2.368357	70.710678	151.268188	150.574835	0.995416	0.995416
11	1.275846	2.368357	70.710678	151.268188	151.783467	1.003406	1.003406
12	1.275846	2.368357	70.710678	151.268188	150.869600	0.997365	0.997365
13	1.275846	2.368357	70.710678	151.268188	151.592146	1.002142	1.002142
14	1.275846	2.368357	70.710678	151.268188	150.994202	0.998189	0.998189
15	1.275846	2.368357	70.710678	151.268188	151.510838	1.001604	1.001604
Step-size parameters Q = 1.00, p = 2.50, and $\alpha = 0.80$; λ option 1							
0	1.000000	1.000000	*********	110.000000	100.000000	1.171573	2.426407
1	1.273034	2.396751	70.710678	151.270881	229.107461	1.519095	1.490414
2	1.275796	2.368857	70.710678	151.268189	129.649103	0.857126	0.856838
3	1.275828	2.368538	70.710678	151.268188	150.929298	0.997779	0.997657
4	1.275838	2.368438	70.710678	151.268188	151.206677	0.999602	0.999547
5	1.275842	2.368402	70.710678	151.268188	151.246980	0.999865	0.999834
6	1.275843	2.368386	70.710678	151.268188	151.258099	0.999936	0.999917
7	1.275844	2.368378	70.710678	151.268188	151.262330	0.999964	0.999949
8	1.275845	2.368373	70.710678	151.268188	151.264296	0.999976	0.999965
9	1.275845	2.368370	70.710678	151.268188	151.265341	0.999983	0.999974
10	1.275845	2.368368	70.710678	151.268188	151.265952	0.999986	0.999979
11	1.275845	2.368367	70.710678	151.268188	151.266336	0.999989	0.999982
12	1.275845	2.368366	70.710678	151.268188	151.266591	0.999990	0.999984
13	1.275845	2.368365	70.710678	151.268188	151.266766	0.999992	0.999986
14	1.275845	2.368365	70.710678	151.268188	151.266892	0.999992	0.999987
15	1.275845	2.368364	70.710678	151.268188	151.266983	0.999993	0.999988

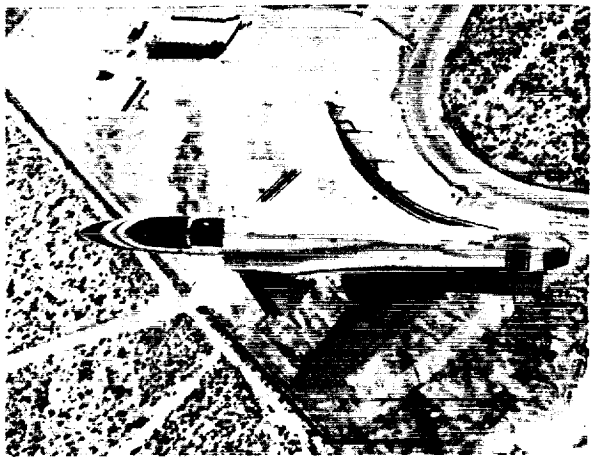


Figure 1.— Grumman/DARPA X-29A advanced technology demonstrator.

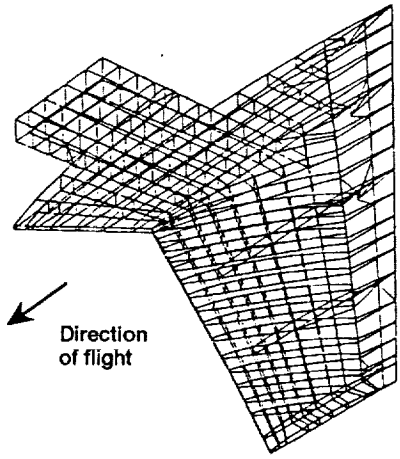


Figure 2.—Isometric view of wing finite element model for forward-swept wing preliminary design.

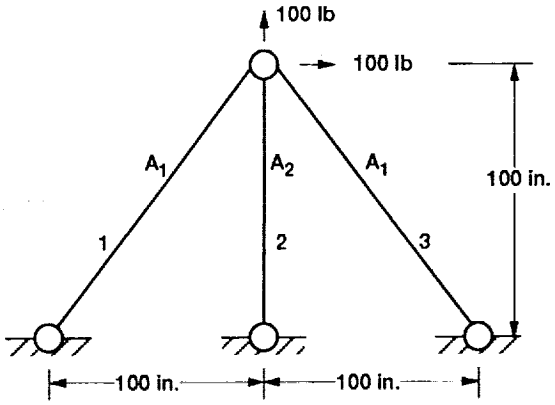


Figure 3.—Three-bar truss.

