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During this second semester of the research I have developed the basic mathematical background and some of the computational tools for exploring the physical processes of current buildup in the layers above the photosphere. This process is the origin of the flare energy and its physic-mathematical understanding is the cornerstone for any attempt to quantify the coarse ideas developed during the first semester. The result consisted in the development of the formalism for solving the coupled gas and current density temporal/spatial 3-dimensional evolution for given velocities at the photospheric boundary (the J-method). The method was tested using two basic 2-dimensional magnetic configurations and we were able to solve analytically the problems posed which include regions with null field and gas pressure effects (some of these problems were never solved before). A presentation was made at the recent American Astronomical Meeting in Columbus, Ohio, and a first paper has been submitted to Physical Review A. These papers (enclosed) present part of the results achieved so far.

Also work has been done in characterizing the magnetic free-energy content of a region as results from the electric currents. This approach shows precisely which is the energy and why it arises, as well as how can it be released. The work gives mathematical foundation to the notions of self- and mutual-inductance and indicates how they can be applied to the solar atmosphere in which complex distributions of spread current systems and current sheets are likely exist as a result of the dynamics previously studied. Further work in applying and presenting these results on the energy content will proceed during the next period of performance of the Grant.

I have also studied a preliminary list of UV lines which may be suitable for Stokes profiles study through space-based future instrumentation. More studies will be carried out in the next period with the aim to narrow down the list of interesting lines and estimate the magnitude and spectra of the polarization signals which are likely to be observed. This research would provide the design constraints for the instrument I intend to propose.

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THE DYNAMICS OF ELECTRIC CURRENTS IN HIGH CONDUCTIVITY PLASMA

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ABSTRACT

We present a method for solving plasma MHD problems arising from the interaction of plasmas with magnetic fields in stellar atmospheres. Our approach, in contrast to previous methods, is not based on solving equations for the magnetic field and plasma velocity but rather studies the evolution of the electric current and density (and the related gas pressure). We have applied the method to several studies involving linearized departures from static, current-free equilibria. The applications show explicit solutions for cases found in astrophysics and to problems encountered with earlier studies where the gas pressure was neglected. The method is particularly well suited for studying situations which involve a transition between high and low plasma-beta regions. It shows precisely how electric currents, and magnetic free-energy, build up in the plasma as a result of the slow stressing of a potential magnetic field configuration. The method also demonstrates how transverse-current waves, a mix of Alfven and magneto-acoustic modes, propagate in a low-beta plasma for any density stratification and background field geometry.

I. Introduction

It is customary to study the behavior of astrophysical plasmas by deriving MHD equations for the magnetic field and the plasma velocity from which the electric current has been eliminated[1,2]. Although these equations are based on a number of simplifications[3] they have been used until present to address a wide variety of problems[4,5]. These methods lead to coupled partial differential equations for the spatial and temporal evolution of the magnetic field and the velocity, and in theory can provide precise solutions to the dynamics. The solution of such equations requires knowledge of boundary and initial conditions. These conditions, in practical applications, are hard to specify and usually involve arbitrary assumptions which make it difficult to establish the roles of the local plasma and the external sources of the magnetic field. Other approaches have been used for ideal plasmas which are based on energy or magnetic flux arguments[6,7,8]. These approaches, also depend critically on the boundary and initial conditions and often have to resort to strong simplifications.

A different approach of circuit analogy studies the sources of the field, the electric currents, treated as circuit elements. This approach has been successful in providing insight into the energetics[9] . However, this approach does not give a full solution to the plasma dynamics. The method we propose constitutes a synthesis of the previous views in which we study in detail the plasma dynamics together with the electric currents. This method gives a full solution to the plasma dynamics and also gives insight into the energetics and allows us to define, when possible, the self- and mutual-inductance of complex current systems.

In this paper we present our approach in which we eliminate the plasma velocity and use equations describing the evolution of the electric current and the plasma density. The equations so derived are very powerful for solving several types of problem found in astrophysics. We describe a few particular applications to the linear analysis of the propagation of disturbances in otherwise static, current-free atmospheres. The case studies we treat in detail involve only regions of low (<<1) and intermediate (~1) plasma-beta. For high-beta regions the plasma dynamics is described by the hydrodynamics and the electric currents (and magnetic fields) can be evaluated straightforwardly.

In our analysis we will show how the magnetic free-energy of the plasma can be obtained directly from the electric currents. This method allows us to obtain the free-energy without resorting to differencing which nearly always leads to large errors when applied to observations which contain significant uncertainties. Moreover, in our formulation the volume integral need not be evaluated over very large volumes but only over the volume in which the relevant currents flow.

For the low beta case we consider two examples involving two-dimensional geometry of the background field. These examples are selected for their simplicity and because they are similar to situations encountered in actual solar magnetic field observations and in addition they give rise to interesting phenomena.

The first example describes the field around a null line which results from the potential field produced by two current systems located outside the domain of interest. Such current systems can be, for instance, two concentric circular current loops with a radius that is very large compared to the dimensions of the region of interest. This null line magnetic configuration is often associated with the production of solar flares, and has been studied by Craig and McClymont[10] and Hassam[11]. These authors neglect gas pressure effects and obtain a fast reconnection regime by matching analytical solutions for the inner resistive core with selected solutions for the ideal, low-beta, outer envelope. Their analysis is based on physical conditions which are quite different to the problems we consider here. In particular we are concerned not with the explosive solar flare phenomena, but rather with the gradual stressing of the magnetic field which can convert the convective energy, which drives the high-beta layers, into magnetic free-energy of the low-beta regions. This energy will then be available for fuelling a wide range of solar and stellar phenomena including flares. We show that while the solutions obtained by these authors are mathematically correct, they represent only a subset of the possible solutions. This is a result of their neglect of the gas pressure which has important consequences to the overall behavior. In our analysis of this case we divide the space surrounding the resistive core into high-beta (>>1) and low-beta

(<<1) regions. We derive the more general eigenfunctions for the outer low-beta region which we match to a solution for the high-beta region through a region of intermediate plasma-beta. We show that the high-beta region is much larger than the resistive core and has very important effects on the plasma behavior, in particular allowing for substantial localized energy storage.

We also present a detailed study of the simple arcade magnetic field configuration, which would result from a single current loop. We determine the eigenfunctions for this configuration, and show how to compute the full solution from specified motions of the footpoints. Our setup is somewhat similar to that of Murata[5], however we study other modes not previously solved in detail. We are able to show the character of the standing oscillations which could be driven by periodic motions of the footpoints, and also how the deformations of the magnetic field and the corresponding electric currents can be produced by non-periodic footpoint motions.

