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Renormalization Group Methods for the Reynolds Stress Transport Equations

R. Rubinstein

1. Motivation and Objective

The Yakhot-Orszag renormalization group is used to analyze the pressure gradient-velocity correlation and return to isotropy terms in the Reynolds stress transport equations. The perturbation series for the relevant correlations, evaluated to lowest order in the ϵ-expansion of the Yakhot-Orszag theory, are infinite series in tensor product powers of the mean velocity gradient and its transpose. Formal lowest order Padé approximations to the sums of these series produce a rapid pressure strain model of the form proposed by Launder, Reece, and Rodi, and a return to isotropy model of the form proposed by Rotta. In both cases, the model constants are computed theoretically. The predicted Reynolds stress ratios in simple shear flows are evaluated and compared with experimental data. The possibility is discussed of deriving higher order nonlinear models by approximating the sums more accurately.

The Yakhot-Orszag renormalization group provides a systematic procedure for deriving turbulence models. Typical applications have included theoretical derivation of the universal constants of isotropic turbulence theory, such as the Kolmogorov constant, and derivation of two equation models, again with theoretically computed constants and low Reynolds number forms of the equations. Recent work has applied this formalism to Reynolds stress modeling, previously in the form of a nonlinear eddy viscosity representation of the Reynolds stresses, which can be used to model the simplest normal stress effects. The present work attempts to apply the Yakhot-Orszag formalism to Reynolds stress transport modeling.

2. Work Accomplished

The modelling of the pressure gradient-velocity correlation and return to isotropy term in the Reynolds stress transport equation remains an area of active research.[1,2,3] Models will be derived here using the Yakhot-Orszag renormalization group[4] partially along the lines of our previous work[5]. The result is a model for the rapid pressure-strain term of the form proposed by Launder, Reece and Rodi[6] (LRR) and a model for return to isotropy of the form proposed by Rotta[7] with theoretically computed constants in good agreement with accepted values. As is usual in investigations of this sort, the priority of Yoshizawa in deriving a pressure strain model analytically[8] must be noted.

The analysis requires some new ideas in renormalization group theory recently introduced by Yakhot et al[9]. As Yakhot et al[9] emphasize, the application of the renormalization group mode elimination formalism to shear flow creates a double perturbation series in powers of ϵ, the parameter of the isotropic theory, and in powers of a dimensionless strain rate, $\eta = SK/\varepsilon$, where K denotes the turbulence kinetic energy, ε denotes the dissipation rate, and S is a measure of the mean strain: in Ref. 9, $S^2 = \left(\frac{\partial U_m}{\partial x_n} + \frac{\partial U_n}{\partial x_m}\right)\frac{\partial U_m}{\partial x_n}$. The present analysis also leads to double expansions of this type, with the powers S^n replaced by tensors $\mathrm{S}^{(n)}$ homogeneous of degree n in the mean velocity gradient ∇U and its transpose. It will be convenient to retain the terminology of Ref. 9 and call this expansion the η-expansion; when the distinction is pertinent, the expansion of Ref. 9 will be called a scalar η-expansion.

The heuristic program of evaluating all scalar amplitudes to lowest order in ϵ has proven successful in the past: apparently, the ϵ-expansion is an asymptotic series with a sum given very nearly by its first term[10]. Unfortunately, there is no analogous basis for truncating the η-expansion. There are fundamental reasons for this distinction between these expansions. The present η-expansion is tensorial: successively higher order terms do not introduce merely numerical corrections, but increasingly complex asymmetries into the theory. Truncation therefore imposes a possibly inappropriate symmetry or other constraint on the model. Thus, in Ref. 5 the η-expansion for the Reynolds stress τ was truncated at second order as suggested by previous work of Yoshizawa[11] and Speziale[12]. Although this type of modelling permits unequal normal stresses in a simple shear flow, it is not maximally asymmetric: for example, in a flow with mean velocity components $U_i(x_1, x_2)$, a cubic model including a term $\tau \sim \nabla U^2 \nabla U^T + \nabla U \nabla U^{T2}$ would permit nonzero τ_{23} in the presence of vanishing $\partial U_2/\partial x_3$ and $\partial U_3/\partial x_2$, an effect which cannot be ruled out in advance.

