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ELECTROMAGNETIC SCATTERING IN CLOUDS

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Techniques used to explain the nature of the optical effects of clouds on the light produced by lightning include a Monte Carlo simulation (10), an equivalent medium approach (6), and methods based on Boltzmann transport theory (4, 7). A cuboidal cloud has been considered using transform methods (7) and a diffusion approximation (4).

Many simplifying assumptions have been used by authors to make this problem tractable. In this report, the cloud will have a spherical shape and its interior will consist of a uniform distribution of identical spherical water droplets. The source will be modeled as a Hertz dipole, electric or magnetic, inside or outside the cloud. An impulsive source is used. Superposition may be employed to obtain a sinusoid within an envelope which describes a lightning event. The problem is investigated by transforming to the frequency domain, obtaining Green's functions, and then using the Cagniard-DeHoop method (3) to symbolically recover the time domain solution.

The Hertz dipole potential may be derived from a wave equation whose source is proportional to the dipole moment per unit volume,

$$\mathbf{P} = N\mathbf{p}, \quad \mathbf{p} = Q\mathbf{s}, \tag{1}$$

where N is the number of dipoles per unit volume and $\mathbf{p}$ is the dipole moment of a single dipole constructed from equal and opposite charges of magnitude Q separated by a distance s oriented along $\hat{\mathbf{s}}$. From Maxwell's equations, it is possible to show that the electric Hertz vector is the solution to

$$\nabla^2\mathbf{\Pi} - \mu\epsilon\frac{\partial^2\mathbf{\Pi}}{\partial t^2} = -\frac{\mathbf{P}}{\epsilon}. \tag{2}$$

The resulting electric and magnetic fields may be written in terms of $\mathbf{\Pi}$ by

$$\mathbf{H} = \epsilon\nabla\wedge\frac{\partial\mathbf{\Pi}}{\partial t}, \quad \mathbf{E} = \nabla\nabla\cdot\mathbf{\Pi} - \mu\epsilon\frac{\partial^2\mathbf{\Pi}}{\partial t^2}. \tag{3}$$

Similarly, for a magnetic dipole,

$$\nabla^2\mathbf{\Pi}_m - \mu\epsilon\frac{\partial^2\mathbf{\Pi}_m}{\partial t^2} = -\mathbf{M}, \quad \mathbf{E} = \mu\nabla\wedge\frac{\partial\mathbf{\Pi}_m}{\partial t}, \quad \mathbf{H} = \nabla\nabla\cdot\mathbf{\Pi}_m - \mu\epsilon\frac{\partial^2\mathbf{\Pi}_m}{\partial t^2}. \tag{4}$$

The symbols are defined as in (9). Polarizations may be induced in a dielectric by the fields. The dipole moments used above are considered to be controlled only by the source. The sources will be taken to be unit pulses (polarizations per unit time),

$$\mathbf{P} = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\hat{\mathbf{p}}, \quad \mathbf{M} = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\hat{\mathbf{p}}_m. \tag{5}$$

Superposition may be used to construct fields due to general temporal and spatial dependencies.

In the following, script variables will denote quantities in the frequency domain. They are related to the time domain variables through

$$\mathbf{E}(\mathbf{r},\, t) = \int_{-\infty}^{\infty} d\omega\vec{\mathcal{E}}(\mathbf{r},\, \omega)e^{-i\omega(t-t')}, \quad \mathbf{H}(\mathbf{r},\, t) = \int_{-\infty}^{\infty} d\omega\vec{\mathcal{H}}(\mathbf{r}, \omega)e^{-i\omega(t-t')}. \tag{6}$$

The transforms of the Hertz potentials are given by

$$\vec{\pi} = \frac{iK}{4\pi\epsilon}h_0^{(1)}(KR)\hat{\mathbf{p}}, \quad \vec{\pi}_m = \frac{iK}{4\pi}h_0^{(1)}(KR)\hat{\mathbf{p}}_m, \quad h_0^{(1)}(x) = \frac{e^{ix}}{ix}, \quad K^2 = \omega^2\mu\epsilon, \tag{7}$$

where $R = |\mathbf{r} - \mathbf{r}'|$, $\mathbf{r}$ is an observation point and $\mathbf{r}'$ is a source point. The electric and magnetic fields due to these dipoles are given by