We formally solve the case of a region of intermediate beta (of order of magnitude unity) in which the pressure effects and the Lorentz forces are both significant. Our formal solution applies for quasi-stationary cases in which the terms containing the partial derivatives of the electric current and the gas density with respect to time are negligible. We show how these regime can be matched to the previous solutions for the null line case.

The examples we have chosen show how our method allows us to derive the behavior of the major plasma parameters in relatively complex vector field configurations, through the solutions to a pair of coupled second order differential equations. These equations which describe the electric current (a vector quantity) and the gas density (a scalar quantity) dynamics, do in principle contain non-linear terms. However, the equations in our formulation are easy to linearize in most cases because the scaling of the terms can be easily estimated a priori, and verified a posteriori.

Although the explicit solutions we derive correspond only to small departures from potential cases, that is to small current buildup, they do display the beginning of the development of stresses in the field and of the storage of magnetic free-energy in the plasma. Our examples correspond to the very beginning of a magnetic evolution when motions within the high-beta plasma, start the flow of electric currents through previously current-free regions. In this stage, there are no fast instabilities and the classical Joule dissipation of the electric currents would be small for conductivities and velocities typical of the solar atmosphere. Therefore, the field stresses and electric currents will grow into a more complicated non-linear regime in which fast instabilities become possible.

II.Plasma electromagnetic.

In the electromagnetic equations we shall, as is usual, neglect displacement-current, magnetization, polarization and many high-frequency plasma phenomena. We further assume a pure non-ideal MHD in which the plasma is quasi-neutral and the electromagnetic equations take the form

$$\begin{aligned} &\nabla . B=0 \\ &\nabla\times B=\frac{4\pi}{c}J \\ &\nabla\times E=\frac{-1}{c}\frac{\partial B}{\partial t} \end{aligned} \tag{1}$$

where the equation for the divergence of the electric field is omitted and the displacement current is neglected. The space-charge is not specified but derives from the condition of zero divergence of the electric current (see Braginski[12]),

$$\nabla . J=0 \tag{2}$$

We invoke the linearity of the electromagnetic equations (1-2) and use the Coulomb gauge to decompose the vector fields into rotational and irrotational (or potential) parts, giving

$$\begin{aligned} &E=E_p+E_r \\ &E_p=\nabla\Phi_e \\ &E_r=\nabla\times A_e \\ &\nabla . A_e=0 \end{aligned} \tag{3}$$

For simplicity we consider scalar electric conductivity, after transformation to the fluid frame, and we neglect transport phenomena that may lead to charge separation (e.g. Hall effect). However, the analysis may easily be extended by including a tensor conductivity, or better yet a tensor resistivity, and the Hall electric field. For the Ohm's law we use

$$\frac{c}{\sigma}J=cE+v\times B \tag{4}$$

This is the first non-linear equation we introduce, and it involves properties of the local plasma.

From the second of equations (1) the potential part of the electric current is

$$J_p=0 \tag{5}$$

and only the rotational part of the current remains, in the present approximation.

The third of equations (1) give the rotational part of the electric field. The potential part of the electric field can be found from equations (2) and (4). We find that the potential part of the electric field, E_p, satisfies the equation

$$\nabla.(cE_p+v\times B)-J_r.\nabla\left(\frac{c}{\sigma}\right)=0 \tag{6}$$

This expression shows that the electric currents needs to be included in determining the electric field (the potential part) only when: conductivity is not too large and varies substantially in the direction along the current. Otherwise, we obtain E_p from the plasma velocity, magnetic field, and the gradient of a potential with zero Laplacian. This electrostatic potential constitutes a boundary condition and depends on the space-charge outside our domain. The external charge is part of the global problem, and is considered as given for our study. The space-charge within our domain can be found from the equation:

$$4\pi c\rho_e = c\nabla . E_p = -\nabla . (v\times B)_p + J_r . \nabla\left(\frac{c}{\sigma}\right) \tag{7}$$

At this point the usual approach is to derive an equation for the magnetic field by taking the curl of the second of equations (1) and replacing equation (3), resulting in

$$\frac{\partial B}{\partial t} + \nabla\times[D_m(\nabla\times B)] = \nabla\times(v\times B) \tag{8}$$

where D_m is the magnetic diffusion coefficient. For homogeneous and isotropic resistivity this equation gives the more usual expression containing the Laplacian of B. The right hand side (RHS) of equation (8) depends on the velocity and may be strongly non-linear. This equation generally has very complicated vector properties which often make it difficult to find the solutions of the equations, and the magnetic field it describes contains the result of all sources of the field including the local and remote sources.

We take a different approach by combining equations (1) to eliminate the magnetic field. The relationship between the electric current and field follows, thus

$$\frac{\partial J_r}{\partial t}+\frac{c^2}{4\pi}\nabla\times(\nabla\times E_r)=0 \tag{9}$$

Then using the Ohm's law (equation [4]) to eliminate the electric field we derive a modified diffusion equation for the electric current J

$$\frac{\partial J_r}{\partial t}+\nabla\times[\nabla\times(D_m J_r)]=\frac{c}{4\pi}\nabla\times[\nabla\times(v\times B)] \tag{10}$$

The equation for the electric current (10) resembles the equation for the magnetic field (8) but has much simpler vector properties because the diffusion coefficient appears directly as a coefficient of the current. Again the double vector product can be replaced by a Laplacian to obtain a classical diffusion equation. However this transformation no longer requires the diffusion coefficient to be homogeneous and isotropic, but also results when the divergence of the term (D_m J) is zero; which is satisfied not only by homogeneous resistivity, but also by constant resistivity along the current lines. The equation for the electric current (10) could also be obtained by taking curl of the equation for the field

(8), and it may be considered as of higher order in the spatial derivatives. However, this is not a disadvantage quite the contrary because the LHS of equation (8) contains a mix of the potential and rotational magnetic fields, and the effects on the plasma dynamics only arise by the curl of the field. While the LHS of equation (10) contains only the electric current which corresponds to a local property of the plasma and not the more abstract magnetic field which reflects contributions from the whole spatial domain. Therefore, equation (10) describes the temporal variation of a quantity similar to other local properties of the plasma such as velocity and density.

The magnetic field appears in equation (10) only in the form of a vector product with the velocity. This magnetic field can be expressed in terms of the currents J and boundary conditions using the second of equations (1). The subsequent equation can be formally solved by transforming it into an integral form. If the integral is performed over the whole infinite domain the potential part of the magnetic field, B_p, must be zero and the magnetic field is the curl of a vector potential, A_b. This vector potential can, in turn, be computed from the total electric current distribution from

$$A_b(r) = \frac{1}{c}\int_{\infty} \frac{J(r')}{|r-r'|}\, dr'^{3} \tag{11}$$

However, in astrophysical problems, there is most often only observational data concerning the magnetic field on the boundaries of the finite region under study. In this case it is more convenient to define both a scalar and vector potentials for the magnetic field. These potentials arise from the currents both outside and inside the domain of interest and satisfy the equations

$$\nabla^2 \Phi_b = 0$$
$$\nabla^2 A_b = \frac{-4\pi}{c} J \tag{12}$$

The scalar potential can be found from the boundary conditions alone, but the vector potential requires the knowledge of the currents inside our domain. These equations can be formally integrated using Green's functions or using image methods.