Although generalizations[13] of the Cayley-Hamilton Theorem limit the number of independent tensors $\mathrm{S}^{(n)}$, anisotropy and asymmetry cannot exist at all without some terms of higher order in η; indeed, truncation at lowest order in η just produces a theory of isotropic turbulence. But the series truncated at any higher order can be unsatisfactory in flow regions in which some components $\left(\frac{\partial U_i}{\partial x_j}\right)K/\varepsilon$ are large. In such regions, the truncated series is dominated by its highest order terms. For the quadratic stress models of Refs. 5, 11, 12, this domination can produce negative normal stresses in the buffer layers of wall bounded flows. Increasing the order of truncation obviously exacerbates this problem.

It follows that finite truncation of the η-expansion is theoretically unsatisfactory. Yakhot et al[9] therefore propose that this expansion must be summed, even if only approximately, and have suggested a prototype summation in a different context. It should be noted that the same issues arise naturally in Yoshizawa's formalism, which also generates infinite series in the mean velocity gradients (and in other quantities as well) for correlations of interest in turbulence modeling. Yoshizawa has concluded independently that summation of this series is essential and has also derived a Reynolds stress transport model by introducing such summations[14].

In this paper, the perturbation series which the Yakhot-Orszag renormalization group generates for the correlation

$$\Pi_{ij} = -\left\langle u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i} \right\rangle \tag{1}$$

is summed by a low order Padé approximation. Coefficients are evaluated to lowest order in the ϵ expansion, but the summation includes effects of all orders in η. The result is essentially identical to the "model 1" proposed by Launder, Reece, and Rodi[6]. An entirely analogous treatment of return to isotropy yields a model of the form proposed by Rotta[7]. Combining these models leads to a preliminary Reynolds stress transport model. The problem of closing the Reynolds stress diffusion terms is addressed. This problem also leads to an infinite sum.

While it is encouraging that renormalization group methods can be used to derive familiar models, the goal of this investigation is not limited to providing theoretical justification for the LRR and Rotta models, which although widely applied are nevertheless deficient in several well-documented respects[1,2,3]. Instead, renormalization group methods together with approximate summation of the η-expansion can be used to derive higher order and nonlinear corrections to these models in a systematic fashion. Explicit development of such models is left to future investigations.

2.1 Analysis of the Pressure Gradient-Velocity Correlation

The analysis will follow Yakhot and Orszag's derivation of turbulence transport models by renormalization group methods[4]. The equation for velocity products is

$$\frac{\partial}{\partial t} u_i u_j + u_p \left(u_i \frac{\partial u_j}{\partial x_p} + u_j \frac{\partial u_i}{\partial x_p} \right) = - \left(u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i} \right)$$
$$+ \nu_0 \nabla^2 u_i u_j - 2\nu_0 \frac{\partial u_i}{\partial x_p} \frac{\partial u_j}{\partial x_p} \tag{2}$$

where ν_0 denotes the kinematic viscosity. The product $-(u_i \partial p/\partial x_j + u_j \partial p/\partial x_i)$ on the right side of Eq. (2) will become the correlation Π_{ij} defined by Eq. (1) following elimination of all fluctuating modes.
Thus, the perturbation series will be written as

$$\Pi = T_0 + T_1 + \cdots$$

where T_n is of order n in $u^<$ and all amplitudes are evaluated to lowest order in ϵ. To lowest order in ϵ and SK/ε

$$T_1 = \frac{2}{5} K \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{3}$$

in agreement with the analysis of Crow.[16]

At the next order in SK/ε, in the high Reynolds number limit,

$$T_2 = -\frac{1}{105} \frac{1}{4} \frac{40}{3} \frac{3}{4} \nu \left[32 \left(\frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_j}{\partial x_p} + 4 \left(\frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_p}{\partial x_j} \right]^{(0)} + (ij)$$
$$= -\frac{1}{21} \nu \left[16 \left(\frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_j}{\partial x_p} + 2 \left(\frac{\partial U_i}{\partial x_p} + \frac{\partial U_p}{\partial x_i} \right) \frac{\partial U_p}{\partial x_j} \right]^{(0)} + (ij) \tag{4}$$

where (0) denotes deviatoric part and (ij) denotes index interchange in the preceding term.

