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THE KAPPA DISTRIBUTION AS A VARIATIONAL SOLUTION FOR AN INFINITE PLASMA

Prepared By:	Michael J. Pangia, Ph.D.
Academic Rank:	Assistant Professor
Institution and Department:	Georgia Southwestern College Department of Geology and Physics
NASA/MSFC:	
Office:	Space Science Laboratory
Division:	Solar-Terrestrial Physics
Branch:	Magnetospheric Physics
MSFC Colleagues:	Dennis L. Gallagher, Ph.D. Craig J. Pollock, Ph.D. Thomas E. Moore, Ph.D.

The basis for this research is the 1991 Summer Faculty Fellowship work by the present author. In review, ***Pangia*** [1992] postulated that the preferred state of a single component infinite plasma is the one that will change the least when perturbed. The plasma distribution of such a state would maximize the plasma wave damping rate, or, minimizing γ where negative γ is the damping rate. This explained the tendency for low values of κ, when fitting plasma distributions to a κ-distribution, which is

$$\frac{A_\kappa}{(1 + v^2/(2\kappa v_\kappa^2))^\kappa} \qquad [1]$$

where A_κ and v_κ are the overall normalization factor and thermal speed parameter, respectively. A more compelling question is why a κ-distribution works so well in fitting both electron and ion distributions. *Hasegawa et al.* [1985] derived that the electron distribution subject to a superthermal radiation field is given by [1], but that the ion distribution is Maxwellian for speeds less than the electron thermal speed. It is also desirable to calculate κ, which is not readily doable from the result of *Hasegawa et al.* [1985]. The present study continues from the ideas developed by ***Pangia*** [1991] to determine what distribution function maximizes the damping rate. A κ-distribution will be the outcome for either electrons or ions, with κ tending toward 3/2.

It is necessary to extend the work of the author to include the general case of a multi-component plasma. For the sake of completeness, this presentation will reproduce also the arguments of the previous work. For a 1-dimensional, collisionless plasma with electrostatic field, E(t,x), the Vlasov and Maxwell equations are

$$\frac{\partial f_s}{\partial t} + v\frac{\partial f_s}{\partial x} + \frac{q_s}{m_s} E \frac{\partial f_s}{\partial v} = 0 \qquad [2]$$

$$\frac{\partial E}{\partial x} = \sum_s 4\pi q_s \int_{-\infty}^{\infty} dv\ f_s \qquad [3]$$

$$\frac{\partial E}{\partial t} = -\sum_s 4\pi q_s \int_{-\infty}^{\infty} dv\ v\ f_s \qquad [4]$$

where $f_s(t,x,v)$ is the reduced distribution function for particles of type s. In use is the normalization that the number of particles of type s that are between x and x+dx with velocities ranging from v to v+dv is $f_s dx\, dv$. Particles of type s have charge q_s and mass m_s.

In the absence of any external field, the steady state solution to the Vlasov-Maxwell equations is $f_s = F_s(v)$ and $E = 0$, where F_s is any positive function normalized to the number density. One might then infer that, in equilibrium, a plasma distribution is virtually likely to be anything. However, the response of the plasma to a perturbation from equilibrium will greatly depend on the function $F_s(v)$. Since plasmas evolve to equilibrium under the influence of plasma waves, a particular steady state could be favored over all the others based on its response to a perturbation. In an actual plasma, waves are usually present, indicating some sort of evolution of the distribution function. Although equilibrium may never be fully obtained, the plasma should be at least tending toward it. Changes in the background distribution function should diminish as equilibrium is approached. Change can possibly be used as the criterion for identifying the preferred steady state of a plasma, because a steady state that appreciably changes after having been perturbed is expected to evolve to a different steady state which is more stationary. Therefore, it will be postulated that the plasma will tend toward the steady state which changes the least when perturbed from equilibrium.

Since the steady state distribution in a plasma where external forces are absent is spatially uniform, each distribution function will be expressed as a sum of a spatially uniform part, $f_s(t,v)$, and the deviation from uniformity, $f'_s(t,x,v)$, with a similar expression for the electric field

$$f_s(t,x,v) = f_s(t,v) + f'_s(t,x,v) \qquad [5]$$
$$E(t,x) = E(t) + E'(t,x) \qquad [6]$$

The uniform electric field, E(t), may be zero, but, in general, it will exist if a zero wave number mode is part of the perturbation or if it develops in time. A spatial average over all space will be used to formally define the uniform part of any function Y(x); namely,

$$\langle Y \rangle_x = \lim_{L\to\infty} \int_{-L}^{L} \frac{dx}{2L} Y(x) \qquad [7]$$

