# SIMPLIFIED COMPUTATIONAL METHODS FOR ELASTIC AND ELASTIC-PLASTIC FRACTURE PROBLEMS

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### SUMMARY

This paper presents an overview of some of the recent (1984-1991) developments in computational/analytical methods in the mechanics of fracture: (i) analytical solutions for elliptical or circular cracks embedded in isotropic or transversely isotropic solids (the crack-plane being at an arbitrary angle to the axis of transverse isotropy), with crack-faces being subjected to arbitrary tractions; (ii) finite-element or boundary-element alternating methods for two- and three-dimensional crack problems; (iii) a "direct-stiffness" method for stiffened panels with flexible fasteners and with multiple cracks, using the alternating method; (iv) multiple-site-damage near a row of fastener holes; (v) analysis of cracks with bonded repair patches; (vi) methods for generation of weight-functions for 2- and 3-D crack problems, and (vii) domain-integral methods for elastic plastic or inelastic crack mechanics.

### INTRODUCTION

The starting point for this overview is the monograph on "Computational Methods in the Mechanics of Fracture" [1], with contributions by several noted researchers. The various articles in that monograph were prepared by individual authors in the 1984-85 time frame, with the material being current mostly as of 1984. In the present paper, some recent advances in computational fracture mechanics in the intervening years are summarized. The coverage of topics is limited to those listed in the "Summary" above; and furthermore the scope of the article is limited by the authors' own interests.

In each of the topics listed in the "Summary", a reasonably self-contained account of the generic issues, and the progress made in addressing them, is provided below.

Elastic and elastic-plastic fracture mechanics methodologies, that are easy to apply, are mandatory in a strategy for assessing the integrity of cracked structures. Straight-forward finite element/boundaryelement analyses of such problems, with detailed numerical modeling of crack-tip singularities, would be prohibitively expensive, even in the linear elastic regime. Furthermore, such straight-forward numerical analyses are useful for a single configuration of crack(s) and loads; and whenever the crack-configuration or the load-configuration changes, the entire analysis must be repeated. Thus, alternate strategies for efficient analyses of structural integrity in the presence of multiple-site-damage are needed. This paper reviews some such possible strategies.

# 1 ANALYTICAL SOLUTIONS FOR ELLIPTICAL OR CIRCULAR CRACKS IN ISOTROPIC OR TRANSVERSELY ISOTROPIC SOLIDS, WITH ARBITRARY CRACK-FACE TRACTIONS

In practice, the actual flaws in three-dimensional structural components are often approximated by elliptical cracks. For this reason, the problem of an embedded elliptical crack in an infinite solid has been the focus of a considerable number of studies.

The authors in [2] have presented a general solution procedure for an embedded elliptical crack in an isotropic infinite solid, subject to arbitrary crack-face tractions. Later, the authors in [3] have refined

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and completed this solution, deriving (i) alternative non-singular forms for linear algebraic equations relating crack-face tractions and potential functions, (ii) a general procedure for the evaluation of the required elliptic integrals, and (iii) a systematic procedure for the evaluation of the partial derivatives of the potential functions. These derivations made it possible to extract the analytical close-form solution for any polynomial order of crack-face tractions. For later convenience, we cite this general solution altogether [2] and [3] as VNA solution. The VNA solution represents a generalization, of the potential representation of [4] and [5].

In the following, we present a brief summary of the VNA solution. Further details of the VNA solution can be found in the cited original papers.

1.1 THE VNA SOLUTION (AN ELLIPTICAL CRACK IN AN ISOTROPIC SOLID)

Suppose that  $x_1$  and  $x_2$  are Cartesian coordinates in the plane of the elliptical crack and  $x_3$  is normal to the crack-plane such that:

$$(x_1/a_1)^2 + (x_2/a_2)^2 = 1, a_1 > a_2 (1)$$

describes the border of the elliptical crack of aspect ratio  $(a_1/a_2)$ . The necessary ellipsoidal coordinates  $\xi_{\alpha}$   $(\alpha = 1, 2, 3)$  are defined as the roots of the cubic equation

$$\omega(s) = 1 - \left(\frac{x_1^2}{a_1^2 + s}\right) - \left(\frac{x_2^2}{a_2^2 + s}\right) - \left(\frac{x_3^2}{s}\right) = 0$$
(2)

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where

$$\infty > \xi_3 \ge 0 \ge \xi_2 \ge -a_2^2 \ge \xi_1 \ge -a_1^2$$

so that the interior of the ellipse is given by  $\xi_3 = 0$  and its boundary by  $\xi_2 = \xi_3 = 0$ .

Let the tractions along the crack-surface be expressed in the form

$$\sigma_{3\alpha}^{(0)} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} A_{\alpha,m-n,n}^{(i,j)} x_1^{2m-2n+i} x_2^{2n+j}; \qquad (\alpha = 1, 2, 3)$$
(3)

so that the values of (i, j) specify the symmetries of the load with respect to the axes of the ellipse. M is an arbitrary integer which is related to the order of the polynomial. The solution corresponding to the load expressed by Eq. (3) can be assumed in terms of the potential functions

$$f_{\alpha} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{M} \sum_{\ell=0}^{k} C_{\alpha,k-\ell,\ell}^{(i,j)} F_{2k-2\ell+i,2\ell+j}; \qquad (\alpha = 1, 2, 3)$$
(4)

where

$$F_{2k-2\ell+i,2\ell+j} = \frac{\partial^{2k+i+j}}{\partial x_1^{2k-2\ell+i} \partial x_2^{2\ell+j}} \int_{\xi_3}^{\infty} [\omega(s)]^{2k+i+j+1} \frac{\mathrm{d}s}{\sqrt{Q(s)}}$$
(5)

and  $Q(s) = s(s + a_1^2)(s + a_2^2)$ . The components of displacement  $u_i$  and stress  $\sigma_{ij}$  in terms of  $f_{\alpha}$  ( $\alpha = 1, 2, 3$ ) are given by

$$u_{1} = (1 - 2\nu)(f_{1,3} + f_{3,1}) - (3 - 4\nu)f_{1,3} + x_{3}(\nabla \cdot \hat{f})_{,1}$$
  

$$u_{2} = (1 - 2\nu)(f_{2,3} + f_{3,2}) - (3 - 4\nu)f_{2,3} + x_{3}(\nabla \cdot \hat{f})_{,2}$$
  

$$u_{3} = -(1 - 2\nu)(f_{1,1} + f_{2,2}) - 2(1 - \nu)f_{3,3} + x_{3}(\nabla \cdot \hat{f})_{,3}$$
(6)

and

$$\begin{aligned} \sigma_{11} &= 2\mu [f_{3,11} + 2\nu f_{3,22} - 2f_{1,31} - 2\nu f_{2,32} + x_3 (\nabla \cdot \hat{f})_{,11}] \\ \sigma_{22} &= 2\mu [f_{3,22} + 2\nu f_{3,11} - 2f_{2,32} - 2\nu f_{1,31} + x_3 (\nabla \cdot \hat{f})_{,22}] \\ \sigma_{12} &= 2\mu [(1 - 2\nu) f_{3,12} - (1 - \nu) (f_{1,23} + f_{2,13}) + x_3 (\nabla \cdot \hat{f})_{,12}] \\ \sigma_{33} &= 2\mu [-f_{3,33} + x_3 (\nabla \cdot \hat{f})_{,33}] \\ \sigma_{31} &= 2\mu [-(1 - \nu) f_{1,33} + \nu (f_{1,11} + f_{2,21}) + x_3 (\nabla \cdot \hat{f})_{,13}] \\ \sigma_{32} &= 2\mu [-(1 - \nu) f_{2,23} + \nu (f_{1,12} + f_{2,22}) + x_3 (\nabla \cdot \hat{f})_{,23}] \end{aligned}$$
(7)

and

$$\nabla \cdot \hat{\boldsymbol{f}} = f_{1,1} + f_{2,2} + f_{3,3} \tag{8}$$

where  $\mu$  and  $\nu$  are the shear modulus and Poisson's ratio.

By successive differentiation, it can be shown from (5) that, since  $\omega(\xi_3) = 0$ ,

$$F_{k\ell} = \int_{\xi_3}^{\infty} \frac{\partial^{k+\ell} \omega^{k+\ell+1}}{\partial x_1^k \partial x_2^\ell} \frac{\mathrm{d}s}{[Q(s)]^{1/2}} \equiv \int_{\xi_3}^{\infty} \partial_1^{k_1} \partial_2^{\ell_1} \partial_3^{m_1} \omega^{k+\ell+1} \frac{\mathrm{d}s}{[Q(s)]^{1/2}} \tag{9}$$

wherein  $(2k - 2\ell + i)$  and  $(2\ell + j)$  in Eq. (24) are replaced by k and  $\ell$  in the above equation and

 $k_1=k, \qquad \ell_1=\ell, \qquad m_1=0$ 

In (9) we have used the additional notation that  $\partial_{\alpha}^{j}$  implies the *j*th partial derivative with respect to  $x_{\alpha}$ . Similarly, the first-order partial derivatives of  $F_{k\ell}$  with respect to  $x_{\beta}$  ( $\beta = 1, 2, 3$ ) can be expressed by

$$F_{k\ell,\beta} = \int_{\xi_3}^{\infty} \partial_1^{k_1} \partial_2^{\ell_1} \partial_3^{m_1} \omega^{k+\ell+1} \frac{\mathrm{d}s}{[Q(s)]^{1/2}}$$
(10)

where

 $k_1 = k + \delta_{1\beta}, \quad \ell_1 = \ell + \delta_{2\beta}, \quad m_1 = \delta_{3\beta}$ 

and  $\delta_{1\beta}$ , etc., are the well-known Kronecker deltas.

In the case of the second- and third-order partial derivatives, we derive:

$$F_{k\ell,\beta\gamma} = \int_{\xi_3}^{\infty} \partial_1^{k_1} \partial_2^{\ell_1} \partial_3^{m_1} \omega^{k+\ell+1} \frac{\mathrm{d}s}{[Q(s)]^{1/2}} + F_{k\ell\beta\gamma}^0 \tag{11}$$

where

$$F_{k\ell\beta\gamma}^{0} = (k+\ell+1)! \frac{x_1^{k_1} x_2^{\ell_1} x_3^{m_1}}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} \{\rho_1^{k_1} \rho_2^{\ell_1} \rho_3^{m_1} [Q(s)]^{1/2}\}_{s=\xi_3}$$
(12)

$$k_1 = k + \delta_{1\beta} + \delta_{1\gamma}, \quad \ell_1 = \ell + \delta_{2\beta} + \delta_{2\gamma}, \quad m_1 = \delta_{3\beta} + \delta_{3\gamma}$$
(13)

$$\rho_{\alpha} = \frac{\partial^2 \omega}{\partial x_{\alpha}^2} = -2/(a_{\alpha}^2 + s), \qquad (\alpha = 1, 2, 3)$$
(14)

in which  $a_3 \equiv 0$ , and

$$F_{k\ell,\beta\gamma\delta} = \int_{\xi_3}^{\infty} \partial_1^{k_1} \partial_2^{\ell_1} \partial_3^{m_1} \omega^{k+\ell+1} \frac{\mathrm{d}s}{[Q(s)]^{1/2}} + \frac{\partial F_{k\ell\beta\gamma}^0}{\partial x_\delta} + G^0$$
(15)

where

$$G^{0} = \frac{(k+\ell+1)![Q(s)]^{1/2}}{(\xi_{3}-\xi_{1})(\xi_{3}-\xi_{2})}\rho_{1}^{k_{1}}\rho_{2}^{\ell_{1}}\rho_{3}^{m_{1}}x_{1}^{k_{1}}x_{2}^{\ell_{1}}x_{3}^{m_{1}} \\ \times \left[\frac{k_{0}(k_{0}-1)}{2\rho_{1}x_{1}^{2}} + \frac{\ell_{0}(\ell_{0}-1)}{2\rho_{2}x_{2}^{2}} + \frac{m_{0}(m_{0}-1)}{2\rho_{3}x_{3}^{2}}\right]_{s=\xi_{3}} \\ k_{0} = k + \delta_{1\beta} + \delta_{1\gamma}; \quad \ell_{0} = \ell + \delta_{2\beta} + \delta_{2\gamma}; \quad m_{0} = \delta_{3\beta} + \delta_{3\gamma} \\ k_{1} = k_{0} + \delta_{1\delta}; \quad \ell_{1} = \ell_{0} + \delta_{2\delta}; \quad m_{1} = m_{0} + \delta_{3\delta}$$
(16)

A systematic procedure for the evaluation of the partial derivatives of  $F^0_{k\ell\beta\alpha}$  in (15) is given in [3]. It is noted that these derivatives are needed (i) in satisfying the boundary conditions on the crack-face and (ii) in evaluating the far-field stresses in the solid containing the elliptical crack which is subject to arbitrary tractions.

It is now seen from (9)-(15) that one needs to evaluate a generic integral of the type:

$$\int_{\xi_3}^{\infty} \partial_1^{k_1} \partial_2^{\ell_1} \partial_3^{m_1} \omega^{k+\ell+1} \frac{\mathrm{d}s}{[Q(s)]^{1/2}}$$
(17)

To accomplish this, we expand  $\omega^{k+\ell+1}$  in terms of  $x_{\alpha}^2$  and carry out the indicated differentiations term by term. Thus, one obtains:

$$\int_{\xi_{3}}^{\infty} \partial_{1}^{k_{1}} \partial_{2}^{\ell_{1}} \partial_{3}^{m_{1}} \omega^{k+\ell+1} \frac{\mathrm{d}s}{[Q(s)]^{1/2}} = (k+\ell+1)! \sum_{p=0}^{k+\ell+1} \sum_{q=0}^{p} \sum_{r=0}^{q} \frac{(-1)^{p}}{(k+\ell+1-p)!} \\ \times \frac{(2p-2q)!}{(p-q)!} \frac{(2p-2r)!}{(q-r)!} \frac{(2r)!}{(r)!} \frac{x_{1}^{2p-2q-k_{1}}}{(2p-2q-k_{1})!} \\ \times \frac{x_{2}^{2q-2r-\ell_{1}}}{(2q-2r-\ell_{1})!} \frac{x_{3}^{2r-m_{1}}}{(2r-m_{1})!} J_{p-q,q-r,r}(\xi_{3})$$
(18)

where

$$J_{p-q,q-r,r}(\xi_3) = \int_{\xi_3}^{\infty} \frac{\mathrm{d}s}{(s+a_1^2)^{p-q}(s+a_2^2)^{q-r}(s+a_3^2)^r [Q(s)]^{1/2}}$$
(19)

In general, the integral indicated in Eq. (19), for a given set of parameters p, q, r, can be evaluated in terms of incomplete elliptic integrals of the first and second kinds, and Jacobian elliptic functions. The derivation of the closed-form expressions involves exorbitant work even for relatively lower-order components of  $J_{p-q,q-r,r}$  (see [6]). Therefore, the derivation of a systematic generic procedure for the evaluation of the elliptic integrals  $J_{p-q,q-r,r}$  was important in the development as well as in the numerical implementation of the VNA solution. A procedure for this has been developed in [3] and is summarized as follows.

Eq. (19) can be rewritten in terms of Jacobian elliptic functions, as:

$$J_{p-q,q-r,r} = \frac{2}{a_1^{2p+1}} \int_0^{u_1} (\operatorname{sn}^{2p} u) (\operatorname{nd}^{2q-2r} u) (\operatorname{nc}^{2r} u) du$$
  
$$\equiv \frac{2}{a_1^{2p+1}} L_{p,q-r,r}$$
(20)

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where

$$\mathrm{sn}^2 u_1 = a_1^2 / (a_1^2 + \xi_3)$$

The following identities for Jacobian elliptic functions are used:

$$sn^{2}u_{1} + cn^{2}u = 1; \qquad k^{2}sn^{2}u + dn^{2}u = 1$$
  

$$dn^{2}u - k^{2}cn^{2}u = k^{\prime 2}; \qquad k^{\prime 2}sn^{2}u + cn^{2}u = dn^{2}u$$
  

$$tnu = snu/cnu; \qquad dcu = dnu/cnu; \qquad cdu = cnu/dnu$$
  

$$ndu = 1/dnu; \qquad ncu = 1/cnu; \qquad sdu = snu/dnu$$
(21)

where

$$k^{2} = (a_{1}^{2} - a_{2}^{2})/a_{1}^{2}; \qquad k'^{2} = 1 - k^{2}$$
 (22)

By using integration by parts in (20), one sees that:

$$L_{p,q-r,r} = \frac{1}{(2r-1)k'^2} \{ (\operatorname{sn}^{2p+1}u) (\operatorname{nc}^{2r-1}u) (\operatorname{nd}^{2q-2r-1}u) |_0^{u_1} + [2(-p+r-1)+2(p-q-r+2)k^2] L_{p,q-r,r-1} + k^2(-2p+2q-3) L_{p,q-r,r-2} \}$$
(23)

Thus, one needs the starting values of  $L_{p,q-r,r-1}$  and  $L_{p,q-r,r-2}$  to evaluate  $L_{p,q-r,r}$ . The lowest-order starting values are:

$$L_{p,q,-1} = \int_{0}^{u_{1}} \operatorname{sn}^{2p} u \operatorname{nd}^{2q} u \operatorname{nc}^{-2} u \, \mathrm{d} u$$
  

$$L_{p,q,-2} = \int_{0}^{u_{1}} \operatorname{sn}^{2p} u \operatorname{nd}^{2q} u \operatorname{nc}^{-4} u \, \mathrm{d} u$$
(24)

The above integrals can be reduced to the forms:

$$L_{p,q,-1} = \frac{1}{k^{2p+2}} \sum_{j=0}^{p} \sum_{\gamma=0}^{1} \frac{(-1)^{j+\gamma+1} k^{\prime 2(1-\gamma)} p!}{(p-j)! j! (1-\gamma)! \gamma!} I_{2(q-j-\gamma)}$$

$$L_{p,q,-2} = \frac{1}{k^{2p+4}} \sum_{j=0}^{p} \sum_{\gamma=0}^{2} \frac{(-1)^{j+\gamma+2} k^{\prime 2(2-\gamma)} p! 2}{(p-j)! j! (2-\gamma)! \gamma!} I_{2(q-j-\gamma)}$$
(25)

where

$$I_{2m} = \int_0^{u_1} \mathrm{nd}^m u \,\mathrm{d}u \tag{26}$$

$$I_{2m+2} = \frac{2m(2-k^2)I_{2m} + (1-2m)I_{2m-2} - k^2 \operatorname{sn} u_1 \operatorname{cn} u_1 \operatorname{nd}^{2m+1} u_1}{(2m+1)k'^2}$$
(27)

For  $2(p - j - \gamma) < 0$  in (44), we find  $I_{-2m} = G_{2m}$ , where

$$G_{2m+2} = \frac{k^2 \mathrm{dn}^{2m-1} u_1 \mathrm{sn} u_1 \mathrm{cn} u_1 + (1-2m)k'^2 G_{2m-2} + 2m(2-k^2) G_{2m}}{(2m+1)}$$
(28)

Thus, finally we see that one needs the following starting values for evaluating the general terms of  $I_{2m+2}$ and  $G_{2m+2}$ :

$$I_{0} = G_{0} = F(u_{1}) = u_{1}$$

$$I_{2} = (1/k'^{2})[E(u_{1}) - k^{2} \operatorname{sn} u_{1} \operatorname{cd} u_{1}]$$

$$G_{2} = E(u_{1})$$
(29)

where  $F(u_1)$  and  $E(u_1)$  are incomplete elliptic integrals of the first and second kinds, respectively.