The next order will produce a term T_3 containing cubic products of velocities $u^<$. In view of the form of the LRR model, it is reasonable to ask whether a term with only one gradient, proportional in the high Reynolds number limit to $\tau\nabla U$ might occur at this order. Such terms do occur, but they cancel. Evaluation of T_3 proves to require expansions of the projection operators to second order, leading instead to terms $\mathbf{S}^{(3)}$ homogeneous of degree three in the mean velocity gradient and its transpose. In general, the term T_n of order n has the form $\mathbf{S}^{(n)}(K/\varepsilon)^n$. As noted in the Introduction, it will be imperative to include effects of all orders in SK/ε in the model, but because the terms T_n involve ever higher order derivatives of the transverse projection operators, they do not have an obvious law of formation. Therefore, an exact summation does not appear feasible.

A simple approximate summation is obtained by introducing the perturbation series[5] for $\overline{u_iu_j}^{(0)}$ in the form

$$\nu\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)=-\overline{u_iu_j}^{(0)}+\sum_{n\geq 2}\mathbf{S}^{(n)}(K/\varepsilon)^n$$

and dropping the quadratic terms. The resulting model,

$$\begin{aligned}\Pi_{ij}=\frac{2}{5}K\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)&+C_{\dagger 1}\left[\overline{u_iu_p}^{(0)}\frac{\partial U_j}{\partial x_p}+\overline{u_ju_p}^{(0)}\frac{\partial U_i}{\partial x_p}\right]^{(0)}\\&+C_{\dagger 2}\left[\overline{u_iu_p}^{(0)}\frac{\partial U_p}{\partial x_j}+\overline{u_ju_p}^{(0)}\frac{\partial U_p}{\partial x_i}\right]^{(0)}\end{aligned}\tag{5}$$

with

$$C_{\dagger 1}=\frac{16}{21}=.7619\qquad C_{\dagger 2}=\frac{2}{21}=.0952\tag{6}$$

agrees with the perturbation series (3) and (4) to terms of order $\mathbf{S}^{(3)}$. However, unlike the explicit quadratic model which results from simply dropping the $O(\mathbf{S}^{(3)})$ terms, this model includes effects of all order in SK/ε. The consequences of this fact will be discussed later. This type of summation has also been applied by Yoshizawa[14]. Eqs. (5) and (6) can be compared with Launder, Reece and Rodi's "model 1", Eq. (5) with the empirically adjusted constants

$$C_{\dagger 1}=.7636\qquad C_{\dagger 2}=.1091\tag{7}$$

In this model, the constants $C_{\dagger 1}$ and $C_{\dagger 2}$ were not chosen independently; instead, to insure some consistency conditions introduced by Rotta[7], Launder, Reece, and Rodi set[6]

$$C_{\dagger 1}=\frac{C_2+8}{11}\qquad C_{\dagger 2}=\frac{8C_2-2}{11}$$

where only the constant C_2 is arbitrary. By eliminating C_2 between these equations, there results

$$8C_{\dot{\tau}1} - C_{\dot{\tau}2} = 6 \tag{8}$$

which is also satisfied by the choice of constants in Eq. (6). The LRR model corresponds to the choice $C_2 = .4$; Eqs. (5) and (6) correspond instead to the choice $C_2 = 8/21 \sim .36$.

The approximate summation used to derive Eq. (5) can be systematically generalized to generate an infinite number of models for Π_{ij}. For example, suppose that the perturbation series for τ is introduced into the cubic terms in the perturbation series instead of in the quadratic terms as above. This substitution will produce a model which can be written symbolically in the form

$$\Pi \sim \mathbf{S}^{(1)} + \mathbf{S}^{(2)} + \tau(\mathbf{S}^{(1)'} + \mathbf{S}^{(2)'})$$

where $\tau\mathbf{S}^{(i)'}$ denotes a sum of matrix products in all possible orders of τ and terms $\mathbf{S}^{(i)}$. The requirement that the original series agree to order $\mathbf{S}^{(4)}$ with the approximation when τ is replaced by its perturbation series determines this approximation uniquely.