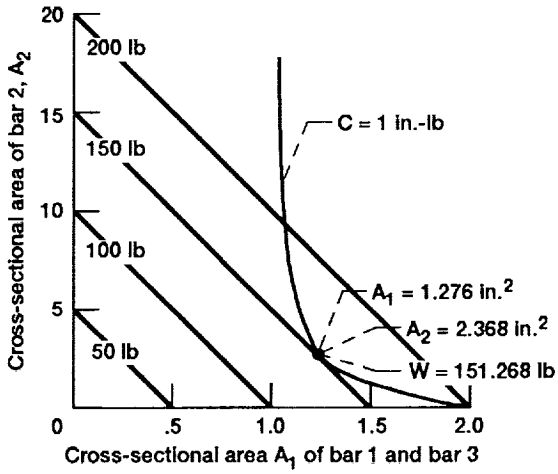
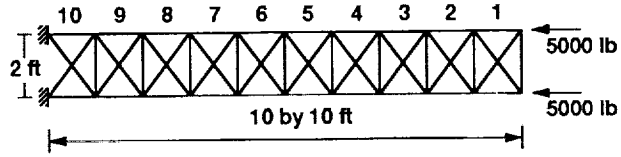


Figure 4.—Design space for three-bar truss.



(a) 50-bar truss column.

(b) Comparison of chord areas.

Figure 5.—Optimum chord areas of 50-bar truss column.

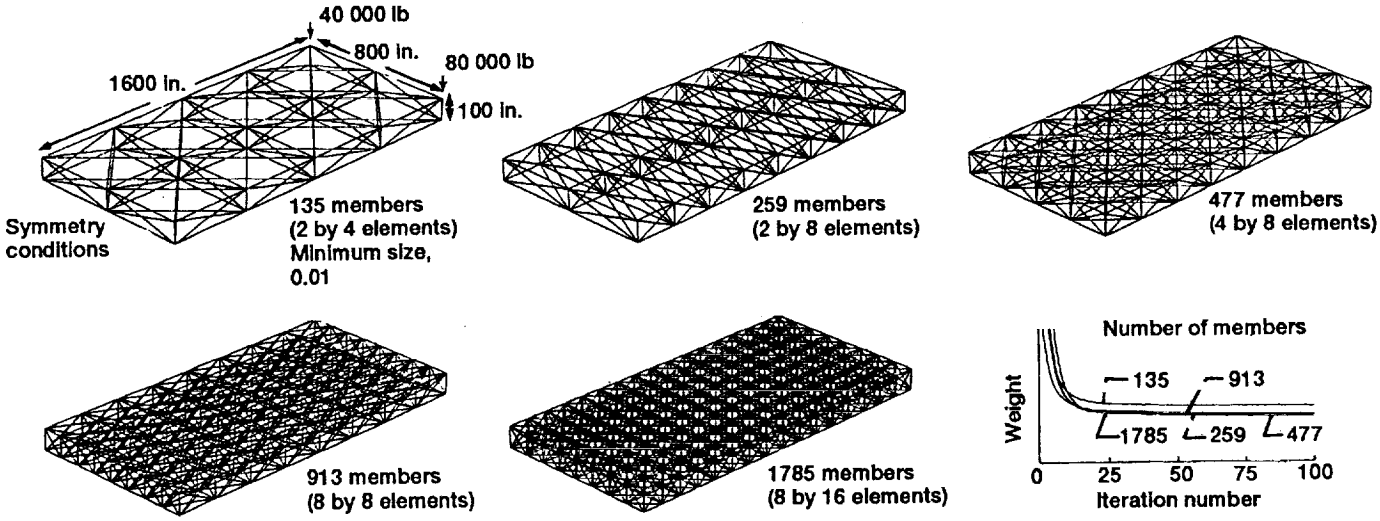


Figure 6.—Structural tailoring with increasing number of variables.

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13. ABSTRACT ***(Maximum 200 words)***

The fundamental concepts of the optimality criteria method of structural optimization are presented. The effect of the separability properties of the objective and constraint functions on the optimality criteria expressions is emphasized. The single constraint case is treated first followed by the multiple constraint case with a more complex evaluation of the Lagrange multipliers. Examples illustrate the efficiency of the method.

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