In some problems with simple geometry authors have resorted to formulating equations for the vector potential A_b instead of solving equation (8). This approach has some advantage for ideal cases in which current dissipation is negligible. However, the vector potential is still a quantity which depends on the total current distribution, as shown by equation (11), and is not a local property of the plasma.

Next we will consider how to derive the velocity from a consideration of the plasma dynamics. This plasma dynamics only arise through the curl of the magnetic field, viz. through the interaction of the local electric current with the remote currents. This emphasizes the significance of our formulation of equation (10).

III. The Plasma dynamics

The dynamics of the plasma is found from the equations

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+\nabla.(\rho v)=0 \\
&\rho\frac{\partial v}{\partial t}+\rho(v.\nabla)v+\nabla p-\vec{g}\rho=F_v+F_1 \\
&F_v=-\nabla\times[\mu(\nabla\times v)]+\nabla(\frac{4}{3}\mu\nabla.v) \\
&F_1=\frac{1}{c}(J\times B)
\end{aligned} \tag{13}$$

which contain both gravity and viscous terms.

The temporal variation of the density can be obtained by combining the first and second of equations (13) to obtain an explicit equation for the density which contains no first-order velocity terms. Thus

$$\frac{\partial^2\rho}{\partial t^2}-\nabla.[v\nabla.(\rho v)+(\rho v.\nabla)v+\nabla p-\vec{g}\rho-F_v-\frac{(J\times B)}{c}]=0 \tag{14}$$

An equation describing the rotational electric current can also be obtained which contains only higher-order velocity terms. Taking the vector product of the second of equations (13) with the magnetic field we derive the expression for the time derivative of the right hand side of equation (10),

$$\frac{\partial(v\times B)}{\partial t}=(v\times\frac{\partial B}{\partial t})+B\times[(v.\nabla)v]+\frac{F_v\times B}{\rho}+ \\ +\frac{B\times\nabla p}{\rho}+(\vec{g}\times B)+\frac{1}{\rho c}[(J\times B)\times B] \tag{15}$$

This equation can be further expanded by combining the last of equations (1) and equation (4) into

$$\frac{\partial B}{\partial t}=\nabla\times(v\times B-\frac{c}{\sigma}J) \tag{16}$$

Substituting we obtain the more explicit relation

$$\frac{\partial(v\times B)}{\partial t}=-v\times[\nabla\times(\frac{c}{\sigma}J)]+v\times[\nabla\times(v\times B)]+B\times[(v.\nabla)v]+ \\ +\frac{F_v\times B}{\rho}+\frac{B\times\nabla p}{\rho}+(\vec{g}\times B)+\frac{1}{\rho c}[(J\times B)\times B] \tag{17}$$

which displays a number of non-linear terms whose importance can be assessed for different problems. Inserting equation (17) into the time derivative of equation (10) we obtain the equation for the electric current, in which the lowest order terms of equation (10) on the velocity have been eliminated,

$$\frac{\partial^2 J}{\partial t^2}+\nabla\times\{\nabla\times\{\frac{\partial(D_m J)}{\partial t}+v\times[\nabla\times(D_m J)]-\frac{c}{4\pi}v\times[\nabla\times(v\times B)]- \\ -\frac{c}{4\pi}B\times[(v.\nabla)v]+\frac{c(B\times F_v)}{4\pi\rho}+\frac{c(\nabla p\times B)}{4\pi\rho}+\frac{c(B\times\vec{g})}{4\pi}+\frac{B\times(J\times B)}{4\pi\rho}\}\}=0 \tag{18}$$

This equation displays only higher-order terms in the velocity except for the third term which contains the product of the current and the velocity. However, this term also contains the magnetic diffusion as a factor and therefore is negligible in most high-conductivity cases.

IV. Linear departures from static equilibria

Let us now apply equations (18) and (14) to describe the electric current and plasma dynamics in some specific situations.

Linearizing the hydrodynamic equations, and neglecting viscosity, the equation of motion (14) for the departures from the equilibrium values becomes

$$\begin{aligned} &\frac{\partial^2 \delta\rho}{\partial t^2} - \nabla^2 \delta p + \vec{g}.\nabla\delta\rho + Q_j = 0 \\ &Q_j = \frac{1}{c}\nabla.\delta\,(J\times B) \end{aligned} \tag{19}$$

Neglecting Joule dissipation the equation for the rotational electric current becomes

$$\begin{aligned} &\frac{\partial^2 J}{\partial t^2} - \nabla\times\{\nabla\times[\frac{(J\times B)\times B}{4\pi\rho_0} + Q_p]\} = 0 \\ &Q_p = c\,[\frac{B\times\nabla\delta p}{4\pi\rho_0} + \frac{\vec{g}\times B}{4\pi\rho_0}\delta\rho] \end{aligned} \tag{20}$$

These two sets of coupled equations, together with the equation which relates the pressure to the other variables, contain the full solutions to the first-order problem. The plasma velocity can be obtained from equations (13), provided that the density, electric current, and magnetic field are known.

The equation for determining the pressure variation results from the energy equation. This equation can be expressed, assuming that the plasma is adiabatic and has a polytropic index gamma, as

$$\frac{\partial^2 \delta p}{\partial t^2}+[\rho_0 \vec{g}+\gamma p_0 \nabla]\,.\,(\frac{-\nabla\delta p+g\delta\rho+\delta F_1}{\rho_0})=0 \tag{21}$$

where we have eliminated the velocity.

The system of equations is rather complicated for cases where the density, temperature and magnetic field are all inhomogeneous. However, equations (19) and (20) have a very simple structure and much can be learned from them even in these complicated cases. In the first place one can easily characterize the acoustic, magneto-acoustic and Alfven modes regardless of geometry and inhomogeneity of the background atmosphere. For both the acoustic and the Alfven modes, the two equations (19) and (20) are decoupled. In the acoustic mode only equations (19) and (21) determine the solutions and give the density and pressure variations by setting $Q_j=0$. In the Alfven mode only equation (20) determines the solutions and $Q_p=0$. Therefore, the equations for propagation of such disturbances become very compact. On the other hand the magneto-acoustic mode arises by the coupling of all three equations, (19), (20) and (21), and this is more complicated. However, in many cases found in astrophysics simplifications can be made which allow for simplifying the equations. For cases where the Alfven velocity is much larger or much smaller than the sound speed either Q_p or Q_j can be neglected and the solution can be found by solving the equation for the dominant variable (J or p, respectively), viz. the Alfven or acoustic mode equation, respectively. Then the equation for the dominated variable (p or J, respectively) is solved by using the already determined force (Q_j or Q_p, respectively).