2.2 The Return to Isotropy Model

The analytical description of return to isotropy is no less controversial than the modeling of the fast pressure strain term[3]. In the usual approach to turbulence modeling, in which correlations generated by Reynolds averaging are closed phenomenologically, this process is considered to result partly from the pressure correlation through a "slow" term independent of the mean flow, and partly from the deviatoric part of the dissipative correlation $\left\langle \nu_0 \frac{\partial u_i}{\partial x_p} \frac{\partial u_j}{\partial x_p} \right\rangle$. From this viewpoint, the analysis in Sect. I is incomplete because it discloses only a term proportional to the mean velocity gradient, but no slow term. The return to isotropy will be derived here by renormalization group methods following a suggestion of Yakhot[17].

From the renormalization group viewpoint, it is natural to investigate the return to isotropy, even independently of the stress transport equation, by writing the perturbation series for

$$u_i \frac{\partial u_j}{\partial t} + u_j \frac{\partial u_i}{\partial t} = \int u_i(\hat{k} - \hat{q})(-i\omega) u_j(\hat{q}) d\hat{q} + (ij) \tag{9}$$

This perturbation series differs from the perturbation series for the Reynolds stresses previously reported[5] only in the occurence of an additional factor $-i\omega$ in all frequency integrals.

The analysis is straightforward. Only the deviatoric terms require attention because the part of the correlation proporational to δ_{ij} contributes to the transport equation for K which has been analyzed by Yakhot and Smith[15]. The lowest order deviator appears at first order in η; to lowest order in ϵ

$$T_1 = \hat{T}_1\Big(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\Big) \tag{10}$$

where

$$\frac{d\hat{T}_1}{dr} = \frac{1}{15}\frac{\mathcal{D}}{\nu\Lambda^2} \tag{11}$$

In view of the form of the Rotta model, it is reasonable to seek terms at the next order proportional to $u_i u_j$. As in Sect. I, such terms do appear, but cancel exactly. This apparently ubiquitous cancellation was also obtained by Smith and Reynolds[18] in an analysis of the ε transport equation. Accordingly, the second order analysis in η produces quadratic terms in the velocity gradients. Finite truncation of this series violates the requirement that return to isotropy be independent of the mean flow. Therefore, we must seek a reasonable approximate summation. The form of the lowest order term given in Eqs. (10) and (11) suggests

$$\Pi'_{ij} = \frac{\hat{T}_1}{\nu}\nu\Big(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\Big) \sim -\frac{\hat{T}_1}{\nu}\overline{u_i u_j}^{(0)}$$

Despite its triviality, this replacement does produce an approximate sum which agrees exactly with perturbation theory to lowest order. It therefore can be considered a type of Padé approximation.
At the infinite Reynolds number asymptotic limit

$$\Pi'_{ij} = -C_R\frac{\varepsilon}{K}\overline{u_i u_j}^{(0)} \tag{12}$$

where, in the Yakhot-Orszag theory, $C_R = \mathcal{D}/\varepsilon \sim 1.6$. Equation (12) is therefore simply the standard Rotta model with Rotta constant ~ 1.6 in agreement with an earlier proposal of Yakhot[17].

A preliminary discussion of higher order summation may be appropriate. By analyzing the spectral dynamics of the return to isotropy, Weinstock[3] concluded that the shear and normal stresses relax at different rates. Although this behavior is obviously not accommodated by the Rotta model, it is consistent with the present theory: the perturbation series for Π' is obtained from the series for τ by multiplying the term of order n by the factor $C_n\varepsilon/K$ for some constant C_n. The C_n are all unequal; therefore, the Rotta model is not exact. Now comparison with the series for τ shows[5] that relaxation of the shear stress is governed by the linear term $\mathbf{S}^{(1)}$, whereas relaxation of the normal stresses is governed by the quadratic term $\mathbf{S}^{(2)}$. Since $C_2 \neq C_1$, these stresses relax at different rates. The difference is suppressed in the Rotta model, which arose in the present formalism by replacing all of the C_n by C_1.