Now, the boundary conditions on the crack-face  $(\sigma_{3\alpha} = \sigma_{3\alpha}^0)$  can be expressed in terms of the potential functions, as follows:

$$\sigma_{33}^{(0)} = -2\mu f_{3,33}$$
  

$$\sigma_{3\alpha}^{(0)} = -2\mu [(1-\nu)f_{\alpha,33} - \nu (f_{1,1\alpha} + f_{2,2\alpha})]\alpha = 1,2$$
(30)

in which the boundary condition for  $f_3$  is uncoupled from  $f_1$  and  $f_2$ . However, if Eqs. (30) are used directly as in [2], finite parts of the singular terms in the equations relating the coefficients C of (4) to coefficients A of Eq. (3) have to be considered. Alternative non-singular forms for the boundary conditions may also be used. Since  $f_{\alpha}$  ( $\alpha = 1, 2, 3$ ) are harmonic functions, it is seen that:

$$f_{\alpha,33} = -f_{\alpha,11} - f_{\alpha,22} \qquad (\alpha = 1, 2, 3) \tag{31}$$

Then, (30a, b) can be rewritten as follows:

$$\sigma_{33}^{(0)} = 2\mu(f_{3,11} + f_{3,22})$$
  

$$\sigma_{3\alpha}^{(0)} = 2\mu[(1-\nu)(f_{\alpha,11} + f_{\alpha,22}) + \nu(f_{1,1\alpha} + f_{2,2\alpha})]$$
(32)

Substituting (3) and (4) into (32a, b), we obtain the following linear algebraic equations, upon comparing coefficients of like powers in the polynomial series. The relation between the parameters A and parameters C can be summarized in a matrix form:

$$\{A\} = \begin{bmatrix} B \end{bmatrix} \quad \{C\} \\ N \times 1 \qquad N \times N \qquad N \times 1$$
 (33)

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where N is the total number of coefficients A or C. For a complete polynomial expressed by (3), the maximum degree of the polynomial  $M_c$  and the number of coefficients N can be expressed, respectively, as  $M_c = 2M + 1$  and  $N = (M + 1)(2M + 3) \times 3$ . For an incomplete polynomial, the maximum degree of polynomial and the number of coefficients depend not only on the parameter M but also on the parameters i and j in (3). Detailed expressions of the components of matrix [B] are given in [3] for Mode I and mixed modes of II and III. A more convenient form for the mixed modes of II and III also can be found in [7].

Once the coefficients C are determined by solving (33) for given loadings on the crack surface, the stressintensity factors corresponding to these loads are computed from the following equation [2].

For the Mode I problem,

$$K_{I} = 8\mu \left(\frac{\pi}{a_{1}a_{2}}\right)^{1/2} A^{1/4} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{M} \sum_{\ell=0}^{k} (-2)^{2k+i+j} (2k+i+j+1)! \\ \times \frac{1}{a_{1}a_{2}} \left(\frac{\cos\theta}{a_{1}}\right)^{2k-2\ell+i} \left(\frac{\sin\theta}{a_{2}}\right)^{2\ell+j} C_{3,k-\ell,\ell}^{(i,j)}$$
(34)

where  $\theta$  is the elliptic angle and

$$A = a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta \tag{35}$$

For the mixed-mode problem of Modes II and III

$$K_{II} = 8\mu \left(\frac{\pi}{a_1 a_2}\right)^{1/2} A^{-1/4} \frac{1}{a_1 a_2} [H_1 a_2 \cos \theta + H_2 a_1 \sin \theta]$$
(36)

$$K_{III} = 8\mu \left(\frac{\pi}{a_1 a_2}\right)^{1/2} A^{-1/4} \frac{(1-\nu)}{a_1 a_2} [H_2 a_2 \cos \theta - H_1 a_1 \sin \theta]$$
(37)

in which

$$H_{1} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{M} \sum_{\ell=0}^{k} (-2)^{2k+i+j} (2k+i+j+1)! \times \left(\frac{\cos\theta}{a_{1}}\right)^{2k-2\ell+i} \left(\frac{\sin\theta}{a_{2}}\right)^{2\ell+j} C_{1,k-\ell,\ell}^{(i,j)}$$
(38)

$$H_{2} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{M} \sum_{\ell=0}^{k} (-2)^{2k+2-i-j} (2k+3-i-j)! \times \left(\frac{\cos\theta}{a_{1}}\right)^{2k-2\ell+1-i} \left(\frac{\sin\theta}{a_{2}}\right)^{2\ell+1-j} C_{2,k-\ell,\ell}^{(1-i,1-j)}$$
(39)

The VNA solution has been implemented by [8, 3] in a new finite-element alternating method for the solution of problems of embedded or surface flaws in complex structural geometries.

## 1.2 AN ELLIPTICAL CRACK IN A TRANSVERSELY ISOTROPIC SOLID

Recently, the authors in [9] have extended the VNA solution procedure to a transversely isotropic case, with arbitrary tractions on the face of an elliptical crack in a transversely isotropic solid, with the crack plane being at an arbitrary angle to the axis of elastic symmetry.

A material is said to be transversely isotropic when it possesses an axis of elastic symmetry such that the material is isotropic in the planes normal to this axis. Let  $\bar{z}$  be the direction of elastic symmetry. Then the stress-strain relations in the  $(\bar{x}, \bar{y}, \bar{z})$  system could be written as,

$$\sigma_{\bar{x}} = C_{11} \frac{\partial u_{\bar{x}}}{\partial \bar{x}} + C_{12} \frac{\partial u_{\bar{x}}}{\partial \bar{y}} + C_{13} \frac{\partial u_{\bar{z}}}{\partial \bar{z}}$$

$$\sigma_{\bar{y}} = C_{12} \frac{\partial u_{\bar{x}}}{\partial \bar{x}} + C_{11} \frac{\partial u_{\bar{y}}}{\partial \bar{y}} + C_{13} \frac{\partial u_{\bar{z}}}{\partial \bar{z}}$$

$$\sigma_{\bar{z}} = C_{13} \left( \frac{\partial u_{\bar{x}}}{\partial \bar{x}} + \frac{\partial u_{\bar{y}}}{\partial \bar{y}} \right) + C_{33} \frac{\partial u_{\bar{z}}}{\partial \bar{z}}$$

$$\tau_{\bar{y}\bar{z}} = C_{44} \left( \frac{\partial u_{\bar{z}}}{\partial \bar{y}} + \frac{\partial u_{\bar{y}}}{\partial \bar{z}} \right)$$

$$\tau_{\bar{z}\bar{x}} = C_{44} \left( \frac{\partial u_{\bar{x}}}{\partial \bar{z}} + \frac{\partial u_{\bar{z}}}{\partial \bar{x}} \right)$$

$$\tau_{\bar{x}\bar{y}} = \frac{1}{2} (C_{11} - C_{12}) \left( \frac{\partial u_{\bar{x}}}{\partial \bar{y}} + \frac{\partial u_{\bar{y}}}{\partial \bar{x}} \right)$$
(40)

 $(\sigma_{\bar{x}}, \sigma_{\bar{y}}, \sigma_{\bar{z}}, \tau_{\bar{y}\bar{z}}, \tau_{\bar{z}\bar{x}}, \tau_{\bar{x}\bar{y}})$  and  $(u_{\bar{x}}, u_{\bar{y}}, u_{\bar{z}})$  are stresses and displacement components in the  $(\bar{x}, \bar{y}, \bar{z})$  system and  $C_{ij}$  are elastic constants, as discussed in [10].

The displacement field  $u_{\bar{x}}$ ,  $u_{\bar{y}}$ , and  $u_{\bar{z}}$  is represented in terms of potential functions  $\phi_j$  (j = 1, 2, 3), such that it satisfies the equilibrium equations expressed in terms of displacements, identically, as follows:

$$u_{\bar{x}} = \frac{\partial}{\partial \bar{x}} (\phi_1 + \phi_2) - \frac{\partial \phi_3}{\partial \bar{y}}$$

$$u_{\bar{y}} = \frac{\partial}{\partial \bar{y}} (\phi_1 + \phi_2) + \frac{\partial \phi_3}{\partial \bar{x}}$$

$$u_{\bar{z}} = \frac{\partial}{\partial \bar{z}} (m_1 \phi_1 + m_2 \phi_2)$$
(41)

$$m_j = \frac{C_{11}n_j - C_{44}}{C_{13} + C_{44}} = \frac{(C_{13} + C_{44})n_j}{C_{33} - C_{44}n_j}, \qquad j = 1, 2$$
(42)

The quantities  $n_1$  and  $n_2$  are the roots of the quadratic equation in n

$$C_{11}C_{44}n^2 + [C_{13}(C_{13} + 2C_{44}) - C_{11}C_{33}]n + C_{33}C_{44} = 0$$
(43)

and  $n_3$  is defined as:

$$n_3 = \frac{2C_{44}}{(C_{11} - C_{12})} \tag{44}$$

Introduction of the following modified coordinate systems  $(x_j, y_j, z_j)$  (j = 1, 2, 3)

$$\begin{aligned} x_j &= \bar{x} \\ y_j &= \bar{y}\cos\theta_j + \frac{\bar{z}}{\sqrt{n_j}}\sin\theta_j \\ z_j &= -\bar{y}\sin\theta_j + \frac{\bar{z}}{\sqrt{n_j}}\cos\theta_j \end{aligned}$$

$$(45)$$

yields the expressions of the elliptical crack in each modified coordinate system

$$\frac{x_j^2}{a_j^2} + \frac{y_j^2}{b_j^2} = 0 \quad \text{in } z_j = 0 \quad (j = 1, 2, 3 \text{ no sum})$$
(46)

where

$$a_{j} = a$$
  

$$b_{j} = b \left( \cos \bar{\theta} \cos \theta_{j} + \frac{1}{\sqrt{n_{j}}} \sin \bar{\theta} \sin \theta_{j} \right)$$
(47)

 $\operatorname{and}$ 

$$\tan \theta_j = \frac{1}{\sqrt{n_j}} \tan \bar{\theta} \tag{48}$$

Here,  $\bar{\theta}$  is the angle between the physical crack axis (z) and the material axis ( $\bar{z}$ ).

Now, each of the potentials  $\phi_j$  (j = 1, 2, 3) can be expressed by the harmonic equations in a set of coordinate  $(x_j, y_j, z_j)$  (j = 1, 2, 3), as:

$$\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2}\right)\phi_j = 0 \qquad (j = 1, 2, 3 \text{ no sum on } j)$$
(49)

Thus, to solve the problem on hand, appropriate potential functions  $\phi_j$  (j = 1, 2, 3) each in a different set of coordinates  $(x_j, y_j, z_j)$  (j = 1, 2, 3) will now be assumed. The necessary ellipsoidal coordinates  $\xi_1^j, \xi_2^j, \xi_3^j$  (j = 1, 2, 3) for a point in the  $x_j, y_j, z_j$  (j = 1, 2, 3) coordinate system are given by the roots of the cubic equation

$$\omega_j(\xi^j) = 0$$
  $(j = 1, 2, 3 \text{ no sum on } j)$  (50)

where

$$\omega_j(\xi^j) = 1 - \frac{x_j^2}{a^2 + \xi^j} - \frac{y_j^2}{b_j^2 + \xi^j} - \frac{z_j}{\xi^j}$$
(51)

and

$$-a^{2} \leq \xi_{1}^{j} \leq -b_{j}^{2} \leq \xi_{2}^{j} \leq 0 \leq \xi_{3}^{j} < \infty$$
(52)

Appropriate expressions of the potentials  $\phi_j$  for the present case of transverse isotropy as discussed above, are assumed as:

$$\phi_j = \sum_{\ell=0}^{M} \sum_{k=0}^{\ell} B_{\ell-k,k}^j F_{\ell-k,k}^j$$
(53)

where,  $B_{\ell-k,k}^{j}$  are unknown coefficients to be determined, M is the highest order polynomial terms considered and  $F_{\ell-k,k}^{j}$  is defined as:

$$F_{k\ell}^{j} = \frac{\partial^{k+\ell}}{\partial x_{j}^{k} \partial y_{j}^{\ell}} \int_{\xi_{3}^{j}}^{\infty} [\omega_{j}(s)]^{k+\ell+1} \frac{\mathrm{d}s}{\sqrt{Q_{j}(s)}}$$
(54)

As can be easily noticed, the above equation is basically the same as Eq. (5) or Eq. (10) in the VNA solution procedure for real  $n_1$  and  $n_2$  Eq. (44). Therefore, the VNA solution procedure can be used to obtain the complete general solution for an transversely isotropic material. However, for complex  $n_1$  and  $n_2$  analytic continuation from the real axis to the complex plane has to be carried out. This solution is given in [9] in more detail.

### **1.3** A CIRCULAR CRACK IN AN ISOTROPIC SOLID

The analytical general solution for a circular crack in an infinite isotropic elastic solid, subject to arbitrary crack-face tractions, is briefly summarized here. The solution was revisited in [11], based on the Fourier-Hankel transform technique developed in [12], which was later generalized in [13]. Although the authors in [13] have derived the general solution for this type of problem, certain portions of the mixed-mode solution were lacking in their final results. Thus, the complete form of the general solution has been recently rederived [11] as follows:

For a penny-shaped crack embedded in an infinite 3-D elastic body, we need to solve the following mixed boundary value problems.

### <u>Mode I</u>

$$\sigma_{rz}(r,\theta,0) = \sigma_{\theta z}(r,\theta,0) = 0 \qquad r \ge 0; \qquad 0 \le \theta \le 2\pi$$
  

$$\sigma_{zz}(r,\theta,0) = p_1(r,\theta) \qquad 0 \le r < a; \qquad 0 \le \theta \le 2\pi$$
  

$$u_z(r,\theta,0) = 0 \qquad r > a; \qquad 0 \le \theta \le 2\pi$$
(55)

(a is the crack radius)

### Modes II and III

$\sigma_{zz}(r,\theta,0)=0$	$r \ge 0$	$0 \le  heta \le 2\pi$	
$\sigma_{rz}(r,\theta,0) = p_2(r,\theta)$	$0 \leq r < a$	$0 \le  heta \le 2\pi$	(56)
$\sigma_{\theta z}(r,  heta, 0) = p_3(r,  heta)$	$0 \leq r < a$	$0 \le  heta \le 2\pi$	
$u_r(r, heta,,0) = u_ heta(r, heta,0) = 0$	r > a	$0 \le  heta \le 2\pi$	

where  $p_{\alpha}(r, \theta)$  ( $\alpha = 1, 2, 3$ ) are given functions describing the distribution of the loads applied to the crack surface.

An appropriate solution for this boundary-value problem can be obtained by expressing the displacement components in terms of three harmonic functions  $\phi_{\alpha}$  ( $\alpha = 1, 2, 3$ ), as

$$u_{r} = (1 - 2\nu)\frac{\partial\phi_{1}}{\partial r} + z\frac{\partial^{2}\phi_{1}}{\partial r\partial z} + \frac{2}{r}\frac{\partial\phi_{2}}{\partial\theta} + 2(1 - \nu)\frac{\partial\phi_{3}}{\partial r} + z\frac{\partial^{2}\phi_{3}}{\partial r\partial\theta}$$

$$u_{\theta} = (1 - 2\nu)\frac{1}{r}\frac{\partial\phi_{1}}{\partial\theta} + \frac{z}{r}\frac{\partial^{2}\phi_{1}}{\partial\theta\partial z} - 2\frac{\partial\phi_{2}}{\partial r} + 2(1 - \nu)\frac{1}{r}\frac{\partial\phi_{3}}{\partial\theta} + \frac{z}{r}\frac{\partial^{2}\phi_{3}}{\partial\theta\partial z}$$

$$u_{z} = -2(1 - \nu)\frac{\partial\phi_{1}}{\partial z} + z\frac{\partial^{2}\phi_{1}}{\partial z^{2}} - (1 - 2\nu)\frac{\partial\phi_{3}}{\partial z} + z\frac{\partial^{2}\phi_{3}}{\partial z^{2}}$$
(57)

The corresponding stress components in terms of the potential functions can be easily obtained by the above equations through the use of the strain-displacement and the stress-strain relations. From the above equations, it is seen that  $\phi_1$  is related with mode I, and  $\phi_2$  and  $\phi_3$  are related with the mixed mode of II and III.