In the following paragraphs we will apply our scheme to solve particular problems. In this way we will show how our method permits for simple solution of two-dimensional cases, for given boundary and initial conditions. This cases have not been solved in detail for the situations discussed in this paper, but conceivably they may also be solved by other specifically tailored methods. We will show that our method is simple, very general, and gives substantial insight into the role of the boundary conditions and the magnetic free-energy buildup in the local plasma.

A. Negligible compression cases.

A simple case is that of an inertial regime where the density and pressure terms in equation (20) are negligible. This may correspond to Alfven modes, or to magneto-acoustic mode motions in a medium where the Alfven velocity is much larger than the sound speed. In either case this requires that the background magnetic field be substantial. In this regime equation (20), with Q_p negligible, dominates and the velocity is simply given by the force balance between the Lorentz force (due to the electric current and magnetic field) and the inertial force, according to

$$\frac{\partial v}{\partial t} = \frac{\delta\,(J\times B)}{\rho_0 c} \tag{22}$$

where the density and the background magnetic field can be arbitrary functions of the coordinates.

Considering a purely transverse (to the magnetic field) current, equation (20) becomes

$$\frac{\partial^2 J_t}{\partial t^2} + \nabla\times[\nabla\times(\frac{B_0^2}{4\pi\rho_0}J_t)] = 0 \tag{23}$$

where the factor multiplying the current is the Alfven velocity V_A. The magnitude of the Alfven velocity can be variable because of either density or background magnetic field inhomogeneities, and in this apparently simple case the equations for the velocity or the magnetic field may have very complicated vector properties because of variations in plasma density and in magnitude and orientation of the background magnetic field. Instead, we find a far simpler equation (23) for the rotational part of the current (the potential part was discussed above, see equation 5).

Let us consider the cases of a magnetic arcade and of a horizontal line of null field, both in a stratified isothermal atmosphere. These cases can be considered as prototypes for many cases found in solar physics and astrophysics[13]. For further simplification we assume that the background current is zero through the domain we solve, i.e. the background field is potential. The two magnetic configurations are depicted in Figures 1 and 2.

The simple magnetic arcade can be constructed using the potential field

$$B_0 = b\nabla arctg(\frac{y}{z}) \tag{24}$$

The case of the null line can be constructed using

$$B_0 = b\nabla(yz) \tag{25}$$

In both cases we only consider regions away from discontinuities or zeroes of the field (which occur at y=z=0). Note that in both cases the magnitude of the field decreases linearly or increases inversely, respectively, with distance from the y=z=0 line. Therefore a cylindrical coordinate system is appropriate for the case of homogeneous background density. We find the equations

$$\frac{\partial^2 J}{\partial t^2}+\frac{b^2}{4\pi\rho_0}\nabla\times[\nabla\times(\frac{J_t}{r^2})]=0 \tag{26}$$

for the simple arcade case, and

$$\frac{\partial^2 J}{\partial t^2}+\frac{b^2}{4\pi\rho_0}\nabla\times[\nabla\times(r^2 J_t)]=0 \tag{27}$$

for the null line case. In both cases $r=(y^2+z^2)^{1/2}$ is the distance to the discontinuity or null line, respectively. Considering separable solutions, assuming that the current flows only along the x direction, and using cylindrical coordinates in which the longitudinal axis is oriented along x, and the angular variable is measured from the vertical (z axis), we find solutions of the form

$$J_x=ae^{i(\omega t+m\phi)}f(r) \tag{28}$$

where a is the amplitude and f(r) satisfies the equation

$$\frac{1}{r}\frac{d}{dr}[r\frac{d(r^{2\alpha}f)}{dr}]+q(r)f=0 \tag{29}$$

with

$$\alpha=-1$$

for the arcade case, and

$$\alpha=1$$

for the null line case. In both cases is

$$q(r)=\omega^2\frac{4\pi\rho_0}{b^2}-m^2\frac{r^{2\alpha}}{r^2} \tag{30}$$

We find the solutions for the two previous cases by performing some transformations. Note that for purely oscillating solutions

$$K=\omega^2\frac{4\pi\rho_0}{b^2}>0 \tag{31}$$

The general solutions for the arcade case are found in terms of Bessel functions because equation (29) can be transformed into a standard Bessel equation[14]

$$\xi^2\frac{d^2g}{d\xi^2}+\xi\frac{dg}{d\xi}+(\xi^2-\nu^2)g=0 \tag{32}$$

by defining the variables

$$\begin{aligned} g&=\frac{f}{r^2}\\ \xi&=\frac{r^2}{2}\sqrt{K} \end{aligned} \tag{33}$$

and the parameter

$$\nu=\frac{m}{2} \tag{34}$$

The behavior of the solution depends on the value of m, but in general it oscillates above and below zero as r increases. The solutions can be expressed as

$$f(r)=r^2 J_{m/2}\left(\frac{r^2}{2}\sqrt{K}\right)$$
$$f(r)=r^2 Y_{m/2}\left(\frac{r^2}{2}\sqrt{K}\right) \tag{35}$$

We can describe exponential growth or damping in time of disturbances from the current-free background by assigning a negative value to K (i.e. imaginary omega), and we obtain the modified Bessel equation[14]. In this case the solutions show a monotonic behavior and no change of sign.

The solution for the null line case is very different and can be expressed as

$$f(r)=\frac{r^{\pm i(K-m^2)^{1/2}}}{r^2} \tag{36}$$

The behavior depends critically on the parameter ($K-m^2$), but the same formal expression can be always be used. If this quantity is larger than zero the solution f(r) oscillates around zero with decreasing amplitude as r increases. This case can also be expressed using trigonometric functions. But, if the parameter ($K-m^2$) is negative, i.e. $m^2>K$, the solution is monotonically increasing or decreasing and does not change sign. The case of exponential grow or decrease in time, viz. K<0, is just another case where the parameter $(K-m^2)<0$.