2.3 Algebraic Reynolds Stress Models

The approximation, due to Rodi[20], of the Reynolds stress transport equation by an algebraic model under the conditions of semi-homogeneous flow (negligible diffusion of τ and τ/K approximately constant) takes the form

$$\frac{P-\varepsilon}{K}\,\overline{u_i u_j}^{(0)} = -\left(\overline{u_i u_p}\,\frac{\partial U_j}{\partial x_p} + \overline{u_j u_p}\,\frac{\partial U_i}{\partial x_p}\right)^{(0)} + \Pi'_{ij} + \Pi_{ij} \tag{13}$$

where Π and Π' depend on τ and ∇U. Explicit solutions for τ can be obtained, at least in principle, for any such approximation[25]. Briefly, one introduces a basis for polynomials in ∇U, and ∇U^T. The basis contains 11 terms of homogeneity order $n \le 5$. Writing τ as a sum of these terms with unknown coefficients and substituting in Eq. (13) leads to the explicit expression

$$\tau/K = \sum H_i^{(m)}\, \mathrm{S}_i^{(n)}\ (\nabla U, \nabla U^T) \tag{14}$$

where $H_i^{(m)}$ is a scalar function of ∇U and ∇U^T such that

$$H_i^{(m)} \sim |\nabla U|^m$$

when $|\nabla U| \to \infty$. The assumptions made on the approximate summations require $m+n=0$; thus, τ/K is bounded when $SK/\varepsilon \to \infty$. For example, the familiar eddy viscosity formula is replaced in Eq. (29) by a term

$$\tau \sim \frac{K^2}{\varepsilon}\, H^{(-1)}\ (\nabla U, \nabla U^T)\ (\nabla U + \nabla U^T)$$

Pope observed[25] that the coefficients $H^{(-n)}$ in Eq. (14) would certainly be intractably complex; although they could be explicitly exhibited by symbolic computation, the result would only pertain to the particular implicit equation for the Reynolds stresses assumed initially in Eq. (13). Therefore, it is equally reasonable just to postulate simple forms for the functions $H^{(-n)}$. This type of modeling could be particularly interesting when applied to the coefficients of the quadratically nonlinear models of Refs. 5, 11, and 12.

2.4 Discussion

The present analysis of the Reynolds stress transport equation, based on the Yakhot-Orszag renormalization group and (tensorial) η–expansion summation as suggested by Yakhot et al.[9], has led to a model transport equation incorporating the well-known LRR and Rotta models. The analysis gives theoretical support both to these models and to the constants sometimes used with them. More significantly, it exhibits the LRR and Rotta models as lowest order approximations, and therefore also supports their replacement with higher order nonlinear models which would be deduced by more accurate approximate summations. The consistency of the analysis with higher order effects like the unequal relaxation rates of shear and normal stresses has been discussed.

3. Future Plans

The nonlinear eddy viscosity representation of the Reynolds stresses

$$\tau_{ij} = \overline{u_i u_j} = \frac{2}{3}K\delta_{ij} - C_\nu \frac{K^2}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$+\frac{K^3}{\varepsilon^2}\left[C_{\tau 1}\left(\frac{\partial U_i}{\partial x_p}\frac{\partial U_j}{\partial x_p}\right)^{(0)} + C_{\tau 2}\left(\frac{\partial U_i}{\partial x_p}\frac{\partial U_p}{\partial x_j} + \frac{\partial U_j}{\partial x_p}\frac{\partial U_p}{\partial x_i}\right)^{(0)} + C_{\tau 3}\left(\frac{\partial U_p}{\partial x_i}\frac{\partial U_p}{\partial x_j}\right)^{(0)}\right] \quad (15)$$

in which K is the turbulence kinetic energy, ε is the dissipation rate, $C_\nu, C_{\tau 1}, C_{\tau 2}, C_{\tau 3}$ are constants, and (0) denotes deviatoric part, was proposed by Yoshizawa[11] in order to model normal stress effects in shear flows by means of an explicit formula for the stresses. The significance of this formula is not limited to this property: Yoshizawa's derivation using a special perturbation expansion, the two-scale direct interaction approximation, showed that the expansion could be continued to any order in the mean velocity gradient and thereby exhibited the Reynolds stress tensor as the result of an infinite number of increasingly complex interactions between the mean velocity field and turbulence. Related expansions are given in Refs. 5, 12.