In order to express general loadings, the applied loads  $p_{\alpha}(r, \theta)$  are expressed by Fourier series as follows:

$$p_{\alpha}(r,\theta) = \sum_{n=0}^{\infty} \frac{\cos n\theta \cdot A_{\alpha n}(r)}{\sin n\theta \cdot B_{\alpha n}(r)} \qquad (\alpha = 1, 2, 3)$$
(58)

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In order to solve the proposed problem, the potential functions are represented by the Fourier-Hankel transform:

$$\phi_{\alpha}(r,\theta,z) = \sum_{n=0}^{\infty} \frac{\cos n\theta}{\sin n\theta} \int_{0}^{\infty} \frac{C_{\alpha n}(s)}{D_{\alpha n}(s)} \frac{1}{s} J_{n}(rs) e^{-sz} \,\mathrm{d}s \qquad (\alpha = 1,2,3)$$
(59)

The substitution of Eqs. (58) and (59) into Eqs. (55), (56) and (57) yields the following relations: Mode I

$$C_{1n}(s) = -\frac{1}{\mu} \sqrt{\frac{s}{2\pi}} \int_0^a J_{n+1/2}(st) \frac{dt}{t^{n-1/2}} \int_0^t \frac{r^{n+1}A_{1n}(r) dr}{(t^2 - r^2)^{1/2}}$$
  

$$D_{1n}(s) = -\frac{1}{\mu} \sqrt{\frac{s}{2\pi}} \int_0^a J_{n+1/2}(st) \frac{dt}{t^{n-1/2}} \int_0^t \frac{B_{1n}(r) dr}{(t^2 - r^2)^{1/2}}$$
(60)

Mode II and III

$$C_{20}(s) = -\frac{1}{\mu} \sqrt{\frac{s}{2\pi}} \int_{0}^{a} J_{3/2}(st) \frac{\mathrm{d}t}{t^{1/2}} \int_{0}^{t} \frac{r^{2} A_{30}(r)}{(t^{2} - r^{2})^{1/2}} \mathrm{d}r$$

$$C_{30}(s) = \frac{1}{\mu} \sqrt{\frac{s}{2\pi}} \int_{0}^{a} J_{3/2}(st) \frac{\mathrm{d}t}{t^{1/2}} \int_{0}^{t} \frac{r^{2} A_{20}(r)}{(t^{2} - r^{2})^{1/2}} \mathrm{d}r$$
(61)

$$C_{2n}(s) = \sqrt{s} \int_{0}^{a} [(\nu - 1)\Phi_{1}^{\star}(t)J_{n-1/2}(st) + \Phi_{2}^{\star}(t)J_{n+3/2}(st)] dt, \quad n \ge 1$$

$$C_{3n}(s) = \sqrt{s} \int_{0}^{a} [\Phi_{1}(t)J_{n-1/2}(st) + \Phi_{2}(t)J_{n+3/2}(st)] dt \quad n \ge 1$$

$$D_{2n}(s) = \sqrt{s} \int_{0}^{a} [(1 - \nu)\Phi_{1}(t)J_{n-1/2}(st) - \Phi_{2}(t)J_{n+3/2}(st)] dt, \quad n \ge 1$$

$$D_{3n}(s) = \sqrt{s} \int_0^a [\Phi^*(t) J_{n-1/2}(st) + \Phi_2^*(t) J_{n+3/2}(st)] dt \qquad n \ge 1$$
(62)

where,

$$\begin{split} \Phi_1(t) &= \frac{-t^{-n+3/2}}{(2-\nu)\mu\sqrt{2\pi}} \int_0^t \frac{r^n [A_{2n}(r) - B_{3n}(r)]}{(t^2 - r^2)^{1/2}} \, \mathrm{d}r \\ \Phi_1^\star(t) &= \frac{-t^{-n+3/2}}{(2-\nu)\mu\sqrt{2\pi}} \int_0^t \frac{r^n [A_{3n}(r) + B_{2n}(r)]}{(t^2 - r^2)^{1/2}} \, \mathrm{d}r \\ \Phi_2(t) &= \frac{\nu}{2} \Phi_1(t) + \frac{t^{-n-1/2}}{2\mu\sqrt{2\pi}} \left\{ \frac{(1+2n)\nu}{2-\nu} \int_0^t r^n [A_{2n}(r) - B_{3n}(r)] \right. \\ & \left. \times (t^2 - r^2)^{1/2} \, \mathrm{d}r + \int_0^t \frac{r^{n+2} [A_{2n}(r) + B_{3n}(r)]}{(t^2 - r^2)^{1/2}} \, \mathrm{d}r \right\} \end{split}$$

$$\Phi_{2}^{\star}(t) = \frac{\nu}{2} \Phi_{1}^{\star}(t) + \frac{t^{-n-1/2}}{2\mu\sqrt{2\pi}} \left\{ \frac{(1+2n)\nu}{2-\nu} \int_{0}^{t} r^{n} [A_{3n}(r) + B_{2n}(r)] \right. \\ \left. \times (t^{2} - r^{2})^{1/2} \,\mathrm{d}r + \int_{0}^{t} \frac{r^{n+2} [B_{2n}(r) - A_{3n}(r)]}{(t^{2} - r^{2})^{1/2}} \,\mathrm{d}r \right\}$$
(63)

Without going into details, the stress intensity factors for all the modes are given by

$$K_{I} = -\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\cos n\theta}{\sin n\theta} \frac{1}{a^{n+1/2}} \int_{0}^{a} \frac{A_{1n}(r)}{B_{1n}(r)} \frac{r^{n+1}}{(a^{2}-r^{2})^{1/2}} dr$$

$$K_{II} = -\frac{2}{\sqrt{\pi}a^{3/2}} \int_{0}^{a} \frac{r^{2}A_{20}(r)}{(a^{2}-r^{2})^{1/2}} dr + \frac{2\mu\sqrt{2}}{a} \sum_{n=1}^{\infty} [\Phi_{1}(a) - \Phi_{2}(a)] \cos n\theta$$

$$-\frac{2\mu\sqrt{2}}{a} \sum_{n=1}^{\infty} [(1-\nu)\Phi_{1}^{*}(a) + \Phi_{2}^{*}(a)] \sin n\theta$$

$$K_{III} = -\frac{2\mu\sqrt{2}}{a} \sum_{n=1}^{\infty} [(1-\nu)\Phi_{1}(a) + \Phi_{2}(a)] \sin n\theta$$

$$-\frac{2}{\sqrt{\pi}a^{3/2}} \int_{0}^{a} \frac{r^{2}A_{30}(r)}{(a^{2}-r^{2})^{1/2}} dr + \frac{2\mu\sqrt{2}}{a} \sum_{n=1}^{\infty} [\Phi_{1}^{*}(a) - \Phi_{2}^{*}(a)] \cos n\theta$$
(64)

# 2 FINITE-ELEMENT AND BOUNDARY-ELEMENT ALTERNATING METHODS FOR 2D AND 3D CRACK PROBLEMS

Consider a 2 or 3 dimensional homogeneous solid containing multiple cracks, as shown in Fig. 1, and assume that the crack-faces are traction-free. In the alternating technique [14], and [15], the stresses in the *uncracked* body are first analyzed, by using a numerical/analytical method such as the finite element/boundary element method, for the given system of external loading. In order to assess the effect of the crack, the tractions at the locations of the cracks in the otherwise uncracked body must be erased.

Assume for simplicity, for now, that there is a single crack. Thus, one has to deal with the problem of a *finite body* with a crack, the faces of which are subject to arbitrary tractions, and the outer boundaries are traction free. Denoting by  $S_c$  the crack-face, and by  $S_{\sigma}$  the external boundary of the *finite-body* V, this problem may be posed as:

Solve: 
$$L(u) = 0$$
 in V;  $B(u) = \overline{t}$  at  $s_c$ ;  $B(u) = 0$  at  $S_{\sigma}$  (65)

where u are the displacements, and L and B are the appropriate differential operators. To solve problem (65) by using the alternating method, the finite body is replaced by an *infinite body*, with stresses going to zero at infinity. The problem of an infinite homogeneous body containing a crack, the faces of which are subject to tractions  $\bar{t}$  as in (65), does have an *analytical* solution often times, and hence need not be solved numerically. The far-field stresses from this *infinite body* solution do not satisfy the condition B(u) = 0 at  $S_{\sigma}$ . Thus, the residual tractions at the boundaries of the finite-body are erased, by first solving an *uncracked* body with these residual boundary tractions, and then erasing the tractions at the location of the crack in the uncracked body. This last problem of erasing crack-face tractions is similar to problem (65) above. This iterative loop is continued until the analytical solution for the infinite body satisfies also the zero-traction condition, B(u) = 0 at  $S_{\sigma}$ , of the finite body. Thus, in the alternating method, problem (65) is recast as:

÷,

$$L(u) = 0 \quad in \quad V_{\infty}; \quad B(u) = \overline{t}^{(i)} \quad at \quad S_c; \quad and \quad B(u) = \overline{t}^{(i)}_{\sigma} \quad at \quad S_{\sigma}$$
(66)

where *i* denotes the *i*th iteration, and  $V_{\infty}$  is the infinite domain. The iteration continues until  $\bar{t}_{\sigma}^{(i)} = 0$ . Let  $\bar{t}^{(i)}$  be the converged value at  $S_c$ , corresponding to which the stress-intensity factors are determined from

the analytical infinite-body solution [15,1]. For multiple cracks, the alternating method is analogous; and is represented schematically in Figs. 1 and 2. In a variety of practical cases, this alternating method has been established to be a very simple and cost-effective tool of analysis [16].



• Alternating Technique for Multiple 2D or 3D Embedded/Surface Cracks

Henceforth we assume, for convenience, that a finite-element method is used. The steps involved in the finite-element alternating method for an embedded crack in a finite body are described below:

1. Solve the *uncracked finite body* under the prescribed external loads by using the finite element method. The uncracked body has the same geometry as the given problem except for the crack.

2. Using the finite element solution, compute the stresses at the location of the crack.

3. Compare the residual stresses calculated in step 2 with a permissible stress magnitude.

4. To create traction-free crack faces as in the given problem, reverse the residual stress at the location of the crack as computed in Step 2 and "least-squares fit" them to polynomials.

5. Obtain the analytical solution to the infinite body with the crack subject to the polynomial loading as in Step 4.

6. Calculate the stress intensity factors for the current iteration, using the above analytical solution.

7. Calculate the residual stresses on external surfaces at the body due to the applied loads on crack-faces, as in step 4. To satisfy the given traction boundary conditions, at the external boundaries, reverse the residual stresses on the external surfaces of the body, and calculate the equivalent nodal forces.

8. Consider the nodal forces in step 7 as externally applied loads acting on the uncracked body.

Repeat all steps in the iteration process until the residual stress on the crack surface becomes negligible. To obtain the final solution, add the stress intensity factors for all iterations.

Since the alternating method is iterative in nature, the finite-element equations may, in general, have to be solved repeatedly for different applied loads, while keeping the stiffness matrix the same. To save computational time, special computational techniques were implemented in [3]. These are explained below.



As seen from above, for the finite-element alternating method, we need to solve the following type of finite-element equations:

$$[K][q^0, q^1, \dots, q^n] = [Q^0, Q^1, \dots, Q^n]$$
(67)

and

$$Q^{i} = Q^{i}(q^{i-1});$$
  $i = 1, 2, ..., n$  (68)

in which the superscript denotes the cycle of iteration, [K] is the global (assembled) stiffness matrix of the *uncracked* body, and remains the same during the iteration process, and  $q^i$  is the nodal displacement vector for *i*th iteration.  $Q^i$  is the nodal force vector for the *i*th iteration and depends on the solution for the previous iteration  $q^{i-1}$  as expressed by Eq. (68).

An efficient equation solver OPTBLOK developed in [17] may be used to save computational time in solving Eq. (67). The solution algorithm is divided into three parts, i.e. (i) reduction of stiffness matrix, (ii) reduction of load vector, and (iii) back substitution. In OPTBLOK the reduction of stiffness matrix is done only once, although the reduction of load vector and back substitution may be repeated for any number of load cases. Thus, denoting the CPU time for each part by  $T_1$ ,  $T_2$ , and  $T_3$ , respectively, the total CPU time T in solving Eq. (67) using OPTBLOK can be expressed as

$$T = T_1 + (n+1)(T_2 + T_3) = (T_1 + T_2 + T_3) + n(T_2 + T_3)$$
(69)

where n is the total number of iterations. Since  $T_1$  is much larger than  $(T_2 + T_3)$ , a substantial reduction in computational time, compared with the case in which Eq. (67) is solved for each iteration [i.e.  $T^* = (n+1)(T_1+T_2+T_3)$ ], may be expected. To illustrate this situation, we consider the example of a set of linear equations with the number of equations of 1960, and half bandwidth of 200, wherein the CPU time for reduction of load vector and back substitution was about 5.6% of the total CPU time  $(T_2 + T_3 \simeq 0.056T)$ . Since, for a typical problem, the present alternating method needs three iterations (n = 3), the additional cost in this case is only about 16.8%, which is considerably smaller than the 300% in the case when Eq. (67) is solved for each iteration.

An efficient procedure was also devised for the calculation of the nodal forces required in step 7 (see also  $\Xi_{2}$ . Eq. (68)). In general, the stress field in a general solution can be expressed by

$$\sigma = \mathbf{P}C\tag{70}$$

where  $\mathbf{P}$  is the basis function matrix for stresses, and C is the vector of unknown coefficients in the general solution which will be determined in step 5. Then, the equivalent nodal forces in step 7 can be computed through:

$$Q_m = -\mathbf{G}_m C \tag{71}$$

≣

and

$$\mathbf{G}_m = \int_{S_m} \mathbf{N}^t \mathbf{n} \mathbf{P} \, \mathrm{d}S \tag{72}$$

where *m* denotes the number for a finite element,  $Q_m$  are nodal forces, N is the matrix of the element shape functions, **n** is the matrix of the normal direction cosines. Although the matrix **P** has the singularity of order  $1/\sqrt{r}$  at the crack-front, the functions in **P** decay very rapidly with the distance from the crack-front. Thus, the matrices  $\mathbf{G}_m$  are calculated only at the external boundary-surface elements which satisfy the condition  $r_{\min} < 5a_1$ , where  $r_{\min}$  is the distance of the closest nodal point of each external boundary-surface element from the center of the ellipse and  $a_1$  is the semi-major axis of the ellipse.

To save computation time, the  $G_m$  matrix can be calculated only once prior to the start of the iteration process. Thus, the equivalent nodal forces  $Q^i$  in each iteration can be evaluated without integration.

## 2.2 3D ALTERNATING TECHNIQUES FOR PART-ELLIPTCAL SURFACE CRACKS

The VNA solution given in 2.1 serves as Solution 1 required in the alternating technique.

Now, some comments concerning the solution of *surface flaw* problems in finite bodies, through the present  $\_$  procedure, are in order. Since the analytical solution for an elliptical crack in an infinite solid is implemented  $\_$  as solution (1), it is necessary to define the residual stresses over the entire crack plane including the  $\_$  fictitious portion of the crack which lies outside of the finite body. Moreover, it is well known that the  $\_$ 

accuracy of the "least-squares" function interpolation inside the interpolated region can be increased with the number of polynomial terms; however, the interpolating curve may change drastically outside the region of interpolation. For these reasons, in [3] numerical experimentation was carried out to arrive at an optimum pressure distribution on the crack surface extended into the fictitious region. For a semi-elliptical crack which lies in the region of  $-a_1 \leq x_1 \leq a_1$  and  $0 \leq x_2 \leq a_2$ , it was concluded that the fictitious pressure, which, for the region of  $-a_2 \leq x_2 \leq 0$ , remains constant in the  $x_2$  direction but varies in the  $x_1$ direction, gives the best result among the several numerical experiments performed in [3], even though the results for other types of assumed pressure in the fictitious region differed only slightly ( $\pm 2\%$ ).

This procedure of fictitious pressure distribution for a semi-elliptical surface crack was successfully used on the analyses of surface cracks, in finite-thickness plates subject to remote tension as well as remote bending [3], and in pressure vessels [18].

Based on the studies in [19], the following "fictitious" stress distribution is recommended for quarterelliptical surface cracks. For the first quadrant  $(x_1, x_2 \ge 0)$  (namely, the actual surface crack), the residual stress can be calculated by the finite element method and is a function of the coordinates  $x_1$  and  $x_2$ . For the other quadrants, the fictitious residual stress is defined as

$$\sigma_{33}^{R} = \begin{cases} \sigma_{33}^{R}(0, x_{2}) & \text{for the second quadrant} \quad (x_{1} \leq 0, x_{2} \geq 0) \\ \sigma_{33}^{R}(0, 0) & \text{for the third quadrant} \quad (x_{1} \leq 0, x_{2} \leq 0) \\ \sigma_{33}^{R}(x_{1}, 0) & \text{for the fourth quadrant} \quad (x_{1} \geq 0, x_{2} \leq 0) \end{cases}$$
(73)

The above alternating method has been successfully applied to the problem of semi-elliptical surface flaws in plates subjected to tension and bending [8, 3], semi-elliptical surface flaws in the meridonal direction at the outer and inner surfaces of pressurized thick and thin cylindrical vessels [18], quarter-elliptical surface flaws emanating from pin-holes in attachment lugs [19], *multiple* coplanar embedded elliptical flaws in an infinite solid subject to arbitrary crack-face tractions [20], and *multiple* semi-elliptical surface flaws in the meridonal as well as circumferential directions in cylindrical pressure vessels [14, 21].

The nature of singularity at the point where the crack-front interests the free-surface is still not yet completely understood. The consensus emerging from the literature of a weaker singuarity (than  $1/\sqrt{r}$ ) at a normal crack/surface interaction has been corraborated recently, in [22]. These authors present two independent numerical analysis techniques for the investigation of some global crack/surface interaction problems. They summarize their findings, thus: "the decays in the energy release rates found as the free surface is approached in the various problems treated are probably not significant from a fracture toughness testing point of view and not of major consequence in cyclic life calculations, although there are some indications that this may not be the case if near-surface residual stress fields are present; and that these variations in energy release rate can be compensated for by relatively minor perturbations in crack-front profiles".