An analysis of the null line problem, although in a very different context, was made by Craig and McClymont[10]. They have not used an equation for the electric current but rather for the velocity and they arrive at solutions which are a subset of those we show, i.e. they correspond to m=0 and K>0. But they address their study to magnetic reconnection which we neglect here because in the conditions we consider it is of little importance (see below). The significance of the reconnection region can be assessed by considering the resistive terms associated to Joule dissipation. Considering the complete equation (18) for the current evolution we compare the dissipation term

$$\frac{\partial(D_m J)}{\partial t}=D_m\omega J \tag{37}$$

with the propagation term

$$V_A^2 J=\frac{b^2 r^{2\alpha}}{4\pi\rho_0}J \tag{38}$$

This comparison permits the definition of a radius R_d at which the Joule dissipation term has the same magnitude as the Lorentz force term

$$R_d^{2\alpha}=\left(\frac{4\pi\gamma p_0}{b^2}\right)\left(\frac{D_m\omega}{V_s^2}\right) \tag{39}$$

Using typical values for the high-conductivity solar atmosphere, $T=10^6$ K and $p=1$ dyne cm^{-2}), and for values of the field gradient and rate of variations from the observed fine-scale photospheric features which drive the field evolution, $b=0.1$ G km^{-1} and $\omega=10^{-3}s^{-1}$, we obtain

$$R_d=0.27\,cm$$

For values of r much larger than R_d, i.e. for all practical values of r, the Joule dissipation is negligible compared with the Lorentz force term and the plasma behavior can be considered as ideal. Analogously, in the arcade case, there is a radius at which the current dissipation becomes as important as the Lorentz force term. The magnitude of this radius can be estimated using equation (44).

Note that the RHS of equation (43) contains two factors, the first corresponds to the radius, R_p, at which the Alfven velocity reaches the same value as the sound speed (viz the plasma-beta becomes of the order of unity). The second is a dimensionless factor, which is independent of the field gradient and much smaller than unity (about 2×10^{-14} in our case). We estimate that the gas pressure effects are significant over the region with $r<R_p$ for the null line case, with $R_p=6.8\ 10^6R_x$ (and for $r>R_p$ for the arcade case, with $R_p=1.5\ 10^{-8}R_d$).

In summary, for cases in the high-conductivity solar plasma and with the kind of magnetic fields and motions commonly observed in the Sun (except perhaps for the most violent events) we expect that the low-beta approach used here will break down long before any non-ideal effects become important. Therefore, the boundary conditions for the low-beta regions are imposed by matching with the surrounding high-beta regions. Current dissipation, and magnetic reconnection, may only take place in a very small core buried inside the high-beta region. In the arcade case the current dissipation will only dominate far outside the arcade, in the high-beta region. In both cases, the dynamics should be obtained independently of the field and then the dissipation effects can be evaluated.

In the case when $r<R_p$ for the null line or $r>R_p$ for the arcade, i.e.

$$\frac{b^2 r^{2\alpha}}{4\pi} < \sqrt{\gamma} p_0 \tag{40}$$

the solutions we have shown are no longer valid and one finds the usual propagation of acoustic perturbations. We estimate that the solutions we show here are valid in the region where $V_A>V_s$, i.e. for $r>R_p$ for the null line case (with the values given above it is R_p=46 km). Therefore, any equations which neglect the gas pressure effects are inappropriate in the limit of very small r, and the outer solutions for the low-beta region have to match those for the high-beta region (instead of those for the central, minute, reconnection region).

For the typical null line case shown here the inner high-beta region is rather small (although far larger than the resistive region) and the sound transit time through this region is much smaller than the characteristic times of the evolution we consider

$$\frac{R_P}{V_s}=0.3s\ll\omega^{-1} \tag{41}$$

Therefore, this central region can be studied using a quasi-steady approach. This approach will be studied in the next section.

Also, using the typical parameters given above we obtain a typical value of K of about 9×10^{-8}, which results a very small number and is almost negligible except for the mode m=0.

Our solutions for the two cases give general analytical solutions for the electric currents which would result in a simple arcade and in a null line case in the corresponding low-beta regions. The solutions shown must be combined with the appropriate coefficients, defined by the boundary and initial conditions, in order to solve particular problems. In most cases these boundary conditions are given by the non-negligible plasma-beta neighboring regions.

As a simple example of how to use the boundary conditions in the simple cases we showed, let us assume that the velocity is well known at the z=0 plane (e.g. from observations). The z>0 region satisfies $V_s<V_A$ and we want to find the electric currents and the

distortion of the field in this region. For the arcade case, at some large value of r, R_p, the previous condition does not apply, and also there must be a lower limit for r, R_m, because otherwise the background potential field would become arbitrarily large. This defines a semi-cylinder, shown in Figure 3, which contains the region where our previous solutions are valid. Let us now assume that the velocity variations are purely horizontal (v_z=0), specified, and continuous at the plane z=0. Let us assume, for simplicity, that the semi-cylindrical boundaries effects on the solution can be ignored. For the case of the null line, one obtains a similar semi-cylindrical domain, except that the roles of R_p and R_m are interchanged (i.e. R_p becomes the lower limit and R_m the upper limit for r).

In these conditions we can obtain the amplitude coefficients from the matching of the specified velocity with our solutions resulting from equation (39)

$$\begin{aligned} v_\omega(z=0, y=+r) &= \frac{+ibr^\alpha}{\omega\rho c}\sum_m a_m e^{im\frac{\pi}{2}} f_m(r) \\ v_\omega(z=0, y=-r) &= \frac{-ibr^\alpha}{\omega\rho c}\sum_m a_m e^{-im\frac{\pi}{2}} f_m(r) \end{aligned} \tag{42}$$

These equation completely define the coefficients a_m and therefore give the full solution for the spatial dependence of all quantities at any given frequency of the driving boundary (z=0).

This calculation becomes simpler due to the smallness of K, which permits expansions of the functions f_m in terms of this parameter. These expansions result in polynomials whose coefficients can be easily determined from the boundary conditions shown.

However, in the previous equation (22) we have neglected the gas pressure and the component of the velocity parallel to the background field remains undefined. But, equation (23) completely defines the current J and we can compute the term Q_j of equation (19), and using this equation (and the energy equation) we can find the density and pressure variations. These variations define the velocity along the field and require additional initial and boundary conditions. Using the previous results we find

$$Q_j = \frac{ab}{\rho c} \frac{f'(r)}{r} e^{i(\omega t + m\phi)} \tag{43}$$

for the simple arcade, and

$$Q_j = \frac{ab}{\rho c} [r f'(r) \cos(2\phi) - i m f(r) \sin(2\phi)] e^{i(\omega t + m\phi)} \tag{44}$$

for the null line case. The solution of the equations in general is very complicated, but we are only interested in the small r and slow variations. In the next section we show how one can fully solve the null line case in the regions close to the high-beta core.