The subscript x on the angular bracket is used to identify it as a spatial average. In regard to [5] and [6], $f_s(t,v) = \langle f_s(t,x,v) \rangle_x$ and $E(t) = \langle E(t,x) \rangle_x$. Taking the spatial average, [2] through [4] become

$$\frac{\partial}{\partial t} f_s(t,v) + \frac{q_s}{m_s}\frac{\partial}{\partial v}\left\{ E(t)\, f_s(t,v) + \langle E'\, f'_s \rangle_x \right\} = 0 \qquad [8]$$

$$0 = \sum_s 4\pi q_s \int_{-\infty}^{\infty} dv\, f_s(t,v) \qquad [9]$$

$$\frac{d}{dt} E(t) = -\sum_s 4\pi q_s \int_{-\infty}^{\infty} dv\, v\, f_s(t,v) \qquad [10]$$

Subtracting [8] through [10] from [2] through [4], respectively, gives

$$\frac{\partial f'_s}{\partial t} + v\frac{\partial f'_s}{\partial x} + \frac{q_s}{m_s}\frac{\partial}{\partial v}\left\{ E(t)\, f'_s + E'\, f_s(t,v) + E'\, f'_s - \langle E'\, f'_s \rangle_x \right\} = 0 \qquad [11]$$

$$\frac{\partial E'}{\partial x} = \sum_s 4\pi q_s \int_{-\infty}^{\infty} dv\, f'_s \qquad [12]$$

$$\frac{\partial E'}{\partial t} = -\sum_s 4\pi q_s \int_{-\infty}^{\infty} dv\, v\, f'_s \qquad [13]$$

Equations [8] through [13] are basically the starting equations for quasi-linear theory [see for example, Swanson, 1989], just extended to include a uniform electric field. Derivable from [8] through [13] are the conservation laws for the system, which are written

$$\frac{d}{dt} \int_{-\infty}^{\infty} dv\, f_s(t,v) = 0 \qquad [14]$$

$$\frac{d}{dt} \sum_s m_s \int_{-\infty}^{\infty} dv\, v\, f_s(t,v) = 0 \qquad [15]$$

$$\frac{d}{dt}\left(\sum_s m_s \int_{-\infty}^{\infty} dv\, v^2 f_s(t,v) + \frac{1}{4\pi} E^2(t) + \frac{1}{4\pi}\langle E'^2 \rangle_x \right) = 0 \qquad [16]$$

Due to the quadratic non-linearity present in the Vlasov equation, an additional simplifying assumption will be made in order to derive a result. Having perturbed a system from equilibrium, it will be assumed that, eventually in its evolution, the predominant wave modes that remain are k=0 and k nearly zero, where k is the wave number, all other modes having been depleted. This assumption is made even if an instability exists for some particular range of modes. The rationale is that non-linear effects will channel wave energy to other modes, most of which damp away. To understand the basic effect of the quadratic non-linearity, consider the spatial dependence of any mode k at some time t. It will have the basic form $e^{ikx} \pm e^{-ikx}$. The product of two modes, k_1 and k_2, will result in modes $k_1 + k_2$, and $k_1 - k_2$. In particular, if $k_1 = k_2$, there will be mode conversion to $2k_1$ and 0, and if k_1 and k_2 are nearly equal, the nearly zero wave mode will result. If k_1 is unstable, the process will convert it to stable modes. Modes that find their way to the instability will have been damped beforehand, amplified during the instability, and, finally, converted back to stable modes. Therefore, all wave activity will experience some damping, either directly or indirectly, during some time in the system's evolution except for the k=0 mode. Due to the initial perturbation having a small amount of total wave energy distributed over all the modes, it is being assumed that even an unstable mode would be depleted of its field intensity through the eventual conversion of all modes to k=0. There comes a stage in the plasma evolution when only the undamped mode, k=0, and the least damped mode, k nearly zero, have survived.

When the plasma reaches the stage of predominantly k=0 and nearly zero wave modes, further amplification of the k=0 mode will be negligible, because only the nearly zero mode, which is decaying, is available to convert to k=0. At this point, E(t), which is the k=0 mode, is well-approximated as an oscillation at the plasma frequency. From [10], $f_s(t,v)$ will oscillate with opposite phase to E(t). Therefore, on a time average, the effects of the k=0 mode on changing the distribution function will be zero. Defining a time average for any function Y(t) by

$$\langle Y\rangle_t = \int_t^{t+T} \frac{dt'}{T} Y(t') \tag{17}$$

where T is the plasma period, the time average of [8] is

$$\frac{\partial}{\partial t}\langle f_s(t,v)\rangle_t = -\frac{q_s}{m_s}\frac{\partial}{\partial v}\langle E'\ f'_s\rangle_{x,t} \tag{18}$$

where the combined subscript x,t on the angular bracket indicates a double average over space and time.