The infinite expansion which contains Eq. (15) can be written symbolically as

$$\tau = KA_0S^0 + \frac{K^2}{\varepsilon}A_1S^1 + \frac{K^3}{\varepsilon^2}\sum_{i\le N_2} A_{i2}S_i^2 + \cdots + \frac{K^{n+1}}{\varepsilon^n}\sum_{i\le N_n} A_{in}S_i^n + \cdots \quad (16)$$

Eq. (16) can be considered a decomposition of the Reynolds stress

$$\tau = \tau^0 + \tau^1 + \cdots \quad (17)$$

where

$$\tau^n = \frac{K^{n+1}}{\varepsilon^n}\sum_{i\le N_n} A_{in}S_i^n = \sum_{i\le N_n}\tau_i^n \quad (184)$$

where S_i^n denotes a symmetric tensor homogeneous of degree n in the mean velocity gradient ∇U and its transpose, N_n denotes the number of linearly independent terms of order n (so in Eq. (15), $N_2 = 3$), and the A_{in} are constants.

In an analysis of the Reynolds stress transport equation by renormalization group techniques[35], we found analogous expansions for the term which governs the return to isotropy,

$$\Pi' = \frac{\varepsilon}{K}\left[\frac{K^2}{\varepsilon}B_1S^1 + \frac{K^3}{\varepsilon^2}\sum_{i\le N_2} B_{i2}S_i^2 + \cdots\right] \quad (19)$$

and for the rapid pressure-strain term

$$\Pi = \frac{2}{5}K\ S^1 + \frac{K^2}{\varepsilon}\left\{C_1\left[S^1\nabla U^T + \nabla U S^1\right]^{(0)} + D_1\left[S^1\nabla U + \nabla U^T S^1\right]^{(0)}\right\}$$

$$+\frac{K^3}{\varepsilon^2}\sum_{i\leq N_2}\left[C_{i2}\left(S_i^2\nabla U^T+\nabla U S_i^2\right)^{(0)}+D_{i2}\left(S_i^2\nabla U+\nabla U^T S_i^2\right)\right]+\cdots \tag{20}$$

Suppose that these sums are introduced into the Reynolds stress transport equation

$$\dot{\tau}=\Pi'+\Pi-P+D \tag{21}$$

where the dot denotes convective derivative, $P=\tau\nabla U^T+\nabla U\tau$ is production and D denotes the diffusion term. In order to obtain a Reynolds stress transport equation, it is necessary to express the sums (19) and (20) in terms of τ and ∇U. Although the coefficients $A,\ B,\ C,\ D$ can be explicitly exhibited to any order in perturbation theory, they do not have an obvious law of formation. Therefore, the sums (19) and (20) can only be approximated by polynomials in τ and ∇U if some hypotheses relating the coefficients is introduced. There is no unique hypothesis of this sort, but the simplest[35] seems to be

$$\begin{aligned}&B_{in}/B_1=A_{in}/A_1, \qquad n\geq 2\\&C_{in}/C_1=D_{in}/D_1=A_{in}/A_1, \qquad n\geq 2\end{aligned} \tag{22}$$

which leads to the Rotta return to isotropy model and to an LRR model for the rapid term. The approximation expressed by Eq. (22) can be compared to the summation introduced in an analogous context by Yakhot et al.[9], and to Padé approximation: it agrees with the perturbation theory of Eqs. (19), (20) to lowest order, but includes effects of all order in ∇U.

By evaluating more terms of the perturbation series explicitly and introducing an equation like (8) for coefficients of higher order, a hierarchy of models could be generated. They would initially be nonlinear in ∇U, as advocated by Speziale[33], but one might introduce the perturbation series for $\tau\cdot\tau$ to obtain a model nonlinear in τ[1,2]. However, the close analogy between Eqs. (19) and (20) and Yoshizawa's expansion (16) suggests a different approach: namely, use Eqs. (17) and (18) to replace S_i^n in Eqs. (19) and (20) by τ_i^n/A_{in}. Substitute these modified expressions and Eq. (17) into the transport equation Eq. (21), treat τ_i^n as having order $|\nabla U|^n$, and separate the terms of like order in $|\nabla U|$ in the standard perturbation theoretic fashion. The result is that the terms τ_i^n in the decomposition (17), (18) themselves satisfy coupled transport equations.