B. Practically incompressible regions

Let us now consider the case in which the velocity variations are very slow and, as before, the magnetic field is almost potential (i.e. the background electric current is zero). We assume here that gravity is negligible and that the time derivative terms in equations (19) and (20) are negligible. The resulting system of equations is

$$\begin{gathered} \nabla . [\nabla(\delta p) - \frac{J \times B}{c}] = 0 \\ \nabla \times \{\nabla \times \{B \times [\nabla(\delta p) - \frac{J \times B}{c}]\}\} = 0 \end{gathered} \tag{45}$$

These equations show that the value between brackets can be solved as a vector function, F. Replacing this into the equations (45) we find that the divergence of the function F is zero, and there is an equation for the part of F perpendicular to B. Any such function defines a solution of the set of equations (45), therefore we can first solve the equations

$$\begin{gathered} \nabla . F = 0 \\ \nabla \times [\nabla \times (B \times F)] = 0 \end{gathered} \tag{46}$$

for the function F, and then find the pressure and electric current using the definition of F.

Using the result for F, the pressure gradient results

$$B.\nabla(\delta p) = F.B \tag{47}$$

and the electric current is given by

$$J_t = \frac{c}{B^2}\,[B\times(\nabla\delta p - F)] \tag{48}$$

where no constraints are posed on the component of J parallel to the magnetic field.

These relations can define the electric current variations across the domain after the pressure variations are solved by using the solutions for the function F and the boundary conditions.

Let us study the central region of the Figure 3, in the case of the null-line shown in the previous section. For this central region we consider two boundaries, a) the z=0 plane, and b) the semi-cylinder of radius R_p. Because we consider homogeneous background density and pressure in the example, the adiabatic energy equation (21) gives simply

$$\delta p = V_s^2 \delta \rho \tag{49}$$

where V_s is the adiabatic sound speed. Using the previously shown estimates for the physical parameters, we find that the domain we consider satisfies the basic conditions for our equations (45). The value of K measures the importance of the time-derivative term in equation (20), and it was shown to be very small. On the other hand, the importance of the time-derivative term in equation (19) can be evaluated from

$$L=\frac{\omega^2 R_p^2}{V_s^2}=9x10^{-8} \tag{50}$$

(This number is identical to K because of the definition of R_p).

Using the second of equations (46) we find that

$$B\times F=\hat{e}_x[C'\ln(r)+\sum_m C_m r^m e^{im\phi}] \tag{51}$$

This equation defines completely the component of F perpendicular to B, in terms of the coefficients C_m given by the boundary conditions at the z=0 plane and (see above), and some properties of the asymptotic physical behavior for either small or large r. In the case we study many solutions (those with m<1, and the logarithmic solution) are non-physical because the force, F, would display singularities at r=0 which are not expected for the problem we are treating.

In order to find the remaining component of the force, the first of equations (46) and a boundary condition must be used. The analysis can be carried with the help of the magnetic field potential and another potential defined in such way that its gradient is perpendicular to the magnetic field, i.e., let

$$\Phi = byz = br^2 \frac{\sin(2\phi)}{2}$$
$$\Psi = \frac{b}{2}(z^2 - y^2) = br^2 \frac{\cos(2\phi)}{2} \tag{52}$$

This set of functions satisfy the conditions

$$B^2 = |\nabla\Phi|^2 = |\nabla\Psi|^2 = b^2 r^2$$
$$\nabla\Phi . \nabla\Psi = \nabla^2\Phi = \nabla^2\Psi = 0 \tag{53}$$
$$\nabla\Phi \times \nabla\Psi = \hat{e}_x (b^2 r^2)$$

(The functions introduced can be interpreted as a curvilinear set of coordinates to replace the cartesian ones.)

The force at any location (except at the central point r=0) can be expressed in terms of the field-aligned and transverse (but in the plane [y,z]) components. Using the functions previously introduced it is

$$F = \alpha\nabla\Phi + \beta\nabla\Psi \tag{54}$$

The transverse component was previously determined, and for a given m (omitting the logarithmic solution) we find

$$\beta = C_m r^{m-2} e^{i(\omega t + m\phi)} \tag{55}$$

Therefore the only remaining problem is to find α by solving the equation

$$\nabla . F = \nabla\alpha . \nabla\Phi + \nabla\beta . \nabla\Psi = 0 \tag{56}$$

Using the previous definitions we find the formal solution

$$\alpha = -\int \frac{\partial\beta}{\partial\Psi} d\Phi + h(\Psi) \tag{57}$$

which contains the boundary condition in the integration constant $h(\Psi)$. (It can also be easily demonstrated that this integration constant also defines the curl of the force.) The solutions in our case have a slightly complicated expression due to the trigonometric expressions involved, we obtain

$$\alpha = \frac{2C_m}{b^2} \int r^{m-6} (\Psi - im\Phi) \, d\Phi + h(\Psi) \tag{58}$$

with

$$\begin{aligned} r &= \frac{2}{b}\sqrt{\Phi^2 + \Psi^2} \\ \phi &= \frac{1}{2} arctg(\frac{\Phi}{\Psi}) \end{aligned} \tag{59}$$

In general the boundary condition can be set by first finding the variation of the magnetic potential along the boundary $\Phi_{bond}(\Psi)$, and second using

$$h(\Psi)=\alpha(\Phi_{bond},\Psi)+\left(\int\frac{\partial\beta}{\partial\Psi}d\Phi\right)_{\Phi_{bond}} \tag{60}$$

In our example, if we assume that the field-aligned component of the force is zero at the plane z=0, we find

$$\begin{aligned}&\Phi_{bond}=0\\&h(\Psi)=\left(\int\frac{\partial\beta}{\partial\Psi}d\Phi\right)_{\Phi=0}\end{aligned} \tag{61}$$

In order to show the details of the application of boundary conditions, let us consider the m=2 mode. Therefore, suppose that from the velocities one has found that the total force at the boundary, plane z=0, one has found that only this mode is present. From equation (55) we obtain

$$\beta_2=C_2e^{i\omega t}\frac{\Psi+i\Phi}{\sqrt{\Psi^2+\Phi^2}} \tag{62}$$

Using equation (57) and the boundary condition (zero vertical force at z=0) we find

$$\alpha_2=C_2e^{i\omega t}\left[\ln\left(\frac{\sqrt{\Psi^2+\Phi^2}+\Phi}{|\Psi|}\right)-\frac{\Phi}{\sqrt{\Psi^2+\Phi^2}}+i\frac{\Psi}{\sqrt{\Psi^2+\Phi^2}}-i\,sign(\Psi)\right] \tag{63}$$

These quantities completely determine the force, but in order to find the specific behavior of the gas pressure and the electric current one more boundary condition has to be applied. Let us consider two extreme cases. In the first we assume that the pressure at the boundary is constant (i.e., not only the vertical but also the horizontal pressure variation is zero. In this case, the pressure anywhere in the domain can be expressed as

$$p=C_2 e^{i\omega t}[2\sqrt{\Psi^2+\Phi^2}-2|\Psi|-\Phi\ln(\frac{\sqrt{\Psi^2+\Phi^2}+\Phi}{|\Psi|}) \\ +i\Phi sign(\Psi)-i\Psi\ln(\frac{\sqrt{\Psi^2+\Phi^2}+\Phi}{|\Psi|})] \tag{64}$$

and the electric current is

$$\frac{J}{c}=\hat{e}_x C_2 e^{i\omega t}[\frac{\sqrt{\Psi^2+\Phi^2}}{\Psi}-2sign(\Psi)-i\ln(\frac{\sqrt{\Psi^2+\Phi^2}+\Phi}{|\Psi|})] \tag{65}$$