Thus the results obtained by the above finite element alternating method based on the VNA solution may be thought of as being of adequate accuracy for most engineering applications. Recently, [23] a more efficient alternating method for the analysis of a group of interacting multiple elliptical cracks was developed, by taking account of geometrical symmetries of crack shapes and location in conjunction with the symmetry of the VNA solution.

Intensive studies of the performance of the finite-element alternating method have been made in [24] for small surface and corner cracks, and in [25] for a part-elliptical surface crack in a cylinder. From the performance studies of the finite-element alternating method, [25] summarized the attractive features of the alternating technique as follows:

(i) The method models only the uncracked solid with finite elements: hence, no special modeling of the crack front is required. In addition, the finite element mesh at the location of the crack, in the uncracked solid, can be completely arbitrary in geometry.

(ii) The method uses the closed-form solution for a crack in an infinite solid which can accommadate arbitrary tractions on the crack surfaces and, therefore, can handle complex loading conditions.

(iii) The stress-intensity factors, including the individual modes, are obtained as part of the solution, in an analytical form, and, hence, post-processing of the output data, as is usually done in the finite-element method, is not needed.

(iv) Several crack configurations could be analysed with a single arbitrary mesh idealization of the uncracked solid, whereas the conventional finite-element method requires a different mesh idealization of the cracked structure for each crack configuration. Thus, this method can efficiently generate very accurate stress-intensity factor weight functions or influence functions, for a variety of crack aspect ratios, in a single computer run.

The above applications of the alternating technique were limited to mode I cases. Recently, a *mixed-mode* alternating finite-element technique in conjunction with the VNA solution (with further improvements in algebraic details), has been developed in [7]. They evaluated the polynomial influence functions for an infinite solid with an elliptical crack subject to shear loading, and for a cantilever beam with a semi-elliptical surface crack subject to end load.

Applications of the finite-element alternating method have been made in [26, 27, 28], for fracture mechanics analyses of various offshore structural components, such as stiffened plate and shells, tethers, or risers. A recent literature survey [29] pointed out that the alternating method is most efficient for stress intensity factor analyses of *planar* surface or embedded flaws in complex geometries such as intersecting tubular structures, etc.

# 2.3 2 D ALTERNATING TECHNIQUES FOR LINE CRACKS

As mentioned earlier, the general solution for a crack subject to arbitrary crack-face tractions i.e. Solution 1 is required. The general solution for an infinite 2D anisotropic body, developed along the lines of [30], is given below. Following the solution procedure in [31], the stress and displacement field can be expressed in terms of two potential  $\phi$  and  $\psi$  as follows:

$$\tau_{xx} = 2 \operatorname{Re}[s_1^2 \phi'(z_1) + s_2^2 \psi'(z_2)]$$
  

$$\tau_{yy} = 2 \operatorname{Re}[\phi'(z_1) + \psi'(z_2)]$$
  

$$\tau_{xy} = -2 \operatorname{Re}[s_1 \phi'(z_1) + s_2 \psi'(z_2)]$$
  

$$u = 2 \operatorname{Re}[p_1 \phi(z_1) + p_2 \psi(z_2)]$$
  

$$v = 2 \operatorname{Re}[q_1 \phi(z_1) + q_2 \psi(z_2)]$$
(74)

where

$$s_{1} = \mu_{1} = \alpha_{1} + i\beta_{1}, \qquad s_{2} = \mu_{2} = \alpha_{2} + i\beta_{2}$$

$$\mu_{3} = \bar{\mu}_{1} \qquad \mu_{4} = \bar{\mu}_{2}$$

$$z_{1} = x + s_{1}y \qquad z_{2} = x + s_{2}y \qquad s_{1} \neq s_{2} \qquad (75)$$

where  $\alpha_j \beta_j$  (j = 1, 2) are real constants.  $\mu_j$   $(j = 1, 2, \dots 4)$  are the roots of the characteristic equation

$$a_{11}\mu_j^4 - 2a_{16}\mu_j^3 + (2a_{12} + a_{66})\mu_j^2 - 2a_{26}\mu_j + a_{22} = 0$$
(76)

where  $a_{ij}$   $(i, j = 1, 2, \dots, 6)$  are the material constants of generalized Hooke's law

$$\varepsilon_x = a_{11}\tau_{xx} + a_{12}\tau_{yy} + a_{16}\tau_{xy}$$
  

$$\varepsilon_y = a_{12}\tau_{xx} + a_{22}\tau_{yy} + a_{26}\tau_{xy}$$
  

$$\gamma_{xy} = a_{16}\tau_{xx} + a_{26}\tau_{yy} + a_{66}\tau_{xy}$$
(77)

The other constants in Eq. (91) are defined as:

$$p_{1} = a_{11}s_{1}^{2} + a_{12} - a_{16}s_{1} \qquad p_{2} = a_{11}s_{2}^{2} + a_{12} - a_{16}s_{2}$$

$$q_{1} = \frac{a_{12}s_{1}^{2} + a_{22} - a_{26}s_{1}}{s_{1}} \qquad q_{2} = \frac{a_{12}s_{2}^{2} + a_{22} - a_{26}s_{2}}{s_{2}}$$
(78)

Suppose a line crack on y = 0  $|x| \le a$  in an infinite plane is inflated by equal and opposite tractions, over the faces of the crack, given by

$$\tau_{yy} - i\tau_{xy} = -[p(t) + is(t)], \qquad |t| \le a$$
(79)

with zero tractions at infinity. Then the potential functions can be written as below:

$$\begin{aligned}
\phi'(z_1) &= \phi'_1(z_1) + \phi'_2(z_1) \\
\psi'(z_2) &= \psi'_1(z_2) + \psi'_2(z_2)
\end{aligned}$$
(80)

where

$$\left(\frac{s_2 - s_1}{s_2}\right) \phi_1'(z_1) = -\frac{X(z_1)}{2\pi i} \int_{-a}^{a} \frac{p(t) dt}{[X(t)]^+(t - z_1)} \left(\frac{s_1 - s_2}{s_1}\right) \psi_1'(z_2) = -\frac{Y(z_2)}{2\pi i} \int_{-a}^{a} \frac{p(t) dt}{[Y(t)]^+(t - z_2)}$$

$$(81)$$

where

$$X(z_1) = (z_1 + a)^{-1/2} (z_1 - a)^{-1/2}$$
  

$$Y(z_2) = (z_2 + a)^{-1/2} (z_2 - a)^{-1/2}$$
(82)

and

$$(s_{1} - s_{2})\phi_{2}'(z_{1}) = -\frac{X(z_{1})}{2\pi i} \int_{-a}^{a} \frac{s(t)dt}{[X(t)]^{+}(t - z_{1})}$$
  

$$(s_{2} - s_{1})\psi_{2}'(z_{2}) = -\frac{Y(z_{2})}{2\pi i} \int_{-a}^{a} \frac{s(t)dt}{[Y(t)]^{+}(t - z_{2})}$$
(83)

We approximate the applied crack-face tractions in the form [30]

$$p(t) + is(t) = -\sum_{n=1}^{N} b_n U_{n-1}(t) \qquad |t| \le a$$
(84)

where  $U_{n-1}(t)$  is the Chebyshev polynomials of the second kind and is defined as

$$U_n = \sin[(n+1)\theta] / \sin\theta \qquad t = a\cos\theta \tag{85}$$

It could be easily shown that:

$$2\phi'(z_1) = \left(\frac{s_2}{s_2 - s_1}\right) \sum_{n=1}^N c_n G_{n-1}(z_1) + \left(\frac{1}{s_1 - s_2}\right) \sum_{n=1}^N d_n G_{n-1}(z_1)$$
  

$$2\psi'(z_2) = \left(\frac{s_1}{s_1 - s_2}\right) \sum_{n=1}^N c_n G_{n-1}(z_2) + \left(\frac{1}{s_2 - s_1}\right) \sum_{n=1}^N d_n G_{n-1}(z_2)$$
(86)

where

$$c_n = \operatorname{real}(b_n)$$
  
 $d_n = i(\operatorname{imag}(b_n))$ 

$$2\phi(z_1) = \left(\frac{s_2}{s_2 - s_1}\right) \sum_{n=1}^N c_n \frac{R_n(z_1)}{n} + \left(\frac{1}{s_1 - s_2}\right) \sum_{n=1}^N d_n \frac{R_n(z_1)}{n}$$
  

$$2\psi(z_2) = \left(\frac{s_1}{s_1 - s_2}\right) \sum_{n=1}^N c_n \frac{R_n(z_2)}{n} + \left(\frac{1}{s_2 - s_1}\right) \sum_{n=1}^N d_n \frac{R_n(z_2)}{n}$$
(87)

 $\operatorname{and}$ 

$$G_{n-1}(z_{\alpha}) = -(z_{\alpha}^2 - a^2)^{-1/2} R_n(z_{\alpha}) \qquad (\alpha = 1, 2)$$
(88)

$$R_n(z_\alpha) = a\{z_\alpha/a - (z_\alpha^2/a^2 - 1)^{1/2}\}^n \qquad (\alpha = 1, 2)$$
(89)

The stress intensity factors  $K_j$  (j = I, II) are defined in a manner consistent with those for isotropic materials.

$$K_{I} = 2\sqrt{2\pi} \left(\frac{s_{2}-s_{1}}{s_{2}}\right) \lim_{z_{1}\to a} (z_{1}-a)^{1/2} \phi_{1}'(z_{1})$$

$$K_{II} = 2\sqrt{2\pi} (s_{2}-s_{1}) \lim_{z_{1}\to a} (z_{1}-a)^{1/2} \phi_{2}'(z_{1})$$
(90)

It is easy to show that

$$K_{I} - iK_{II} = -\sqrt{\pi a} \sum_{n=1}^{N} b_{n} \quad \text{for} \quad x = a$$
  
$$K_{I} - iK_{II} = \sqrt{\pi a} \sum_{n=1}^{N} (-1)^{n} b_{n} \quad \text{for} \quad x = -a$$
(91)

# **3** THE CURRENT STATE-OF-THE-ART: "THE DISPLACEMENT COMPATIBILITY APPROACH" FOR CRACKED, STIFFENED FUSELAGE PANELS

The methods that are currently used for the fracture analysis of stiffened panels are based on the socalled "displacement-compatibility approach" as developed in [32, 33], and others see, for instance, [34]. The "displacement-compatibility" approach is schematically illustrated in Fig. 3. This method can be classified as a general "flexibility matrix" approach in structural mechanics. In this approach, the effects of the fasteners on the sheet are represented by a series of concentrated forces on the sheet, which may contain multiple cracks, and the sheet is subjected, in addition, to the far-field hoop stress [labelled henceforth as problem A].

The stringers, in turn, are acted upon by a series of concentrated forces, equal in magnitude and opposite in direction to those in problem A. The dislacement compatibility between the sheet and the stiffener, at each fastener location, is enforced, by taking into account the fastener flexibility, as:

$$v_{sheet} = v_{stiffener} + v_{fastener} \tag{92}$$

### (at each fastener location)

Problem A above is again broken-up, by the superposition principle, into 3 problems, labelled here as problems B,C and D, respectively. Problem B is that of a sheet, subjected to hoop membrane stress, and containing multiple cracks. Problem C is that of the uncracked sheet subjected to a series of concentrated forces as in Problem A, at the fastener locations. Problem D is that of a sheet containing multiple cracks, each of which is subjected to tractions that are equal in magnitude, but opposite in direction, to the residual tractions existing at the locations of the cracks in an otherwise uncracked sheet, as in Problem C.

$$v_{sheet} = v_B + v_C + v_D \tag{93}$$

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$$v_b + v_C + v_D = V_{stiffener} + v_{fastener} \tag{94}$$

(at each fastener location)



Fig. 3

In the methods used in [32, 33] and [34], one needs: (i) analytical solutions for displacements in the stringer at a given fastener location due to concentrated forces at any other fastener location; (ii) analytical solutions for displacements in the sheet, at a given fastener location, for the Problem B; (iii) analytical solution for displacements, in Problem C, at any given fastener location, due to a set of concentrated forces at any other fastener location; and (iv) analytical solutions for displacements in the sheet, at a given fastener location, for Problem D. It should be noted that not all of these analytical solutions are readily available in the literature; and, when available, they are not suitable for the multiple crack case. When the displacement solutions (i) through (iv) above are used in Equation (94), one obtains the linear system of equations:

$$C \cdot \underline{F} = \underline{A} \tag{95}$$

or

where C is the "flexibility matrix" as assembled from solutions (i) through (iv) above,  $\underline{F}$  is the vector of fastener forces (at all fastener locations) acting on the sheet, and  $\underline{A}$  is the given vector that accounts for fastener flexibility, etc. In equation (95) the vector  $\underline{F}$  is the unknown.

Once Eq. (95) is solved for the vector  $\underline{F}$ , the problem of the fracture of the sheet alone, i.e., Problem A above, is well-posed [i.e., the far-field hoop stress, as well as the fastener reaction forces, are known]. This Problem A is broken up into Problems B, C, and D as above. The stress singualrities at the crack-tips arise only in *Problems B and D*. Thus, the stress-intensity factor at a crack-tip can be written as:

$$K = K_B + K_D \quad (at \ each \ crack - tip) \tag{96}$$

It should be recognized that the key element in the approach described above, is the assembly of the matrix C in Eq. (95) from the various analytical solutions as described under (i) through (iv) earlier. Thus, the approach is by no means a trivial one for adaptation on a modern personal digital computer. Further, in writing Eq. (96), Problems B and D are assumed to correspond to *infinite* domains in [33] and [34]. Thus, finite-dimension correction factors are ignored, which is not necessarily a conservative approximation.

# **3.1** A SIMPLE FINITE ELEMENT ALTERNATING METHOD FOR STIFFENED PANELS WITH MULTIPLE CRACKS

A straight-forward finite element modeling (including the crack-tip singularities) of a cracked, stiffened, fuselage panel, especially when used for a parametric study during the service/design phase of the aircraft, is prohibitively expensive [32]. In order to circumvent this, displacement compatibility methods wherein the panel is assumed to be infinite, have been developed in [32, 33] and [34], among others.

In the following, a very simple alternative to the current state-of-the-art displacement compatibility approach, is presented. The proposed method essentially involves two steps: (i) evaluation of the fastener reaction forces on the sheet through a very course finite element model, wherein the details of the crack-tip singularities are not modeled; and (ii) the application of the finite-element-alternating method, employing the same finite element mesh as in stage (i) for the *unstiffened*, *uncracked* panel, with the applied membrane hoop stress, and the fastener reaction forces as solved in stage (i), to determine the fracture parameters. The present method may be classified as the now-standard finite element stiffness (or displacement) method.

### Stage (i) Solution

The skin is discretized using a very coarse finite element mesh, such that the fastener locations are taken to be the nodes of the finite element mesh. The axial deformation of the stiffener is modeled by using the conventional "truss-type" elements. Since the fastener shear forces are usually offset from the stiffener neutral axis, an out-of-plane bending is also induced in the stiffeners. The out-of-plane bending deformation of the stiffener, between two fasteners, is given by the elementary beam theory:

$$\delta_b = C\left(\frac{ML}{E_{St}I}\right) = a_b F \tag{97}$$

where:

C = distance from the neutral axis of the stiffener to the point of action of the fastener shear force I = stiffener cross-sectional inertia

L = distance between two fasteners, or the length of the stiffener "truss" element

E = Young's modulus of stiffener material

 $M=F\cdot C$  ; where F is the force in the truss element

Note that  $\delta_b$  is in the same direction, as the stiffener axial force, F. From (97), it is seen that:

$$a_b = \left(\frac{C^2 L}{E_{St} I}\right) \tag{98}$$

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The axial stretch of the stringer is given by:

$$\delta_a = \frac{F}{E_{St}A} \equiv (a_S)F \quad ; \quad a_S = \frac{1}{E_{St}A} \tag{99}$$

We use the total axial deformations at the ends of the stringer, at the points where the stringer is attached to the skin, as the generalized degrees of freedom for the stringer. For these degrees of freedom, the stiffness matrix of the stringer element is given by:

$$K_{St}^{e} = \frac{1}{(a_{b} + a_{s})} \qquad \frac{-1}{(a_{b} + a_{s})} \qquad (100)$$
$$\frac{-1}{(a_{b} + a_{s})} \qquad \frac{1}{(a_{b} + a_{s})}$$

The flexibility of the fasteners has been found [33] to be an important factor that influences the stressintensity factors for a crack in the stiffened skin. If Q is the shear force acting on the fastener, the shear deformation of the fastener can be represented by the empirical relation [33].

$$\delta_F = \frac{1}{E_{Sh}D} \left[ A + C \left( \frac{D}{B_1} + \frac{D}{B_2} \right) \right] F \equiv a_F Q \tag{101}$$

where

 $E_{Sh}$  = modulus of sheet material D = rivet diameter  $B_1$  and  $B_2$  = thicknesses of joined sheets A = 5.0 for Al rivets and 1.66 for steel fasteners C = 0.8 for Al rivets and 0.86 for steel fasteners

The "stiffness" of the rivet in shear is thus given by:

$$K_F = \frac{E_{Sh}D}{\left[A + C(\frac{D}{B_1} + \frac{D}{B_2})\right]}$$
(102)

consider, for simplicity (but without any loss of generality), that the skin is discretized into finite elements, with nodes being only at the locations of the fasteners; and likewise, the stringers are discretized into finite elements with nodes being only at the fastener locations. Let the number of fasteners be N. Let the number of stringer elements be  $N_{St}$ ; and the number of sheet elements be  $N_{Sk}$ . Let the generalized displacements of the skin at the nodes of the finite element mesh be denoted as  $q_{sk}$ ; and those of the stringers at the nodes be denoted by  $q_{st}$ . Then the total strain energy of the stiffened fuselage skin, with flexible fasteners, is given by:

$$W = \sum_{ele=N_{Sk}} \frac{1}{2} q_{Sk}^{t_e} K_{Sk}^e q_{Sk}^e + \sum_{N_{St}} \frac{1}{2} q_{St}^{t_e} K_{St}^e q_{St}^e + \sum_{N} \frac{1}{2} K_F (q_{Sk} - q_{St})^2$$
(103)

where  $q_{Sk}^{e}$  is the vector of nodal displacements of a skin element; and  $q_{st}^{e}$  is the vector of nodal displacements of a stringer (stiffener) element.