For simplicity, let us write an approximate system for the τ^n instead of for the τ_i^n and assume the most elementary scalar diffusion model. Then the the single transport equation for τ would be replaced by a system

$$\begin{aligned}\dot{\tau}^n&=-C_R^n\tau^n+C_1^n\left(\tau^{n-1}\nabla U^T+\nabla U\tau^{n-1}\right)^{(0)}\\&+C_2^n\left(\tau^{n-1}\nabla U+\nabla U^T\tau^{n-1}\right)^{(0)}+C_D^n\frac{K^2}{\varepsilon}\nabla^2\tau^n, n\geq 1\end{aligned} \tag{23}$$

Since $\tau=\sum\tau^n$, the system (6) should be constrained to satisfy Crow's condition and to contain the exact production term following summation over n. Making the coefficients $C_R^n, C_1^n, C_2^n, C_D^n$, independent of n reduces the system to a model of the LRR form for $\tau = \sum \tau^n$.

This conclusion also follows from Leslie's analysis[34] of the direct interaction (DIA) equations for shear flow. Leslie suggested a perturbative solution for the (tensor) correlation function and Green's function

$$\begin{aligned} U &= U^0 + U^1 + \cdots \\ G &= G^0 + G^1 + \cdots \end{aligned} \tag{24}$$

where U^n and G^n are of the order $| \nabla U |^n$, and observed that this expansion is simultaneously an expansion in powers of the mean strain, and a decomposition into symmetry types of increasing complexity. This is also a feature of the expansion (16). Substitution of Eq. (20) into the equations of the direct interaction approximation gives a coupled system for the U^n and G^n in standard fashion. Then in principle, by integrating each equation of this system over all wavenumbers and introducing the definitions

$$\tau^n = \int U^n d\,\mathbf{k}$$

we could attempt to obtain coupled transport equations for the τ^n. Unfortunately, the derivation of equations for single-point quantities from DIA is not entirely straightforward, and more heuristic methods like two-scale DIA[8,14] and renormalization group are required.

The simplest model of this type is a two component model,

$$\tau - \frac{2}{3} K I = \tau^1 + \tau^2$$

with

$$\begin{aligned} \dot{\tau}^1 =& - C_R^1 \tau^1 - \frac{4}{15} K \left(\nabla U + \nabla U^T \right) + \left(C_1^1 - 1 \right) \left(\tau^2 \nabla U^T + \nabla U \tau^2 \right)^{(0)} \\ &+ C_2^1 \left(\tau^2 \nabla U + \nabla U^T \tau^2 \right)^{(0)} + C_D^1 \nabla \frac{K^2}{\varepsilon} \nabla \tau^1 \\ \dot{\tau}^2 =& - C_R^2 \tau^2 + \left(C_1^2 - 1 \right) \left(\tau^1 \nabla U^T + \nabla U \tau^1 \right)^{(0)} \\ &+ C_2^2 \left(\tau^1 \nabla U + \nabla U^T \tau^1 \right)^{(0)} + C_D^2 \nabla \frac{K^2}{\varepsilon} \nabla \tau^2 \end{aligned} \tag{25}$$

The appearance of τ^2 in the equation for τ^1 is required by the production term constraint which is violated by direct truncation of the system (20) at the second order. The properties of this model for simple shear flow, in which $\partial U_i / \partial x_j = S \delta_{i1} \delta_{j2}$ follow by setting

$$\tau^1 = \tau_{12} \begin{bmatrix} & 1 & \\ 1 & & \\ & & 0 \end{bmatrix} \qquad \tau^2 = \begin{bmatrix} \tau_{11} & & \\ & \tau_{22} & \\ & & \tau_{33} \end{bmatrix}^{(0)} \tag{26}$$

Then C_R^1 describes the return to isotropy of the shear stress, and C_R^2 the return to isotropy of the normal stresses. These relaxation rates can be unequal in this theory, an effect predicted by Weinstock's analysis[3] of the spectral dynamics of the return to isotropy. Weinstock suggested further that the individual normal stresses relax at different rates: this effect is not accommodated by the present model, but could occur in a model in which τ^2 is divided into the three tensor components $\tau_1^2, \tau_2^2, \tau_3^2$ of Eq. (18).