Note that this solution contains very large currents close to the separatrix surfaces, at $z^2=y^2$. The direction of the current is opposed on both sides of the separatrix. Also, currents flow at the boundary, and below (z<0) and these are the responsible for the plasma dynamics. In the other extreme, we may consider the case when the pressure varies over the boundary and some terms of equation (64) are omitted or changed. In this case terms can be added to the pressure which depend only on Ψ and the electric current changes (to maintain the condition set on the force F). This shows that in this case the dynamics is given by both gas pressure and electric current. Dynamics given purely by gas pressure (with zero electric current) is only possible for vertical pressure gradient imposed at the boundary.

The example studied here in some detail is just intended to illustrate the application of our method to relatively simple two-dimensional cases. However, the more complicated three-dimensional cases can be treated in an analogous way, although the equations become more complicated and difficult to solve.

VI. Conclusions

We have shown a formulation for solving MHD problems which develops equations for the plasma density and electric current. This formulation differs from the customary method of solving the equations for the velocity and the magnetic field. Our proposed method displays vector properties which make it easier to find solutions in geometrically complicated problems. Our method is specially well suited for studying the interaction between low- and high-beta regions which are common in plasmas and lead to very interesting observed phenomena. Our approach permits to compute directly the electric currents from physical considerations, and differs from the usual in which the magnetic field is derived and then the currents may be computed by differentiating the field. The direct study of the electric currents gives insight into the way they are generated and evolve in space and time in typical astrophysical situations.

Our formulation is applied to some typical cases found in astrophysics, and we show the explicit analytic solutions. We have considered two cases in which the velocities and electric currents are small, and the Alfven velocity given by the background potential field is much larger than the sound speed. The two cases studied in detail correspond to the inner region of a simple magnetic arcade, and the outer region around a null line. These two situations have been studied before[5,10,11] in a different context. Here we show a general formulation to the solution of the problem of the dynamics of the plasma and magnetic energy in stellar atmospheres resulting from driving motions of the inner regions plasma. This method is based on the fact that usually some quantities such as plasma velocity and pressure can be estimated over a given boundary from observations. We show two detailed solutions for one normal mode in the null line case. This example shows in full detail how to apply our formalism and the proper boundary conditions for fully solving all variables. The general solutions shown for the arcade and the null line cases are expressed in terms of modes which are neither pure Alfven nor magneto-acoustic modes. We show that in the limit of low plasma beta ($V_A >> V_s$) the distinction between these modes is irrelevant, and much more practical solutions can be found using our formulation.

The applications we have developed here in detail are basically two-dimensional. In this case we find relatively simple analytical expressions for describing all quantities. However, our formulation is particularly interesting for more complicated full three-dimensional problems, in which our approach of dealing with the density and electric current (and pressure) has significant advantage over the previous approaches. This is because we solve the electromagnetic directly for the electric current and gas pressure, which are local plasma related quantities, and we rely on observable boundary conditions, e.g., the velocity and pressure at the boundary. This basic problem is very important in astrophysics because the magnetic energy supply to the outer layers (dependent on the electric currents development in these layers) of solar-type stars is supposed to be the main heating mechanism. Also this energy is supposed to accumulate in the corona, under certain conditions, and release explosively in solar and stellar flares and mass ejections.

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FIGURE CAPTIONS

Fig. 1- The potential magnetic field for a simple arcade is the upper half of the circular lines. This field would result from an infinitely long line current, located along the x axis (perpendicular to the plane of the figure).

Fig. 2- The potential field for an infinitely long null line. This field configuration can be used for approximating the field in the region which surrounds the zero field line. Such field configuration may result from two line currents at a large distance form the region of interest. The configuration is not appropriate at large distances from the line because the magnetic field increases linearly with this distance and this would lead to non-reasonable field values.

Fig. 3- The region in which the solutions obtained here are valid. For the simple arcade the small radius is R_m, and the large radius is R_p. For the null line case the small radius is R_p, and the large radius is R_m. Typical values are, for the small radius 10^2 km, and for the large radius 10^4 km.