Let  $q_{Sk}$  be the master vector of nodal displacements of the skin;  $K_{Sk}$  the assembled nodal stiffness matrix of the skin;  $q_{St}$  be the master vector of nodal displacements of the stringers; and  $K_{St}$  the assembled nodal

stiffness matrix of the stringers. Let  $K_F$  be the "assembled" (diagonal) stiffness matrix of the fasteners; i.e.,

$$N \times N$$

Then, W of Eq. (103) can be written as:

$$W = \frac{1}{2} q_{Sk}^{t} (K_{Sk} + K_{F}) q_{Sk} + \frac{1}{2} q_{St}^{t} (K_{St} + K_{F}) q_{St} - q_{Sk}^{t} K_{F} q_{St}$$
(105)

If the fastener flexibility is ignored, then  $q_{Sk} \equiv q_{St}$ ; and equation (105) reduces to:

$$W = \frac{1}{2} q_{Sk}^{t} (K_{Sk} + K_{St}) q_{Sk}$$
(106)

The potential of the external forces (the hoop stress in the fuselage) may be represented, in general, as:

$$U = (\boldsymbol{q}_{Sk}^{t} \boldsymbol{Q}_{Sk} + \boldsymbol{q}_{St}^{t} \boldsymbol{Q}_{St}) \tag{107}$$

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Let  $\pi = W - U$ . The finite element equations that arise from the vanishing of the variation of  $\delta \pi$  (ie.,  $\delta \pi = 0$ ) are given by:

$$(K_{Sk} + K_F)q_{Sk} - K_Fq_{St} = Q_{Sk}$$

$$(108a)$$

and

$$(K_{St} + K_F)q_{St} - K_Fq_{Sk} = Q_{St}$$

$$(108b)$$

for the case of flexible fasteners. Eqs. (108) may be rearranged as:

$$(K_{Sk} + K_F) \quad (-K_F) \qquad q_{Sk} \qquad Q_{Sk} \tag{109}$$

$$-(K_F)$$
  $K_{St}+K_F$   $q_{St}$   $Q_{St}$ 

After the imposition of appropriate boundary conditions,  $q_{Sk}$  and  $q_{St}$  can be solved for, from Eq. (109). Once  $q_{Sk}$  and  $q_{St}$  are solved for, the *reactions* of the *stiffeners* on the skin, at the locations of the fas eners, can be easily calculated as:

$$F_{stiffener} = K_F(q_{St} - q_{Sk}) \equiv Q_{St} - K_{St}q_{St}$$
(110)

with care being exercised to determine the direction of these reactive forces.

Once the effects of the stringer (with flexible rivets) on the skin are determined, one can consider the freebody diagram of the *cracked skin alone* [See Fig. 4]; the skin being subject to the far-field hoop stresses, and the stringer reaction forces. The stress-intensity factors for the multiple cracks in the skin, subjected to these forces, may now be determined in the stage (ii) alternating solution.





Solve the <u>uncracked sheet</u> using very coarse



FEM.

[Pressurized crack only]

Fig. 4

# Stage (ii) solution

In the finite element alternating method to determine the k-factors for multiple cracks in the skin [after the effects of the stiffeners and fasteners are isolated as in stage (i) above], one may employ the same finite element mesh as in stage (i), to model the *uncracked*, *unstiffened* free-body of the skin alone. In stage (ii), as explained in Section 2 of this paper, one needs to know: (a) the tractions to be erased at the locations of the cracks in an otherwise uncracked skin; and (b) the analytical solution for a crack subjected to arbitrary tractions in an infinite body.

# 3.2 AN EXAMPLE PROBLEM ANALYZED BY THE ALTERNATING METHOD

To illustrate the application of the proposed "direct-stiffness" finite element alternating methodology, for analyzing cracked stiffened fuselage panels, the example problem as shown in Fig. 5 has been analyzed. The problem involves a stiffened panel, with a central crack and a broken stiffener, with the crack being symmetrically located with respect to the broken stiffener. For the purpose of illustration of the types of finite element meshes that may be sufficient in the present methodology, the fastener flexibility is ignored in the analysis. A typical finite element mesh that is used is shown in the inset of Fig. 5. The stress-intensity magnification factors for various crack lengths are shown in Fig. 6, which also shows the convergence of the results with mesh refinement. It is seen that the results are insensitive to the finite element mesh size, with acceptable results being obtained for a  $(16 \times 3)$  mesh. It is also seen that the stress-intensity magnification factor decreases as the crack-tip approaches the stiffener.

### 4 MSD NEAR A ROW OF FASTENER HOLES

A typical multiple site damage near a row of fastener holes is illustrated schematically in Figs. 7 and 8. In example 4.2, the crack was supposed to be much larger in size as compared to the rivet diameter; and thus, the detailed distribution of stresses, on the hole-surface in the skin, due to fastener reaction forces, was not a factor in determining the fracture parameters near the crack-tip. In fact, in Section 3, these reaction forces were simply treated as concentrated forces. However, in a typical MSD situation as depicted in Figs. 7 and 8, the fastener interaction stresses on the hole-surface in the skin are likely to play a significant role in the fracture parameters for the cracks emanating from fastener holes in the skin.

The fastener reaction forces can still be determined for the lap-splice joint configuration of Figs. 7 and 8, through the direct-stiffness finite element method presented above in Section 4. Once these reaction forces, treated as concentrated forces in Section 3, are determined, one may use the known elasticity solution, to approximate the detailed stress-field on the hole-surface in the skin, that is equivalent to these concentrated forces. Under the action of these fastener interaction stress-fields, the stress-intensity factors for MSD near fastener holes can be determined using the alternating method described earlier. Such methods are currently being developed at Georgia Tech.



Fig. 5

**HANNES** 





Fig. 6



Fig. 7



# 5 ANALYSIS OF CRACKS WITH BONDED REPAIR PATCHES

Repair of cracked structures with bonded composite laminate patches appears to be a cost-effective and reliable way of enhancing the fatigue life of fuselage panels [35]. The load-transfer to the composite patch is the primary cause for lessening the stress- intensity near the crack-tip in the main panel, and thus improves its fatigue life. This reduction in the stress-intensity for a crack in the main panel depends on the laminate properties, its thickness, the properties of the adhesive material, and the thickness of the adhesive layer. Analytical solutions have been obtained for: (i) the case of an infinite isotropic material panel with a crack, bonded to an infinite orthotropic panel, and (ii) the case of an infinite isotropic panel with a crack emanating from a hole, bonded to an infinite orthotropic panel. These analytical solutions have been repeated for orthotropic patches of infinite width and finite height (perpendicular to the crack-line in the isotropic panel), with the edges of the patch being parallel to the crack-axes. These analytical solutions, when implemented in the finite-clement alternating method, would provide useful design tools for *designing composite patches* of *arbitrary shape* to arrest further growth of a crack emanating from stress-concentrations in the main panel. Such a design tool, for implementation on a personal computer, is being developed at Georgia Tech.

# 6 WEIGHT-FUNCTIONS FOR 2 AND 3-D ELASTIC CRACK PROBLEMS

The concept of weight functions for elastic crack problems dates back to the work in [36] and [37] see also [38]. The "weight function" may generally be viewed as the appropriately normalized rate of change of displacements (at the surface where tractions are applied, or in the domain where body forces are applied) due to a unit change in the crack length for a *reference state* of loading. The practical importance of the concept of the weight functions lies in the fact that, when the weight functions are evaluated from a *(perhaps simple) reference state of loading*, then the stress-intensity factors for any arbitrary state of loading can be computed by using an integral of the weight function for the reference state at the

same point.

The energy-release due to a unit crack-extension in a cracked elastic body, subject to a system of surface tractions, (We assume, for simplicity, that the surface  $S_u$  where non-zero displacements are prescribed, is zero. One can easily generalize the ensuing discussion to the situation when  $S_u$  is nonzero.), and body forces, is given by:

$$\mathcal{G} = \int_{S_t} t_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \,\mathrm{d}S + \int_V f_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \,\mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}a} \int_V W \,\mathrm{d}V \tag{111}$$

where  $t_i$  are tractions applied at the surface  $S_t$ ;  $f_i$  are body forces in the domain V;  $u_i$  are displacements, and W is the strain-energy density (internal energy in mechanical work). Eq. (111) may be written as:

$$\mathcal{G} + \int_{S_i} \frac{\mathrm{d}t_i}{\mathrm{d}a} u_i \,\mathrm{d}S + \int_V \frac{\mathrm{d}f_i}{\mathrm{d}a} u_i \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}a} \{\int_{S_i} t_i u_i \,\mathrm{d}S + \int_V f_i u_i \,\mathrm{d}V - \int_V W \,\mathrm{d}V\}$$
(112a)

or

$$\mathcal{G} da + \int_{S_i} dt_i u_i dS + \int_V df_i u_i dV = -d\pi$$
(112b)

Let the reference load state be characterized by a parameter  $\lambda$ . Thus,

$$\mathcal{G}\,\mathrm{d}a + \mathrm{d}\lambda q(\lambda) = -\mathrm{d}\pi\tag{113}$$

where

$$q(\lambda) = \int_{S_t} \hat{t}_i u_i \,\mathrm{d}S + \int_V \hat{f}_i u_i \,\mathrm{d}V \tag{114}$$

where,  $dt_i = d\lambda \hat{t}_i$ ;  $df_i = d\lambda \hat{f}_i$ ; and, in general, in a nonlinear elastic problem, the generalized displacement q is a nonlinear function of  $\lambda$ . Equation (113) implies that:

$$\left(\frac{\partial \mathcal{G}}{\partial \lambda}\right)_{a} = \left(\frac{\partial q}{\partial a}\right)_{\lambda} \tag{115}$$

Consider a linear-elastic homogeneous solid, that is in general *anisotropic*, and consider the case when the crack is at an arbitrary angle to the material directions, and under a general mixed mode loading. The energy release rate,  $\mathcal{G}$ , for a mixed-mode crack in a monoclinic anisotropic solid may be written as:

$$\mathcal{G} = AK_I^2 + BK_{II}^2 + CK_I K_{II} \tag{116}$$

where

$$A = -\frac{\pi}{2}a_{22}\operatorname{Im}\left(\frac{\mu_{1}+\mu_{2}}{\mu_{1}\mu_{2}}\right)$$
  

$$B = \frac{\pi}{2}a_{11}\operatorname{Im}(\mu_{1}+\mu_{2})$$
  

$$C = \frac{\pi}{2}\left\{-a_{22}\operatorname{Im}\left(\frac{1}{\mu_{1}\mu_{2}}\right)+a_{11}\operatorname{Im}(\mu_{1}\mu_{2})\right\}$$
(117)

where  $a_{ij}$  are material constants in the relation  $\varepsilon_i = a_{ij}\sigma_j$  (i, j = 6) and  $\mu_j$  are the complex roots of the characteristic equation,  $a_{11}\mu_j^4 - 2a_{16}\mu_j^3 + (2a_{12} + a_{66})\mu_j^2 - 2a_{26}\mu_j + a_{22} = 0$ . See [31] for further details. In the case of isotropy, (118) reduces to

$$A = \frac{1}{H}; \quad B = \frac{1}{H}; \quad C = 0$$
  

$$H = E/(1 - \nu^2) \quad \text{plane strain;} \quad H = E \quad \text{plane stress}$$
(118)

We now consider the simultaneous action of 2 load-systems on the cracked body. The load system is:

$$(\lambda^1 \hat{t}_i^1 + \lambda^2 \hat{t}_i^2)$$
 at  $S_i$ ; and  $(\lambda^1 \hat{f}_i^1 + \lambda^2 \hat{f}_i^2)$  in  $V$  (119)

When (119) is used in (113) one obtains:

$$\mathcal{G} da + d\lambda^R C_{Rn} \lambda^n = -d\pi; \qquad R, n = 1, 2$$
(120)

or

$$\frac{\partial \mathcal{G}}{\partial \lambda^R} = \frac{\mathrm{d}C_{Rn}}{\mathrm{d}a} \lambda^n \tag{121}$$

and

$$\frac{\mathrm{d}C_{Rn}}{\mathrm{d}a} = \int_{S_i} \hat{t}_i^n \frac{\mathrm{d}\hat{u}_i^R}{\mathrm{d}a} \mathrm{d}S + \int_V \hat{f}_i^n \frac{\mathrm{d}\hat{u}_i^R}{\mathrm{d}a} \mathrm{d}V$$
  

$$R, n = 1, 2 \text{ load cases } i = 1, 2, 3$$
(122)

The K-factors under the combined mode loading are

$$K_I = \hat{K}_I^m \lambda^m; \qquad K_{II} = \hat{K}_{II}^m \lambda^m \tag{123}$$

[sum on m = 1,2] and

$$\mathcal{G} = A\hat{K}_{I}^{m}\hat{K}_{I}^{n}\lambda^{m}\lambda^{n} + B\hat{K}_{II}^{m}\hat{K}_{II}^{n}\lambda^{m}\lambda^{n} + C\hat{K}_{I}^{m}\hat{K}_{II}^{n}\lambda^{m}\lambda^{n}$$
(124)

[sum m, n = 1, 2]. Using (124) in (121), and observing that the resulting equation is valid for arbitrary  $\lambda^1$  and  $\lambda^2$  one obtains:

$$(2A+C)\hat{K}_{I}^{R}\hat{K}_{I}^{n} + (2B+C)\hat{K}_{II}^{R}\hat{K}_{II}^{n} = \frac{\mathrm{d}C_{nR}}{\mathrm{d}a}$$

$$= \int_{S_{i}} \hat{t}_{i}^{n}\frac{\mathrm{d}\hat{u}_{i}^{R}}{\mathrm{d}a}\mathrm{d}S + \int_{V}\hat{f}_{i}^{n}\frac{\mathrm{d}\hat{u}_{i}^{R}}{\mathrm{d}a}\mathrm{d}V$$

$$R, n = \text{load cases}$$
(125)

Let R be a known reference load-state, for which the solution, i.e.,  $\hat{K}_{I}^{R}$ ,  $\hat{K}_{II}^{R}$ , and  $(d\hat{u}_{i}^{R}/da)$  are known, and n is an arbitrary load-state for which the mixed mode factors  $K_{I}^{n}$  and  $K_{II}^{n}$  are to be computed. Eq. (126) is thus a single equation governing the two unknowns  $K_{I}^{n}$  and  $K_{II}^{n}$ . By writing Eq. (126) for two known and linearly independent reference states, two equations for two unknowns  $K_{I}^{n}$  and  $K_{II}^{n}$  can be obtained. The solution of these equations can be seen to be:

$$K_{I} = \frac{\hat{K}_{II}^{R2}\lambda^{m}}{K^{R}(2A+C)} \frac{\mathrm{d}C_{mR1}}{\mathrm{d}a} - \frac{\hat{K}_{II}^{R1}\lambda^{m}}{K^{R}(2A+C)} \frac{\mathrm{d}C_{mR2}}{\mathrm{d}a}$$
(no sum on m) (127)

$$K_{II} = \frac{\hat{K}_{I}^{R1}\lambda^{m}}{K^{R}(2B+C)} \frac{\mathrm{d}C_{mR2}}{\mathrm{d}a} - \frac{\hat{K}_{I}^{R2}\lambda^{m}}{K^{R}(2B+C)} \frac{\mathrm{d}C_{mR1}}{\mathrm{d}a}$$
(no sum on m) (128)

and

$$K^{R} = \hat{K}_{I}^{R1} \hat{K}_{II}^{R2} - \hat{K}_{I}^{R2} \hat{K}_{II}^{R1}$$
(129)

where  $(dC_{mR1}/da)$  etc. are defined from (122) by replacing R by R1 etc. It is seen that for  $K^R$  to be *nonzero*, the reference states should not be both of either Mode I or Mode II. Furthermore, in a general

anisotropic body with an arbitrarily oriented crack, the reference states R1 and R2 can be taken to either loads on external surfaces, or tractions on the crack-faces themselves.

Thus, to evaluate the mixed-mode load factors for any reference state m, one only needs the appropriately normalized weight-functions,  $(du_i^{R1}/da)$  and  $(du_i^{R2}/da)$ . In the following we discuss some recent work on computational methods for these weight functions, for anisotropic or isotropic materials.

6.1 WEIGHT FUNCTIONS, USING FINITE ELEMENT/BOUNDARY ELEMENT MODELS OF ONLY UNCRACKED STRUCTURES

Recently [15, 39] and [40, 41] simple methods for computing weight functions were developed using finite element or boundary element models of only the *uncracked* structure.

It is worth noting that, due to the complications of the fundamental solutions (for a point load) for a general anisotropic medium, the boundary element method is not convenient for application to the anisotropic soilds. However, it is well known that the Galerkin finite element method does not have this restriction. The authors in [15, 39] use the finite-element alternating method (for general anisotropic solids) and the authors in [40, 41] use the boundary-element alternating method (for isotropic solids) in computing the weight functions.

While the load-systems are, in general, considered to be at  $S_t$  and in V, it is convenient to consider only the complementary problem of tractions on the crack-face alone, and consider the case, for simplicity, when the body forces are zero. Thus, hence forth we treat  $S_t$  to be the crack-face alone. It is seen that the weight-functions  $(du_i^R/da)$  on the crack-face will be singular (of the  $r^{-1/2}$  type, where r is the distance from the crack-tip). Thus, special quadrature rules are needed to integrate the quantity  $t_i^m$   $[du_i^R/da]$  on the crack-face [15, 39].

The following solution procedure is adopted to compute the weight-functions for an embedded or edge crack in a general anisotropic, finite-dimensional structure, when the crack is oriented arbitrarily with respect to the material axes of anisotropy.

(A) Consider two different reference states: one a normal pressure (say constant) on the crack-face and the second a shear traction (say constant) the crack-face. These two load-states are labelled R1 and R2 respectively. Henceforth it is understood that the following steps are carried out for states R1 and R2 respectively.