The inequality of the Rotta constants in the present model can be used to overcome a defect of models of the LRR type, that in semi- homogeneous flows in which $a_{ij} = \tau_{ij}^{(0)}/K$ is approximately constant, the ratio a_{11} is too small whereas a_{12} is too big[10]. Following Speziale[19], write Eq. (21) as a system of equations for the ratios a_{ij} and set $\dot{a}_{ij} = 0$. There results,

$$2a_{12}\eta = \left(C_R^1 - 1\right) - \left\{\left(C_R^1 - 1\right)^2 - 4\eta^2\left[\left(C_1^1 - 1\right)a_{22} + C_2^1 a_{11} - \frac{4}{15}\right]\right\}^{1/2}$$

$$a_{11} = \frac{\left[-\frac{4}{3}\left(C_1^2 - 1\right) + \frac{2}{3}C_2^2\right]\eta a_{12}}{\left(-C_R^2 + 1\right) + \eta a_{12}} \tag{27}$$

$$a_{22} = \frac{\left[\frac{2}{3}\left(C_1^2 - 1\right) - \frac{4}{3}C_2^2\right]\eta a_{12}}{\left(-C_R^2 + 1\right) + \eta a_{12}}$$

When $P/\varepsilon = -\eta a_{12} = 1$,

$$a_{12}^2 = \frac{4}{15}\frac{1}{C_R^1} - \frac{C_1^1 - 1}{C_R^1}a_{22} - \frac{C_2^1}{C_R^1}a_{11}$$

$$a_{11} = \left[-\frac{4}{3}\left(C_1^2 - 1\right) + \frac{2}{3}C_2^2\right]/C_R^2 \tag{28}$$

$$a_{22} = \left[\frac{2}{3}\left(C_1^2 - 1\right) - \frac{4}{3}C_2^2\right]/C_R^2$$

Eq. (28) shows that setting $C_R^2 < C_R^1$ both increases a_{11} and decreases a_{12} and thus improves the agreement between theory and experiment. The trend required here is consistent with Weinstock's findings[3] that the Rotta constant for the normal stresses should be smaller than the shear constant and should take values close to 1.0.

The inequality of the coefficients describing the rapid terms affects the behavior of the model under rapid distortion. The rapid distortion analysis of passively strained turbulence predicts that a one-component limit state is reached in which $\tau_{ij} = 2K\delta_{i1}\delta_{j1}$, $\eta = SK/\varepsilon \rightarrow \infty$ and P/ε is finite[21]. Whether or not this limit can in fact occur as an asymptotic state, a stress model should accommodate it because it can exist approximately in steady flows as a "spatial transient" in strongly inhomogeneous regions of very high convective or diffusive transport[17]. Moreover, models of the LRR form do not capture the transient evolution of rapidly distorted flows well. Let us assume that all model coefficients are functions of the "state" of turbulence, following Shih and Lumley[2]; the precise parametrization of the state will be left to future investigations. From Eq. (13), it is evident that $P/\varepsilon = -\eta a_{12}$ will be of order η in the one-component state in which $a_{11} = 4/3, a_{22} = -2/3$ unless

$$-\frac{2}{3}\left(C_1^1 - 1\right) + \frac{4}{3}C_2^1 = \frac{4}{15}$$

in this state. But Eq. (13) also shows that $a_{11} = 4/3, a_{22} = -2/3$ requires

$$C_1^2 \sim 0, \qquad C_2^2 \sim 0$$

in this state. These conditions are inconsistent in a model in which $C_1^1 = C_1^2$ and $C_2^1 = C_2^2$, but are clearly consistent with the present proposal.

The advantages of this model are consistency with a systematic perturbation theory, the possibility of unequal relaxation rates for normal and shear stresses in relaxing strained turbulence, improved agreement with experimental data for universal ratios in simple shear flows, and the possibility of accommodating the one-component limit of rapid distortion theory.

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