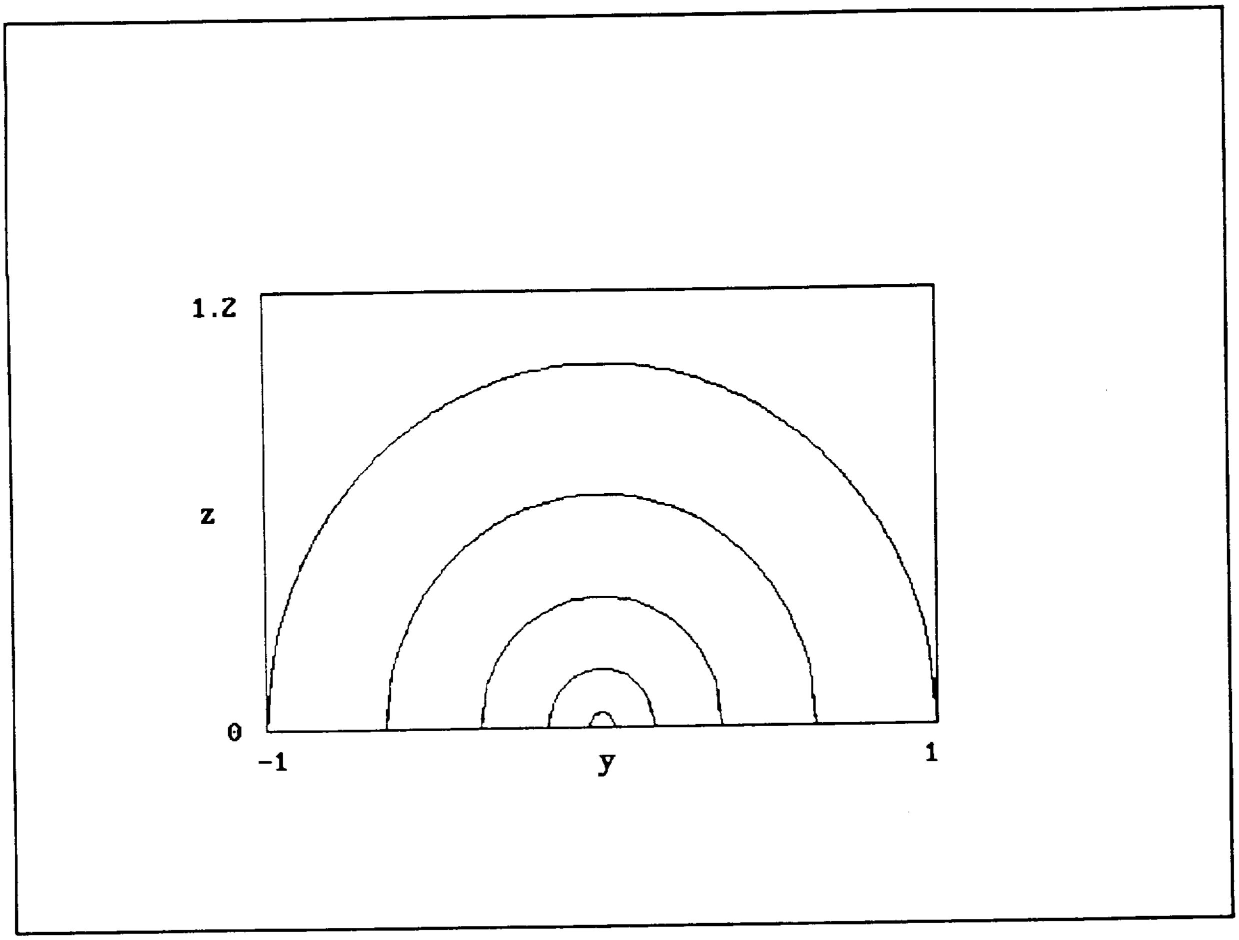

Figure 1

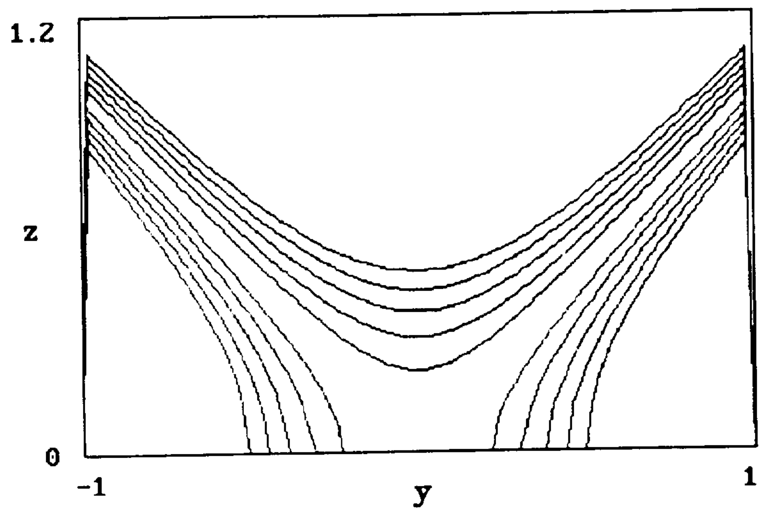

Figure 2

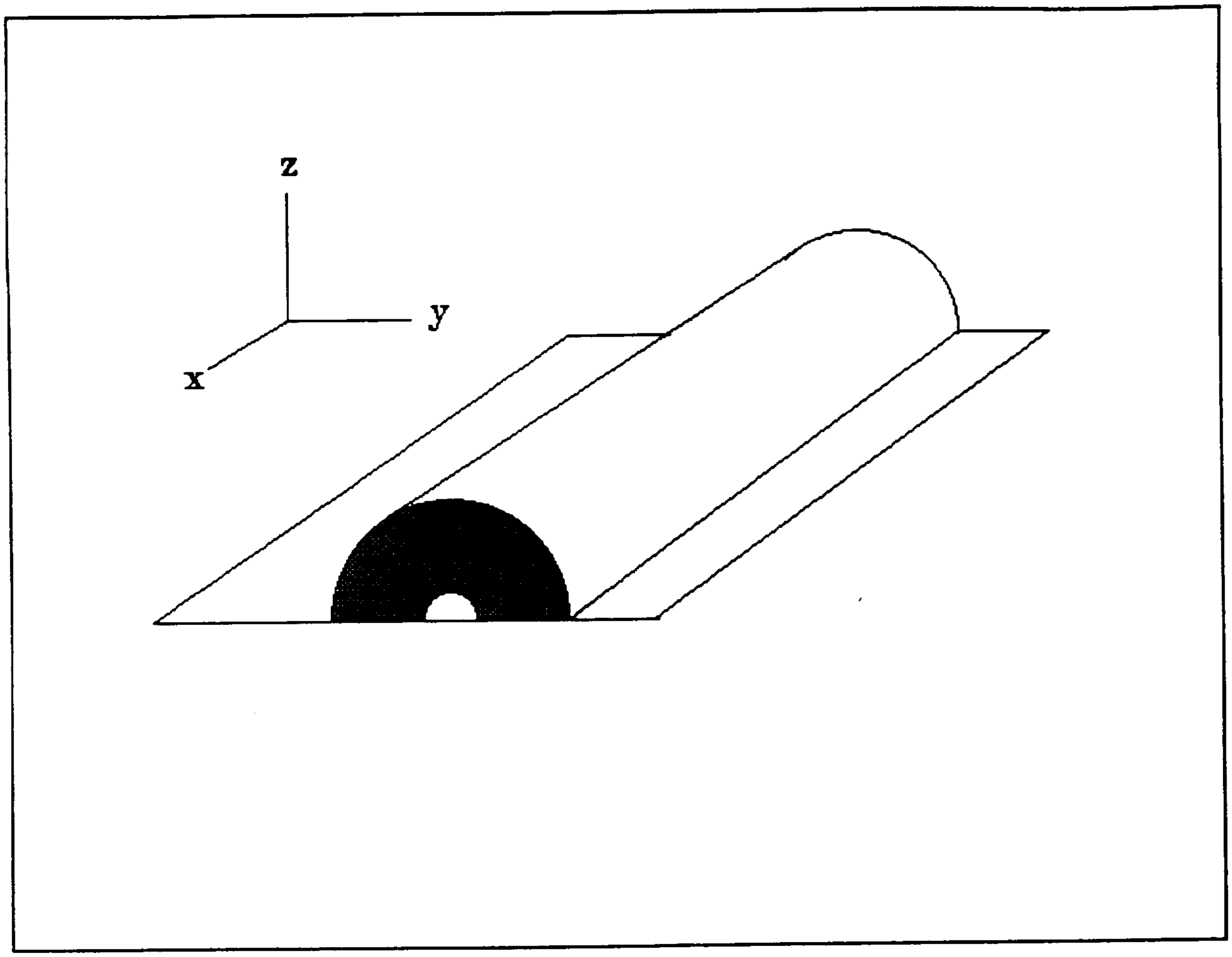

Figure 3.