(B) The start with, treat the problem as one of an *infinite domain*. As discussed in Section 2.3 of this paper, expand the applied tractions on the crack face in the form [30]:

$$\tau_{yy} - i\tau_{xy} = -[p(t) + is(t)] = \sum_{n=1}^{N} b_n U_{n-1}(t) \qquad |t| \le a$$
(130)

where  $U_{n-1}(t)$  is the Chebyshev polynomial of the second kind, defined as:

$$U_n = \sin[(n+1)t]/\sin\theta; \qquad t = a\cos\theta \qquad (131)$$

and  $b_n$  are the parameters determined by curve-fitting. For this applied loading on the crack-face in an infinite anisotropic body, the solution for the K-factors, far-field stresses, and crack-face displacements, can be derived [30], as:

$$K_I - iK_{II} = \pm \sqrt{\pi a} \sum_{n=1}^{N} b_n \qquad x = \pm a$$
 (132)

$$u_x = 2 \operatorname{Re}[p_1\phi(z_1) + p_2\psi(z_2)]$$
  

$$u_y = 2 \operatorname{Re}[q_1\phi(z_1) + q_2\psi(z_2)]$$
(133)

$$\tau_{xx} = 2 \operatorname{Re}[s_1^2 \phi'(z_1) + s_2^2 \psi'(z_2)] \tau_{yy} = 2 \operatorname{Re}[\phi'(z_1) + \psi'(z_2)]$$

$$\tau_{xy} = -2 \operatorname{Re}[s_1 \phi'(z_1) + s_2 \psi'(z_2)]$$
(134)

$$2\phi(z_1) = \left(\frac{s_2}{s_2 - s_1}\right) \sum_{n=1}^{N} \operatorname{Re}(b_n) \frac{R_n(z_1)}{n} + \frac{1}{(s_1 - s_2)} \sum_{n=1}^{N} i \operatorname{Im}(b_n) \frac{R_n(z_1)}{n}$$
$$2\psi(z_1) = \left(\frac{s_1}{s_1 - s_2}\right) \sum_{n=1}^{N} \operatorname{Re}(b_n) \frac{R_n(z_2)}{n} + \frac{1}{(s_2 - s_1)} \sum_{n=1}^{N} i \operatorname{Im}(b_n) \frac{R_n(z_2)}{n}$$
(135)

For further details of the definitions of various parameters in (134), see [30] (note the coordinate system: x along the crack,  $x = \pm a$  for the crack-tips, and y normal to the crack).

(C) Compute the K-factors for step (B) using (132).

(D) Compute the crack-face displacements for step (B) using (133). Note that, once the coefficients  $b_n$  of (130) are known, the crack-face displacements  $u_x$  and  $u_y$  from (135) and (133) are known in an *analytical* form, with their dependence on the crack length being explicitly known (see [30] for details).

(E) Compute the tractions at the boundaries of the given finite-dimensional structure, using (134). Call these residual boundary-traction system as T (recall that steps (B) onwards are repeated for reference systems R1 and R2 of step (A)).

(F) From the analytical expressions for  $u_i$  at the crack-face as determined in step (D), determine the analytical expression for  $(du_i/da)$  by differentiating  $u_i$  w.r.t. a. Note that  $(du_i/da)$  will be infinite at the crack-tip  $x = \pm a$ .

It is important to remember that the present step (F) is still based on an analytical solution. No finite element or boundary-element models, and no virtual-crack extensions, and no finite-difference methods are used in computing  $(du_i^R/da)$ .

(G) Now consider the finite element model of the uncracked structure of the given geometry, and anisotropic material [In the case of *isotropic* material, a boundary-element model (with only the boundary being descritized) of the uncracked structure is far more efficient [40, 41]. Apply the reverse of the traction system T as determined in step (E) above, on the boundaries of the uncracked structural model. From the finite-element (or boundary element) solution, find the tractions at the location of the crack in the uncracked structure, and label this traction system as  $R_c$ .

(H) Reverse the system  $R_c$  on the crack faces. Go back to step (B), and repeat steps (B), (C), (D), (E), and (F), for this system  $R_c$  on the crack face in an *infinite domain*. Repeat steps (B) to (H) until convergence is obtained, i.e., the traction system T in step (E) is negligible.

(J) The weight-functions for a finite-dimensional structure of the given geometry, and given crack-oriented, are obtained by summing up all the values of  $(du_i^{R1}/da)$  [and  $du_i^{R2}/da$ ] obtained in step (F) for all iterations until convergence is established.

A number of problems has been solved in [15, 39] and [40, 41] to demonstrate the ease and accuracy of the above procedures.

It is important to note, the crack is not numerically modelled at all. The dependence of crack plane displacements on crack length as evaluated in steps (D) and (F), for infinite domains, is explicitly known; and thus the weight-functions are evaluated in the above alternating method, in an analytical sense without using numerical differentiations. Furthermore, the above mentioned procedures have been documented in the work of [15, 39] to work very well for anisotropic materials with arbitrarily oriented cracks.

For a two-dimensional anisotropic problem, it is possible to develop a boundary element method for mixedmode crack analysis, wherein a straight crack is explicitly included in the formulation and not modeled by boundary elements, by using the fundamental solutions for an infinite cracked anisotropic plate. This was developed by [42]. This boundary integral equation is:

$$C_{ij}U_j = \int_{\partial\Omega} \left[ u_{ji}^{\star} t_j - t_{ji}^{\star} u_j \right] \mathrm{d}A \tag{136}$$

It should be noted that the crack surface,  $S_c$ , is not a part of  $\partial\Omega$  in (136), as the crack is explicitly accounted for in the fundamental solution. By differentiating (136) w.r.t. a, one obtains the integral equations for  $(du_j/da)$ :

$$C_{ij}(\mathrm{d}u_j/\mathrm{d}a) = \int_{\partial\Omega} \left[ u_{ji}^{\star} \frac{\mathrm{d}t_j}{\mathrm{d}a} - t_{ji}^{\star} \frac{\mathrm{d}u_j}{\mathrm{d}a} + \frac{\mathrm{d}u_{ji}^{\star}}{\mathrm{d}a} t_j - \frac{\mathrm{d}t_{ji}^{\star}}{\mathrm{d}a} u_j \right] \mathrm{d}A$$
(137)

By taking the limit on the left-hand-side to  $\partial\Omega$ , one can solve the boundary integral equation for the unknown values of  $(du_j/da)$  and  $(dt_j/da)$  at  $\partial\Omega$ . Once these data at  $\partial\Omega$  is known, Eq. (137) simply becomes an integral relation for the interior values of  $(du_j/da)$ . Since the crack surface is interior to  $\partial\Omega$  in the formulation, the crack-surface weight functions can be determined from (137). Such procedures have been reported, along with some examples, in [43]. An advantage of this procedure is its ability to decouple the vector components of crack tip behavior easily.

### 6.2 WEIGHT-FUNCTIONS BY USING DIRECT FINITE ELEMENT/BOUNDARY ELEMENT MOD-ELING OF THE CRACKED STRUCTURE

The earlier class of modeling used only F.E.M. or B.E.M. models of the *uncracked* structure, while the crack was accounted for in some *analytical* fashion. If the material is nonhomogeneous, or if the crack exists in a complicated structural construction such as a bi-material plate or a stiffened plate, etc., the direct *numerical* modeling of the crack itself is unavoidable, to determine the weight functions. We discuss here some advances made recently in this direction. The discussion is limited to the case of *isotropy*.

Consider the analytic relation

$$\mathcal{G} = \int_{S_i} t_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \,\mathrm{d}S + \int_V f_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \,\mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}a} \int_V W \,\mathrm{d}V \tag{138}$$

(wherein the existence of  $S_u$  with nonzero values of prescribed  $u_i$  is ignored, for simplicity). In the context of a finite element method, wherein the *cracked structure is modeled* directly by finite elements, Eq. (138) may be written as:

$$\mathcal{G} = \mathbf{Q}\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}a} - \mathbf{q}\mathbf{K}\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}a} - \frac{1}{2}\mathbf{q}\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}a}\mathbf{q} = -\frac{1}{2}\mathbf{q}\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}a}\mathbf{q}$$
(139)

where Q is the generalized nodal force vector (due to applied loading at arbitrary  $S_t$  and in V); q is the nodal displacement vector, and K the stiffness matrix. Eq. (139) follows from (138), since, at equilibrium,  $Kq \equiv Q$ .

When a finite element mesh is used near the crack tip, a small change in crack length, by da, affects only the elements in a core immediately surrounding the crack tip. This is the basic idea behind the stiffness derivative method [44 and 45]. Let the small domain near the crack tip be  $V_{\varepsilon}$ . Thus;

$$\mathcal{G} = -\frac{1}{2} \boldsymbol{q}_{\varepsilon} \frac{\mathrm{d}\mathbf{K}_{\varepsilon}}{\mathrm{d}a} \boldsymbol{q}_{\varepsilon} \tag{140}$$

where ( )<sub> $\varepsilon$ </sub> indicates the quantity ( ) in the region  $V_{\varepsilon}$ . In (140), (dK<sub> $\varepsilon$ </sub>/da) is determined by the finite difference relation:

$$\frac{\mathbf{K}_{\varepsilon}(a + \Delta a) - \mathbf{K}_{\varepsilon}(a)}{\Delta a}$$

Eq. (140) can be applied to the reference state, to determine the K-factors for the reference state. However, if the reference state is of mixed mode loading, for isotropic materials,  $\mathcal{G} = (K_I^2 + K_{II}^2)/H$ , and a mode separation is necessary. Thus within the core-region  $V_{\varepsilon}$  [which is certainly much smaller than the cracked structure], one may decompose  $q^{\varepsilon}$  into mode I and mode II parts; by using the relations:

$$\left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \left\{ \begin{array}{c} u_1^I \\ u_2^I \end{array} \right\} + \left\{ \begin{array}{c} u_1^{II} \\ u_2^{II} \end{array} \right\} = \frac{1}{2} \left\{ \begin{array}{c} u_1 + u_1' \\ u_2 - u_2' \end{array} \right\} + \frac{1}{2} \left\{ \begin{array}{c} u_1 - u_1' \\ u_2 + u_2' \end{array} \right\}$$
(141)

w

and

Thus

quantity at a point p in the upper portion of the cracked plate, and ()' is the respective quantity at a point p' which is the mirror image in the crack axis of p. Thus  $q_{\varepsilon} = q_{\varepsilon}^{I} + q_{\varepsilon}^{II}$ .

where 1 and 2 are directions along and normal to the crack axis, respectively, and where ( ) denotes a

$$\frac{(K_I^R)^2}{H} = -\frac{1}{2} q_{\varepsilon}^I \frac{\mathrm{d}\mathbf{K}_{\varepsilon}}{\mathrm{d}a} q_{\varepsilon}^I$$
(142)

where 
$$q_{\varepsilon}^{I}$$
 is the vector of appropriate nodal displacements  $u_{i}^{I}$ , etc. Now we consider the problem of determining the weight functions for the reference state of mixed mode loading. To this end consider the finite element equilibrium equation for the entire *cracked structure* loaded under mixed mode reference load:

 $\frac{(K_{II}^R)^2}{H} = -\frac{1}{2} q_{\varepsilon}^{II} \frac{\mathrm{d} \mathbf{K}_{\varepsilon}}{\mathrm{d} q} q_{\varepsilon}^{II}$ 

$$\mathbf{K}\boldsymbol{q} = \boldsymbol{f} \tag{144}$$

For a fixed-loading, the weight-functions everywhere in the structure can be derived, from (154), as:

$$\mathbf{K}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\boldsymbol{a}} = -\frac{\mathrm{d}\boldsymbol{K}_{\varepsilon}}{\mathrm{d}\boldsymbol{a}}\boldsymbol{q} \equiv \frac{\mathrm{d}\boldsymbol{K}_{\varepsilon}}{\mathrm{d}\boldsymbol{a}}\boldsymbol{q}_{\varepsilon} \tag{145}$$

From a solution of (145),  $(\frac{du_i}{da})$  is determined for the reference state (which can, in general, be a mixed-mode loading), at all nodes at  $S_t$ , and in V (including the crack-face). The above method, for a pure mode I problem was presented by [46].

Now, we present a simple extension of the "stiffness-derivative" weight-function evaluation method for mixed-mode problems. In as much as the stiffness matrix (K) and its derivative (dK/da) are evaluated once and for all for the structure, Eq. (144) can be solved for two reference states, at least one of them mixed-mode, R1 and R2; Eqs. (142) and (143) can be solved for mixed-mode K-factors for the two reference states, i.e.,  $K_{I}^{R1}$ ,  $K_{II}^{R2}$ ,  $K_{I}^{R2}$ , and  $K_{II}^{R2}$ . Likewise, Eq. (145) can be solved for the two different reference states, to determine  $(du_{i}^{R1}/da)$  and  $(du_{i}^{R2}/da)$  everywhere in the structure (i.e. at  $S_{t}$ , in V, and on the crack-face) as desired. For any other arbitrary state of mixed-mode loading, Eqs. (127) and (128) may be used for determining the mixed-mode K-factors. Note that this simple procedure leads to weight-functions everywhere in the structure (external surfaces, crack faces, and within the body) as may be desired.

On the other hand, the authors in [47], instead of using the procedure as discussed in the above paragraph for mixed-mode problems, proceed to consider only pure-mode I and pure-mode II weight functions,  $(du_i^I/da)$  and  $(du_i^{II}/da)$ , by decomposing the displacement everywhere in the cracked structure (not only in  $V_{\varepsilon}$ ) using Eq. (141). If the weight-functions are sought at the external boundary, pairing the points on the external boundary, and their mirror images is geometrically impossible, for arbitrary-shaped structures, with arbitrarily oriented cracks. They [47] consider the following equations:

$$\mathbf{K}\frac{\mathrm{d}\boldsymbol{q}^{I}}{\mathrm{d}\boldsymbol{a}} = -\frac{\mathrm{d}\mathbf{K}_{\epsilon}}{\mathrm{d}\boldsymbol{a}}\boldsymbol{q}^{I} \tag{146}$$

and

$$\mathbf{K}\frac{\mathrm{d}\boldsymbol{q}^{II}}{\mathrm{d}\boldsymbol{a}} = -\frac{\mathrm{d}\mathbf{K}_{\varepsilon}}{\mathrm{d}\boldsymbol{a}}\boldsymbol{q}^{II} \tag{147}$$

Note that  $q^{I}(q^{II})$  is simply a vector of appropriate displacements  $u_{i}^{I}(u_{i}^{II})$  at each node in the structure, which may not always be geometrically feasible.

Another method for weight-functions which obviates the need for a finite difference evaluation of  $(dK_{\epsilon}/da)$ is based on the equivalent domain-integral method for evaluating the energy-release rate. This method

(143)

has recently been developed in [48] and [15, 39]. As discussed in [49, 50], the energy-release rate in a 2-D elastic problem can be written (as the equivalent domain integral representation of the *J*-integral), as:

$$\mathcal{G} = -\frac{1}{F} \int_{V_{\bullet}} \left[ W \frac{\mathrm{d}s}{\mathrm{d}x_1} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \frac{\partial s}{\partial x_j} \right] \mathrm{d}V$$
(148)

where  $V_s$  is any arbitrary region near the crack-tip  $[V_s$  is much smaller than the total region V]; s is any arbitrary but continuous function which is equal to 1 at the crack-tip, and goes to zero at the boundary of  $V_s$ ; F = 1 in two dimensional crack problems; W is the stress-work density, and  $u_i$  are displacements.

Suppose that the region  $V_s$  in (148) is taken to be the same as the region  $V_{\varepsilon}$  considered in Eq. (140) [Even otherwise, if  $V_{\varepsilon}$  is smaller than  $V_s$ ; since (dK/da) may be taken to be zero in the region  $V_s - V_{\varepsilon}$ ; one may rewrite (140) as  $\mathcal{G} = -\frac{1}{2}q_s(dK_s/da)q_s$ , without loss of generality]. Suppose that one introduces a finite element interpolation:

$$u_i = N^k u_i^k \qquad K = 1, \cdots N \text{ nodes}$$
  
=  $Nq \qquad i = 1, 2$  (149)

Then,

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}N^k_{,j}u^k_i$$
(150)

Also, we introduce the finite element interpolations,

$$s = N^k s^k \tag{151}$$

Using (149–151) in (148), one has:

$$\mathcal{G} = -\frac{1}{F} \int_{V_s} \left[ \frac{1}{2} N_{,j}^k N_{,1}^L - N_{,1}^k N_{,j}^L \right] \sigma_{ij} u_i^k s^L ] dV$$
  
$$\equiv -\frac{1}{2} T_i^k u_i^k \qquad k = 1, \cdots, N \text{ nodes} \quad i = 1, 2$$
(152)

$$\equiv -\frac{1}{2} Q_s^{\star t} \cdot q_s \tag{153}$$

where the definition of  $Q_s^{\star}$  is apparent. Thus, when  $V_s \equiv V_{\varepsilon}$ ; comparing (153) with (140), one has:

$$\frac{\mathrm{d}\mathbf{K}_{\varepsilon}}{\mathrm{d}a}\boldsymbol{q}_{\varepsilon} \equiv \boldsymbol{Q}_{\varepsilon}^{\star} \tag{154}$$

Note that  $Q_{\varepsilon}^{\star}$  is computed from a simple integral over  $V_{\varepsilon}$  as in (152a) and (153). Eq. (154) shows that a finite-difference evaluation of  $(d\mathbf{K}_{\varepsilon}/da)$  as in Eq. (140) can be avoided if the identity in (154) is used, and the energy-release-rate can be computed using (153).

For a fixed reference loading (which can in general be of the mixed-mode type), the weight functions everywhere in the structure (including at the external boundary,  $S_t$ , the crack-face, or in V), can now be obtained, using (145):

$$\mathbf{K}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\boldsymbol{a}} \equiv -\frac{\mathrm{d}\boldsymbol{K}_{\varepsilon}}{\mathrm{d}\boldsymbol{a}}\boldsymbol{q}_{\varepsilon} \equiv \boldsymbol{Q}_{\varepsilon}^{\star} \tag{155}$$

where  $Q_{\varepsilon}^{\star}$  is computed from the domain-integral over  $V_{\varepsilon}$  as apparent from (153).