The nearly zero mode, which is given by E'(t,x), will be in a regime where its evolution and effects are describable by quasi-linear theory. In the quasi-linear approximation, products of E' and f'_s are retained in [8], but neglected in [11]. In search of the distribution that changes the least, a restriction will be made to distributions that change only slightly. Such a subclass of distributions can be insured to exist by making the perturbation small enough. Specifically, based on [16], the initial wave field must satisfy the condition

$$E^2(t) + \langle E'^2\rangle_x \ll 4\pi\sum_s m_s \int_{-\infty}^{\infty} dv\ v^2\ f_s(t,v) \tag{19}$$

at t=0. Then E(t) will be small, and the term E(t) f'_s will be negligible in [11], reducing it to

$$\frac{\partial f'_s}{\partial t} + v\frac{\partial f'_s}{\partial x} + \frac{q_s}{m_s}E'\frac{dF_s}{dv} = 0 \tag{20}$$

where $f_s(t,v)$ was replaced by $F_s(v)$ since it only slightly varies. Equations [12] and [20] are the common linearized equations for electrostatic waves. The solution is that, for a stable mode of wave number k, E' oscillates with frequency ω and exponential decays with damping rate |γ| (negative γ corresponds to damping) given by [Nicholson, 1983]

$$\varepsilon(\omega+i\gamma,k) = 1 - \sum_s 4\pi q_s^2/(k^2 m_s) \int_C dv \frac{dF_s/dv}{v - \frac{\omega+i\gamma}{k}} = 0 \tag{21}$$

where the integration is in the complex v-plane along the Landau contour, and ε is the dielectric function. With E' exponentially decaying at a damping of $|\gamma|$, [18] says that, on the average, the change in $f_s(t,v)$ decays at least as fast. Consequently, the distribution that changes the least will be the one that damps the nearly zero wave mode the most. From [21], one seeks the F_s that maximizes $|\gamma|$, or minimizes γ. Mathematically, the problem at hand is under the classification of Calculus of Variations, where F_s is varied slightly and a minimum in γ is sought.

The functions about which one varies have to be physical. To insure this, constraints must be imposed. The conservation laws, [14] through [16], require that

$$\int_{-\infty}^{\infty} dv\, F_s = n_s \; ; \; \sum_s m_s \int_{-\infty}^{\infty} dv\, v\, F_s = n\, P \; ; \; \sum_s m_s \int_{-\infty}^{\infty} dv\, v^2\, F_s = n\, T \qquad [22]$$

where, n_s, n, P, and T are, respectively, the number density of particle type s, the total number density of all particle types, the total plasma momentum, and the total plasma temperature,all of which are constants. The existence of these velocity moments imply that the high speed dependence of F_s must be less than $1/|v|^3$. This puts a limit on damping rate, because, for small k, γ is proportional to the dF_s/dv evaluated at high velocity ω/k [Nicholson, 1983]. Therefore, the damping is maximized for the function that has the steepest descending slope, which is when the high speed dependence of F_s tends to $1/|v|^3$. In regard to [1], κ tending to 3/2 would maximize damping.

To find the function for all velocity, additional constraints must be imposed to keep F_s physical. One condition is that F_s must be positive. Another condition arises from the fact that each distribution function F_s will consist of a definite number of plasma components, Λ_s, which is determined by the origin of the plasma. Therefore, it is a fixed property of the system. Defining $F_{s,c}$ as the distribution function for component c of particle type s, the total distribution for particle type s is

$$F_s = \sum_{c=1}^{\Lambda_s} F_{s,c} \qquad [23]$$

The number of components, Λ_s, is determined by counting the humps in F_s. By constraining $F_{s,c}$ to be positive with only one hump, F_s is insured to be physical. This is done by maximizing the following integral

$$\int_{-\infty}^{\infty} dv\, (F_{s,c})^r \qquad [24]$$

where the exponent r has to be determined, where $r \neq 1$, so that maximizing [24] is independent from the first equation in [22]. The problem is now well defined, whereby γ is minimized in [21] subject to the constraints in [22] while maximizing [24]. A solution for arbitrary and small variations in $F_{s,c}$ only exists if k approaches zero. The answer is that each component is given by a κ-distribution with appropriate flow speed, where $\kappa=1/(1-r)$ with $3/2 < \kappa < \infty$, and κ tending to 3/2 gives the absolute minimum in γ.

References

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