Equation (155) is solved for two arbitrary reference states [which are both not either of Mode I or of Mode II type, with loading being either on the external boundary, or on the crack-face] to find  $(du_i^{R1}/da)$  and  $(du_i^{R2}/da)$  that are required in Eqs. (123) and (124) in order to compute, the mixed-mode K-factors for any other given arbitrary load-state. Note that in (155), K is computed only once; and  $Q_{\epsilon}^{\epsilon}$  is computed

separately for each reference state. However, examining (152a) and (153) it is seen that the only quantity that is different in integral for  $Q_{\varepsilon}^{\star}$  in the two reference states is  $\sigma_{ij}$  in  $V_{\varepsilon}$ .

In order to use (127) and (128) to compute the mixed-mode K-factors for any given arbitrary load-state is the only additional informations needed are the mixed-mode K-factors  $(K_I^{R1}, K_I^{R2})$ , and  $(K_{II}^{R1}, K_{II}^{R2})$  for the two reference states. For isotropic materials, the mode-decomposition of the energy-release rate of Eq. (152a) can be accomplished by decomposing the displacement, strain, and stress fields in the core region  $V_{\varepsilon}$  near the crack tip [V<sub>s</sub> is much smaller than V, the total domain]. The displacement decomposition is already given in (151), while the stress decomposition can be written as:

$$\begin{cases} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{cases} = \begin{cases} \sigma_{11}^{I} \\ \sigma_{22}^{I} \\ \sigma_{12}^{I} \end{cases} + \begin{cases} \sigma_{11}^{II} \\ \sigma_{22}^{II} \\ \sigma_{12}^{II} \end{cases} \equiv \frac{1}{2} \begin{cases} \sigma_{11} + \sigma_{11}' \\ \sigma_{22} + \sigma_{22}' \\ \sigma_{12} - \sigma_{12}' \end{cases} + \frac{1}{2} \begin{cases} \sigma_{11} - \sigma_{11}' \\ \sigma_{22} - \sigma_{22}' \\ \sigma_{12} + \sigma_{12}' \end{cases}$$
(156)

where, () and ()' are quantitites at a point p in the "upper side" of the crack, and at a point p' which is a mirror image of p in the crack-plane. Thus, for a reference state, the individual nodal intensities are computed from:

$$\frac{(K_I^R)^2}{H} = -\frac{1}{F} \int_V \left[ W^I \frac{\partial s}{\partial x_1} - \sigma_{ij}^I \frac{\partial u_i^I}{\partial x_1} \frac{\partial s}{\partial x_j} \right] \, \mathrm{d}V \tag{157}$$

 $\operatorname{and}$ 

$$\frac{(K_{II}^R)^2}{H} = -\frac{1}{F} \int_V \left[ W^{II} \frac{\partial s}{\partial x_1} - \sigma_{ij}^{II} \frac{\partial u_i^{II}}{\partial x_1} \frac{\partial s}{\partial x_j} \right] dV$$
(158)

where

$$W^{I} = \frac{1}{2}\sigma^{I}_{ij}u^{I}_{ij}; \qquad \qquad W^{II} = \frac{1}{2}\sigma^{II}_{ij}u^{II}_{ij}$$
(159)

This completes the algorithm for determining the mixed mode K-factors for an arbitrary given loading, using the weight functions for reference states, derived from the equivalent domain integral method, which avoids the need for a finite-difference evaluation of the stiffness derivative  $(d\mathbf{K}/da)$  as in the approaches for Mode I problem given in [46].

Special variational technique for determining directly the weight functions that are singular in the vicinity of the crack-tip (crack front) (and hence have unbounded strain-energy) has been presented in [51]. This variational technique handles both traction and mixed boundary conditions. A finite element implementation of the variational principle has also been given in [51] and this leads to a unified approach in the direct finite element computation of weight functions for all three fracture modes. The authors in [52] have presented weight functions for a semi-infinite crack in a full space of arbitrary anisotropy; in particular, the results for monoclinic solids are presented in closed form. Using such a solution, they [52] applied the variational technique in [51] to determining weight functions for homogeneous and piecewise homogeneous anisotropic (where crack tips do not terminate at the material interface) bodies. Employing the weight functions obtained, they also evaluated the stress intensity factors of a matrix crack in an idealized model of a fiber-reinforced composite laminate under curing conditions. The authors in [53] have recently introduced antiplane strain weight functions for an interface notch in an isotropic bi-crystal and in [52] the same variational technique to determine these notch-interface weight functions was used. Generalizing the weight function concepts [36], the authors in [54] have developed higher order weight functions for calculating power expansion coefficients of an elastic field in a two-dimensional body in the absence of body forces. Integration formulas for the expansion coefficient, in analogy to those for stress intensity factors, are given for interior points and crack tips. Some of these expansion coefficients at an interior point can be related to the image force of a discrete dislocation and those for the crack tip correspond to important fracture parameters as discussed in [54].

# 6.3 WEIGHT FUNCTIONS AND INFLUENCE FUNCTIONS FOR 3-D CRACK PROBLEMS

As aptly noted in [55], "since fracture and fatigue are demonstrably three-dimensional, the technology base needed to describe these processes must be of the same dimensionality if only to describe these events, let alone predict them". Thus, simple numerical methods of engineering interest are in need of development for 3-dimensional problems. Along with the domain integral methods, the weight function approaches to 3-D problems present interesting possibilities in this regard.

Here, we consider, for simplicity, only mode I crack problems in 3-D isotropic elastic solids. For isotropic solids, containing cracks of arbitrary shape, under mode I loading, the author in [37] has derived the counterpart of Eqs. (127) and (128) [for mode I], as follows:

$$\int_{\Gamma_c} \frac{2}{H} [K_I K_I^R \delta \ell] d\Gamma = \int_{S_i} t_i \delta_\ell u_i^R dS + \int_V f_i \delta_\ell u_i^R dV$$
(160)

where  $\Gamma_c$  is the crack-front;  $K_I$  is the stress-intensity factor (which varies along  $\Gamma_c$ ) for any arbitrary loading  $t_i$  at  $S_t$  and  $f_i$  in V;  $K_I^R$  is the stress-intensity factor (which varies along  $\Gamma_c$ ) for the reference loading  $t_i^R$  at  $S_t$  and  $f_i^R$  in V;  $S_t$  is the loaded-surface (which may be taken to be the crack-face, without loss of generality);  $\delta \ell$  is a smooth function along d $\Gamma$  denoting the infinitesmal advance of the crack in a direction locally normal to  $\Gamma$ ;  $\delta_\ell u_i^R$  will denote the first-order variation in  $u_i^R$  to a change in the crack-front i.e.,  $\delta_\ell u_i^R$  is a function of the location in  $S_t$  and V, and H is a material constant.

The author in [56] has presented results for the first order variation of an elastic displacement field associated with the arbitrary incremental planar advance of the location of the front of a half-plane crack in a loaded elastic full space, and also discussed the relation of such results to a 3-D weight function theory, and derived an expression for the distribution of the mode-I K-factor for a slightly curved crack-front. Later this work was extended [57] to the mixed mode case.

Recently analytical results for 3-D weight functions were presented [58], for a penny-shaped and a half-plane crack in an elastic full-space, under mixed-mode conditions. Employing these results, and the variational technique in [51], the authors in [54] have determined the Mode I weight functions for both penny-shaped and elliptical cracks in finite bodies.

Here, our objective is to discuss crack-surface weight functions for embedded or surface cracks of the elliptical geometry, i.e., we treat  $S_i$  to be the crack-face in (160) and ignore the body forces. For surface or corner flaws of semi- or quarter-elliptical geometry respectively, engineering theories of fatigue crack-growth are often based on the consideration of the K-factors at the major and minor axis locations (x = a, and y = b), respectively. Thus, one often thinks of a "two-parameter" characterization of the K-factor variation along the crack-front, for the given arbitrary loading. There are two alternative approaches for the above "two-parameter" characterization. One is directly in terms of the "local" values (or values at major and minor axis points on the ellipse)  $K_I^*$  and  $K_I^{**}$ ; and the other is in terms of "local weighted average" values along specified portions of the crack front,  $\bar{K}_I^*$  and  $\bar{K}_I^{**}$ , defined as:

$$(\bar{K}_{I}^{\star})^{2} = \frac{1}{\delta A_{1}} \int_{\Gamma_{c}} K_{I}^{2} \delta \ell \,\mathrm{d}\Gamma$$

$$(\bar{K}^{\star\star})^{2} = \frac{1}{\delta A_{2}} \int_{\Gamma_{c}} K_{I}^{2} \delta \ell \Gamma \qquad (161)$$

where  $K_I$  and  $\delta \ell$  are as in Eq. (160), and  $\delta A_1$  and  $\delta A_2$  are changes in crack area due to a virtual change in the length of the major and minor axes, respectively. The weight functions for these weighted average values,  $\bar{K}_I^*$  and  $\bar{K}_I^{\star\star}$  have been defined in [59], and used for residual life estimations of complex structures in [59] and [60]. In the following we sketch a procedure for determining the weight functions directly for  $K_I^*$  and  $K_I^{\star\star}$ .

Suppose for the reference load state, the K-factor variation,  $K_I^R(\Gamma)$  is determined from the finite element alternating technique as discussed in Section 3 of this paper. Thus, an analytical expression for  $K_I^R$  as a function of the crack-front coordinate  $\phi$ , is known, wherein the appropriate coefficients are determined from the finite element alternating method. We consider the case of the given loading, and introduce a trial solution for  $K_I$  for this loading, in terms of its values,  $K_I$  at x = a (denoted as  $K_I^*$ ) and  $K_I$  at y = b, denoted as  $K_I^{\star\star}$ . In as much as  $\delta \ell$  in Eq. (160) is arbitrary, we introduce two such trial variations: (i) one wherein the major axis is extended by (da) such that the equation of the ellipse is  $(x/(a + da))^2 + (y/b)^2 = 1$ ; and (ii) the other wherein the ellipse is  $(x/a)^2 + (y/(b + db))^2 = 1$ . Thus,  $\delta \ell$  is given in terms of (da) or (db) and the elliptical angle. Also, as discussed in Section 3, the crack surface displacements for the reference load state are known in the form of analytical expressions wherein the coefficients are determined from the finite element alternating method. Thus, for  $\delta \ell$  as in case (i) and (ii), the first-order variations in  $u_i^R$  at  $S_t$  can be determined from analytical expressions where coefficients are numerically determined in the alternating method. Thus, when the known expressions for  $K_I^R(\Gamma)$ ,  $\delta \ell(\Gamma)$ ; and  $\delta_\ell u_i^{(R)}$  are used, and the two-parameter trial function for  $K_I$  (for the given state) are used in Eq. (160), one obtains 2 algebraic equations, (one for each of the two cases of  $\delta \ell$  listed above) governing the two unknowns  $K_I^*$  and  $K_{II}^{\star\star}$ . Thus, one obtains a weight-function representation for the stress-intensity factors at the major and minor axes of the elliptical (or part-elliptical) flaw under the given state of mode I loading.

Using the VNA analytical solution for an embedded elliptical crack in an infinite body as detailed in Section 2 of this paper, an analytical expression was derived [61] for the first order variation of the crackface displacement field due to a geometrical perturbation in the major and minor axes of an elliptical crack. Using this, the above described weight-function representation was developed [61] for the K-factors at the major and minor axes of the (part)-elliptical crack, in the forementioned 2 parameter characterization.

If  $K_I$  variation under the action of an arbitrary crack-face traction is needed all along the crack-front, the *influence function* concept is more useful. For a given surface flaw in given structural geometry, one can generate a stress-intensity factor variation all along the crack-front, for a given polynomial loading on the crack-face, using the alternating method described in Section 3. If the given arbitrary crack-face pressure is then decomposed into individual polynomial variations multiplied by an appropriate constant, then the K-factor variation for the given load can easily be determined by a (weighted) linear superposition of the various influence functions. This has been done for several problems [18, 21].

In general, the advantage of the weight function concept over that of the influence functions is that the former can be used for localized crack-face forces, including point forces, while the latter is more suited for more distributed crack-face tractions of the polynomial type.

# 7 DOMAIN INTEGRAL METHODS FOR COMPUTING FRACTURE PARAMETERS IN 3-DIMENSIONAL CRACK PROBLEMS, UNDER ARBITRARY HISTORIES OF LOADING

It is well-known that energetic methods, such as based on the *J*-integral and other crack-tip integral parameters, play an important role in elastic-plastic and inelastic fracture mechanics, under arbitrary histories of loading (see, for instance, [1]). For elastic crack problems, in three-dimensions, the evaluation of the *J*-integral (in Mode I problems), the stiffness derivative method [44, 62] and the virtual crack extension method [45] proved to be quite useful. Various extensions to, and a certain variety of improvements of, these methods were recently presented in [63]; [64]; [65]; [66]; [49, 50, 67]. These later methods are currently labeled as the "Domain Integral Methods" for the computation of crack-tip integral parameters in nonelastic fracture under arbitrary load histories. In the following, a brief description of these domain-integral methods, as applicable to the analysis of mixed-mode behaviour of arbitrary shaped cracks (surfaces of discontinuity) in 3-D structures, with elastic-plastic or inelastic material behaviour, and under arbitrary loading, is given.

For elastic problems, the energy release rate per unit crack-extension (in a self-similar fashion) is given by:

$$J_1 = \mathcal{G} = \int_{S_i} t_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \mathrm{d}S + \int_V f_i \frac{\mathrm{d}u_i}{\mathrm{d}a} \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}a} \int_V W \,\mathrm{d}V \tag{162}$$

(assuming that  $S_u$ , where non-zero  $u_i$  are prescribed, is zero). Eq. (162) is valid for mixed-mode loadings, and in 3-D problems, it is understood that  $(du_i/da)$  represents a first-order change in  $u_i$  due to a *local* perturbation in the crack-front, and  $J_1$  is the *local* energy-release rate. In a finite element model, Eq. (162) may be written as:

$$J_1 = Q \frac{\mathrm{d}q}{\mathrm{d}a} - \frac{\mathrm{d}}{\mathrm{d}a} (\frac{1}{2} q K q)$$
(163)

≡

$$= -\frac{1}{2}q\frac{\mathrm{d}K}{\mathrm{d}a}q \tag{164}$$

$$= -\frac{1}{2}q_o \frac{\mathrm{d}K_o}{\mathrm{d}a}q_o \tag{165}$$

Eq. (164) follows from (163) since Q = qK at equilibrium and Eq. (165) follows from (164) since a change in crack length can be seen to affect the stiffness matrix of only a small-core of material, in the domain  $V_o$ , near the crack front. The evaluation of  $dK_o/da$  is usually accomplished through a finite difference method [44, 45]. The definitions of  $J_2$  and  $J_3$ , which involve the more mathematical concepts of translation of the crack, in  $x_2$  and  $x_3$  directions [1], are not, in general, amenable to calculation through the above stiffness-derivative methods.

The basic advantages of the domain-integral method can be seen from the following, to be: (i) they allow simple computation of all the 3 components  $J_k$  of the vector J integral; and (ii) in the case of  $J_1$ , the need for a finite difference evaluation of  $(dK_o/da)$  is obviated.

In a general 3-D problem, for arbitrary material behaviour and loading, one may define the J-integral vector components as:

$$J_{k} = \lim_{\epsilon \to 0} \int_{\Gamma_{\epsilon}} \left[ W n_{k} - \sigma_{ij} \frac{\partial u_{i}}{\partial x_{k}} n_{j} \right] d\Gamma$$
(166)

where W is the density of *stress-work* in arbitrary loading and arbitrary material behaviour;  $\sigma_{ij}$  are stresses,  $u_i$  are displacements, and  $n_k$  are components of the unit normal vector to the surface of the tube at points on contour  $\Gamma_{\varepsilon}$ . In principle it is possible to define  $J_k$  in any coordinate system, but for the purposes of prediction of crack behaviour, it is more convenient to have a local crack-front coordinate system  $x_1, x_2, x_3$ :  $x_1$  is normal to the crack front and lies in the plane of the crack surface,  $x_2$  is orthogonal to  $x_1$  and the crack surface, and  $x_3$  is tangential to the crack-front and in the crack-plane. We introduce the equivalent definition of the near-tip J-integral along the surface of the tube, as:

$$J_k \Delta = \lim_{\substack{\epsilon/\Delta \to 0 \\ \Delta \to 0}} \int_{A_\epsilon} \left[ W n_k - \sigma_{ij} \frac{\partial u_i}{\partial x_k} n_j \right] \mathrm{d}A, \qquad k = 1, 2$$
(167)

where  $A_{\varepsilon}$  is the surface of a cylinder, with the centerline along the crack-front, its radius being  $\varepsilon$ , and the length of the generator being  $\Delta$  along the crack-front. This definition is more convenient for numerical applications. Likewise, one may define the energy-release components for symmetric and anti-symmetric deformation modes, as:

$$G_{I}\Delta = \lim_{\substack{\epsilon/\Delta \to 0 \\ \Delta \to 0}} \int_{A_{\epsilon}} \left( W^{I}n_{1} - \sigma_{ij}^{I} \frac{\partial u_{i}^{I}}{\partial x_{1}} n_{j} \right) \mathrm{d}A$$
(168)

$$G_{II}\Delta = \lim_{\substack{\epsilon/\Delta \to 0 \\ \Delta \to 0}} \int_{A_{\epsilon}} \left( W^{II} n_1 - \sigma_{ij}^{II} \frac{\partial u_i^{II}}{\partial x_1} n_j \right) \mathrm{d}A$$
(169)

and

$$G_{III}\Delta = \lim_{\substack{\epsilon/\Delta \to 0 \\ \Delta \to 0}} \int_{A_{\epsilon}} \left( W^{III} n_1 - \sigma_{3j} \frac{\partial u_3}{\partial x_1} n_j \right) \mathrm{d}A \tag{170}$$

where the deformation field is decomposed, locally near each differential segment of the crack-front, into symmetrical and skew-symmetric parts about the crack-plane locally, as follows:

$$\{u\} = \{u^{I}\} + \{u^{II}\} + \{u^{III}\}$$

$$= \frac{1}{2} \begin{cases} u_{1} + u_{1}' \\ u_{2} - u_{2}' \\ u_{3} + u_{3}' \end{cases} + \frac{1}{2} \begin{cases} u_{1} - u_{1}' \\ u_{2} + u_{2}' \\ 0 \end{cases} + \frac{1}{2} \begin{cases} 0 \\ u_{3} - u_{3}' \end{cases}$$

$$(171)$$

$$\{\sigma\} = \{\sigma^{I}\} + \{\sigma^{II}\} + \{\sigma^{III}\}$$

$$= \frac{1}{2} \begin{bmatrix} \sigma_{11} + \sigma_{11}' \\ \sigma_{22} + \sigma_{22}' \\ \sigma_{33} + \sigma_{33}' \\ \sigma_{12} - \sigma_{12}' \\ \sigma_{23} - \sigma_{23}' \\ \sigma_{31} - \sigma_{31}' \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \sigma_{11} - \sigma_{11}' \\ \sigma_{22} - \sigma_{22}' \\ 0 \\ \sigma_{12} + \sigma_{12}' \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \sigma_{33} - \sigma_{33}' \\ 0 \\ \sigma_{23} + \sigma_{23}' \\ \sigma_{31} + \sigma_{31}' \end{bmatrix}$$

$$(172)$$

$$u'_i(x_1, x_2, x_3) = u_i(x_1, -x_2, x_3)$$
(173)

$$\sigma'_{ij}(x_1, x_2, x_3) = \sigma_{ij}(x_1, -x_2, x_3) \tag{174}$$

# 7.1 TRANSFORMATION OF DISPLACEMENTS, STRAINS AND STRESSES TO THE CRACK FRONT COORDINATE SYSTEM

We can simplify many developments if this transformation is performed prior to the calculation of J- and G components. Let  $X_1, X_2, X_3$  be a global Cartesian coordinate system; and  $x_1, x_2, x_3$  be the crack front coordinate system for a particular point along the crack front. For the definition of a crack front coordinate system at any point, it is sufficient to have the direction cosines for a unit vector along  $x_1$ 

$$X_p = \{X_{p1}, X_{p2}, X_{p3}\}$$
(175)

and for a unit vector along  $x_3$ 

$$Z_p = \{Z_{p1}, Z_{p2}, Z_{p3}\}$$
(176)

Then it is easy to define the orientation of  $x_2$  as

$$Y_p = Z_p X X_p \tag{177}$$

$$Y_{p1} = Z_{p2}X_{p3} - Z_{p3}X_{p2}$$

$$Y_{p2} = Z_{p3}X_{p1} - Z_{p1}X_{p3}$$

$$Y_{n3} = X_{n1}X_{n2} - Z_{n2}X_{n1}$$
(178)

We define the coefficients of a transformation matrix  $a_{ij}$  as:

$$\begin{array}{ll} a_{11} = X_{p1} & a_{12} = X_{p2} & a_{13} = X_{p3} \\ a_{21} = Y_{p1} & a_{22} = Y_{p2} & a_{23} = Y_{p3} \\ a_{31} = Z_{p1} & a_{32} = Z_{p2} & a_{33} = Z_{p3} \end{array}$$
(179)

The transformation of coordinates, displacements, strains, and stresses can be done as follows:

$$x_i = a_{ij}X_j \tag{180}$$

$$u_i = a_{ij} u_i^g \tag{181}$$

$$\varepsilon_{ij} = a_{ip}a_{jq}\varepsilon_{pq}^g \tag{182}$$

$$\sigma_{ij} = a_{ip}a_{jq}\sigma_{pq}^g \tag{183}$$

Ξ

Here the superscript g stands for the values in global coordinate system.

# **7.2** EDI-TECHNIQUE FOR $J_1$ , $J_2$ and G CALCULATION

After a point by point coordinate transformation (180), the crack front is straight. Let us consider the segment of crack front and the volume around this segment inside a *larger* cylindrical domain V. V is the volume of the larger cylinder,  $V_{\varepsilon}$  is the volume of small cylinder of radius  $\varepsilon$  around the crack front segment, A is the cylindrical surface of V;  $A_{\varepsilon}$  is the cylindrical surface of  $V_{\varepsilon}$  and  $A_1$ ,  $A_2$  are side surfaces of V. Note

that at any differential segment along the crack-front, the considered domain V is still much smaller than the overall dimensions of the structure.

Then, in general, we can redefine the near-tip parameters  $J_k$  and  $G_{III}$  as:

$$J_k f = -\int_{A-A_{\epsilon}} \left( W n_k - \sigma_{ij} \frac{\partial u_i}{\partial x_k} n_j \right) s \, \mathrm{d}A \tag{184}$$

$$G_{III}f = -\int_{A-A_{\epsilon}} \left( W^{III} n_1 - \sigma_{3j} \frac{\partial u_3}{\partial x_1} n_j \right) s \, \mathrm{d}A \tag{185}$$

Here,  $s = s(x_1, x_2, x_3)$  is an arbitrary but continuous function which is equal to zero on A; and nonzero (equal to 1) on  $A_{\varepsilon}$ ; and f is the area under the s-function curve along the segment of the crack-front under consideration.

Using the divergence theorem, we have the following representation of  $J_k$ :

$$J_{k}f = -\int_{V-V_{\epsilon}} \left( W \frac{\partial s}{\partial x_{k}} - \sigma_{ij} \frac{\partial u_{i}}{\partial x_{k}} \frac{\partial s}{\partial x_{j}} \right) dV - \int_{V-V_{\epsilon}} \left[ \frac{\partial W}{\partial x_{k}} - \frac{\partial}{\partial x_{j}} \left( \sigma_{ij} \frac{\partial u_{i}}{\partial x_{k}} \right) \right] s dV + \int_{A_{1}+A_{2}} \left( W n_{k} - \sigma_{ij} \frac{\partial u_{i}}{\partial x_{k}} n_{j} \right) s dA \qquad k = 1, 2$$
(186)

This expression represents a further variant of the virtual crack extension method, but the elimination of the actual process of virtual crack extension during the development of (186) allows us to use any s-function for the calculation of  $J_k$ . Thus, we have a new and computationally more appealing interpretation of the VCE approach.

In the case of the presence of nonelastic (thermal and plastic) deformations we can define W as

$$W = \int \sigma_{ij} \,\mathrm{d}\varepsilon_{ij} \tag{187}$$

$$\epsilon_{ij} = \epsilon^{e}_{ij} + \epsilon^{p}_{ij} + \epsilon^{t}_{ij} \tag{188}$$

where  $\varepsilon_{ij}^e$ ,  $\varepsilon_{ij}^p$ , and  $\varepsilon_{ij}^t$  are elastic, plastic, and thermal parts of strains. Assuming that the stresses have an elastic potential, i.e.,

$$\sigma_{ij} = \frac{\partial W^e}{\partial \varepsilon^e_{ij}} \tag{189}$$

the second term of (186) can have the form

$$(J_k f)_2 = -\int_{V-V_{\epsilon}} \left( \frac{\partial W}{\partial x_k} - \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial x_k} \right) s \, \mathrm{d}V$$
  
$$= -\int_{V-V_{\epsilon}} \left( \frac{\partial W^p}{\partial x_k} - \sigma_{ij} \frac{\partial \varepsilon_{ij}^{pt}}{\partial x_k} \right) s \, \mathrm{d}V$$
(190)

Here we used equilibrium equations (in the absence of body forces), and introduced the definitions:

$$W^p = \int \sigma_{ij} \,\mathrm{d}\varepsilon_{ij}^{pt} \tag{191}$$

$$\varepsilon_{ij}^{pt} = \varepsilon_{ij}^{p} + \varepsilon_{ij}^{t} \tag{192}$$

It is evident that in the absence of nonelastic strains the second term of (186) is equal to zero. If the s function is equal to zero on faces  $A_1$  and  $A_2$ , then the third term of (186) will be equal to zero as well.

Considering  $G_{III}$  for the linear elastic case (in the absence of body forces), we can have from equation (185),

$$G_{III}f = -\int_{V-V_{\epsilon}} \left( W^{III} \frac{\partial s}{\partial x_{1}} - \sigma_{3j} \frac{\partial u_{3}}{\partial x_{1}} \frac{\partial s}{\partial x_{j}} \right) dV - \int_{V-V_{\epsilon}} \sigma_{3j} \frac{\partial \varepsilon_{ij}^{t}}{\partial x_{1}} s \, dV + \int_{A_{1}+A_{2}} \left( W^{III} n_{1} - \sigma_{3j} \frac{\partial u_{3}}{\partial x_{1}} n_{j} \right) s \, dA$$
(193)

The third terms of (186) and (193) can be simplified if the faces  $A_1$  and  $A_2$  are orthogonal to the crack front  $(n_1 = n_2 = 0 \text{ on } A_1 \text{ and } A_2)$ .

Again, in Eq. (193), the second term is equal to zero if  $\varepsilon_{ij}^t = 0$  and the third term is absent if s = 0 on  $A_1$  and  $A_2$ . We note that the domain integral algorithms analogous to those in (186) can be developed directly for the energy-release-rate quantities  $G_I$  and  $G_{II}$  also.

It is now easy to see the advantage of the domain-integral method over the "stiffness derivative" method for the computation of the first component of J, i.e.,  $J_1$ , for *elasto-static problems*. If in Eq. (186), the function s is taken to be zero at  $(A_1 + A_2)$ ; and if body-forces are zero, in *elasto-static* problems the second and third terms on the r.h.s. in (186) vanish identically, and  $J_1$  can be written as:

$$J_{1} = -\lim_{\varepsilon \to 0} \int_{V-V_{\varepsilon}} \left[ W \frac{\partial s}{\partial x_{1}} - \sigma_{ij} \frac{\partial u_{i}}{\partial x_{1}} \frac{\partial s}{\partial x_{j}} \right] \mathrm{d}V$$
(194)

We compare Eq. (165) of the stiffness derivative method to the above Eq. (194). Without loss of generality, we assume that the domain  $V_o$  of Eq. (165) to be the same as the domain  $(V - V_{\varepsilon})$  of (194). In (165), it is clear that the integral is quadratic in  $q_o$ , say of the form  $q_o \cdot \mathbf{A}_o \cdot q_o$ . Thus the domain integral method gives *directly* the matrix  $\mathbf{A}_o$  [which is equivalent to  $d\mathbf{K}_o/da$ ] without using the finite difference method to evaluate  $(d\mathbf{K}_o/da)$  as in (165) of the virtual crack extension or the stiffness derivative method.

### 7.3 CHOICE OF s-FUNCTION

It is natural to use a parametric representation of function s inside any element as:

$$s = N^{I} s^{I}$$
 (I = 1...20) (195)

where  $N^{I} = N^{I}(\xi, \eta, \zeta)$  — quadratic shape functions, I is the node number. We suppose summation over repeated indices. Then the s-function should be defined in terms of (195) by using 1 or 2 elements in  $x_3$ direction for the crack front disk. Usually it is not useful to have a s-function more complicated than a linear function in the radial direction. Several simple functions s are discussed below:

(a) The disk has two elements in  $x_1$  and in  $x_3$  directions respectively. The function s can be defined on the small tube of radius  $\varepsilon$ , for both the elements along  $x_3$  (with  $\zeta$  being the natural coordinate along the crack front segment), as:

El. 1 : 
$$s = \frac{1}{2}(1+\zeta)$$
  
El. 2 :  $s = \frac{1}{2}(1-\zeta)$ 

The area under the s-function curve along the meridian of the surface of the small tube or on the crack-front  $(\varepsilon = 0)$  is equal to

$$f = \frac{1}{2}(\Delta_1 + \Delta_2)$$

(b)

El. 1 : 
$$s = \frac{1}{2}(\zeta^2 + \zeta)$$
  
El. 2 :  $s = \frac{1}{2}(\zeta^2 - \zeta)$   
 $f = \frac{1}{3} + (\Delta_1 + \Delta_2)$ 

(c)

$$s = (1 - \zeta^2)$$
  
$$f = \frac{2}{3}\Delta$$

(d)

$$s = 1$$
  
$$f = \Delta$$

Assuming a s-function that is linear in the r-direction we have its value for a particular point:

$$s = s_o \frac{r - r_\varepsilon}{r_f - r_\varepsilon}$$

where  $s_o$  - the value of s function at the point  $r = r_{\varepsilon}$ , r is the distance of the point in question from the surface of small tube,  $r_{\varepsilon}$  is the radius of the small tube,  $r_f$  - the outer radius of crack front disk. In practice it is often useful to have one element in the r-direction for the disk and to employ degenerate quarter-point singular elements around the crack tip.

## 7.4 FIRST TERM OF $J_k$

Using the parametric representation of displacements

$$u_i = N^J u_i^J \tag{196}$$

where i is the direction of crack front coordinate system and the superscript J is the node number, it is possible to have such an expression for the calculation of the first term of J-integral of Eq. (186):

$$(J_k f)_1 = -\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \left( W \frac{\partial N^L}{\partial x_k} s^L - \sigma_{ij} \frac{\partial N^M}{\partial x_k} \frac{\partial N^L}{\partial x_j} u_i^M s^L \right) \det(j) \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta \tag{197}$$

where det(j) is the determinant of Jacobi matrix. An effective procedure of computing  $J_k$  with several types of s-functions consists of a separate  $2 \times 2 \times 2$  integration of the expression

$$R_{k}^{L} = -\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left(W\frac{\partial N^{L}}{\partial x_{k}} - \sigma_{ij}\frac{\partial N^{M}}{\partial x_{k}}\frac{\partial N^{L}}{\partial x_{j}}u_{i}^{M}\right)\det(j)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\tag{198}$$

and defining a scalar product:

$$(J_k f)_1 = R_k^L s^L \tag{199}$$

### 7.5 THE SECOND TERM OF $J_k$

The main difficulty in integrating (190) arises from the fact that we know precise enough values of strain energy density, and strain, only at the  $2 \times 2 \times 2$  Gauss Integration points. A possible way of integrating the derivative of such functions is to obtain the derivative at the center of the element and to perform a one-point integration.

Consider a 20-node element in local coordinate system  $\xi_i$ 

$$\xi_1 = \xi, \quad \xi_2 = \eta, \quad \xi_3 = \zeta$$

Let's assume that we know that values of the function F only at the integration points as  $F^{(I)}$ . Using a parametric representation, it is possible to write

$$F^{(J)} = L^{I(J)} F^I (200)$$

where  $F^{I}$  are unknown values of F at corner nodes 1...8,  $L^{I}$  are linear shape functions for corner nodes,  $L^{I(J)}$  are values of shape functions at integration points (J).

The inversion of (200) gives

$$F^{I} = (L^{I(J)})^{-1} F^{(J)}$$
(201)

The coefficients of the extrapolation matrix  $(L^{I(J)})^{-1}$  are:

$$(L^{I(J)})^{-1} = \begin{bmatrix} A & B & C & B & B & C & D & C \\ & A & B & C & C & B & C & D \\ & & A & B & D & C & B & C \\ & & & A & C & D & C & B \\ & & & & A & B & C & B \\ sym. & & & & A & B & C \\ & & & & & & & A & B \\ & & & & & & & & A \end{bmatrix}$$
(202)

$$A = \frac{5+3\sqrt{3}}{4} \quad B = -\frac{\sqrt{3}+1}{4} \quad C = \frac{\sqrt{3}-1}{4} \quad D = \frac{5-3\sqrt{3}}{4} \tag{203}$$

Now, from (202) we can calculate the derivative at the center of the element:

$$\frac{\partial F}{\partial x_k} = \frac{\partial L^I}{\partial x_k} (L^{I(J)})^{-1} F^{(J)}$$
(204)

where, the derivatives  $\partial L^{I}/\partial x_{k}$  should be calculated for  $\xi = \eta = \zeta = 0$ .

7.6 THIRD TERM OF  $J_k$  OF EQ. (196)

For simplicity consider the disk with  $A_1$  and  $A_2$  being orthogonal to the crack front segment. Then

 $n_1 = n_2 = 0,$   $n_3 = 1$  on  $A_1$  $n_1 = n_2 = 0,$   $n_3 = -1$  on  $A_2$ 

$$(J_k f)_3 = -\int_{A_1 + A_2} \sigma_{i3} \frac{\partial u_i}{\partial x_k} n_3 s \,\mathrm{d}A \tag{205}$$

If  $A_1 = A_2$ 

$$(J_k f)_3 = -\int_{A_1} \Delta\left(\sigma_{i3} \frac{\partial u_i}{\partial x_k}\right) s \,\mathrm{d}A \tag{206}$$

where  $\Delta(F) = (F)_{A_1} - (F)_{A_2}$ .

Assuming that every function is linear in the  $x_3$  direction and using the values of functions at integration points  $\zeta = \pm (1/\sqrt{3})$ , it is possible to define  $\Delta F$  as

$$\Delta F = \sqrt{3} \left[ F\left(\zeta = \frac{1}{\sqrt{3}}\right) - F\left(\zeta = -\frac{1}{\sqrt{3}}\right) \right]$$

Then

$$(J_k f)_3 = -\int_{-1}^1 \int_{-1}^1 \sqrt{3}\Delta \left(\sigma_{i3} \frac{\partial u_i}{\partial x_k}\right)^{(T)} s \det(J) \,\mathrm{d}\xi \,\mathrm{d}\eta \tag{207}$$

where  $\Delta F^{(T)} = F(\zeta = \frac{1}{\sqrt{3}}) - F(\zeta = -\frac{1}{\sqrt{3}}).$ 

Examples of three-dimensional  $J_1$  computations using the domain-integral methods may be found in [50, 66]. Application of the "domain-integral" type evaluation of the crack-tip integral parameters in viscoplastic dynamic crack propagation at fast speeds, has been discussed in [68].

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