$$\vec{\mathcal{E}} = \frac{iK}{4\pi\epsilon}\left\{\left(iK - \frac{1}{R}\right)(\hat{\mathbf{p}}\cdot\nabla)\nabla R + \left(-K^2 - \frac{2iK}{R} + \frac{2}{R^2}\right)[(\hat{\mathbf{p}}\cdot\nabla)R]\nabla R + K^2\hat{\mathbf{p}}\right\} h_o^{(1)}(KR),$$
$$\vec{\mathcal{H}} = \frac{K\omega}{4\pi}\left(iK - \frac{1}{R}\right)(\nabla R \wedge \hat{\mathbf{p}}) h_o^{(1)}(KR), \qquad [8]$$

where $\vec{\mathcal{E}}_m$ and $\vec{\mathcal{H}}_m$ are obtained from $\vec{\mathcal{H}}$ and $\vec{\mathcal{E}}$ by replacing $\frac{K\omega}{4\pi}$ by $-\frac{K^3}{4\pi\omega\epsilon}$ and $\frac{iK\omega}{4\pi\epsilon}$ by $\frac{iK}{4\pi}$. The electric and magnetic field vectors may be written in terms of Hansen's functions (5) as

$$\vec{\mathcal{E}} = \frac{iK^3}{4\pi\epsilon}\sum Q_n^m\left[\mathbf{M}_{mn}^3(K,\mathbf{r})\mathbf{M}_{-mn}^1(K,\mathbf{r}') + \mathbf{N}_{mn}^3(K,\mathbf{r})\mathbf{N}_{-mn}^1(K,\mathbf{r}')\right]\cdot\hat{\mathbf{p}},$$
$$\sum \equiv \sum_{n=1}^{\infty}\sum_{m=-n}^{n}, \quad Q_n^m = \frac{(2n+1)(-1)^m}{n(n+1)}, \quad r > r' \qquad [9]$$

for an electric dipole source; the magnetic field is obtained from $\vec{\mathcal{E}}$ by replacing $\frac{iK^3}{4\pi\epsilon}$ by $\frac{K^2\omega}{4\pi}$ and interchanging $\mathbf{M}_{-mn}^1(K,\,\mathbf{r}')$ and $\mathbf{N}_{-mn}^1(K,\,\mathbf{r}')$. The superscript 1 denotes a Hansen function containing regular (Bessel type) functions; 3 refers to outgoing (Hankel type) functions. Analogous forms for fields due to magnetic sources are obtained by replacing the constants to the left of the summations by $-\frac{K^4}{4\pi\epsilon\omega}$ and $\frac{iK^3}{4\pi}$ respectively and interchanging $\mathbf{M}_{-mn}^1(K,\,\mathbf{r}')$ and $\mathbf{N}_{-mn}^1(K,\,\mathbf{r}')$ again. If $r < r'$, the corresponding vectors in [9] are interchanged.

If the medium inside the cloud is specified by $K(\mu,\,\epsilon)$ and that outside the cloud by $k(\mu_o,\,\epsilon_o)$ and the cloud is taken to be a sphere of radius a, the fields may be written as $\vec{\mathcal{G}}_e^e = \vec{\mathcal{E}}^s + \vec{\mathcal{E}}^i$, $\vec{\mathcal{G}}_m^e = \vec{\mathcal{E}}_m^s + \vec{\mathcal{E}}_m^i$, $\vec{\mathcal{G}}_e^h = \vec{\mathcal{H}}^s + \vec{\mathcal{H}}^i$, $\vec{\mathcal{G}}_m^h = \vec{\mathcal{H}}_m^s + \vec{\mathcal{H}}_m^i$, where the superscript on the $\vec{\mathcal{G}}$'s indicates whether the Green's function corresponds to the electric or magnetic field and the subscript indicates the type of source. The superscripts i and s refer to the source (i) or the scattered field (s). Green's functions for external excitation (1), which were obtained by enforcing continuity of the tangential components of the fields across the boundary, may be easily modified to work for sources inside a sphere. Explicit results for an electric dipole source are given by

$$\tilde{\mathcal{G}}_e^e = \frac{iK^3}{4\pi\epsilon}\begin{cases} \sum Q_n^m\Big\{\mathbf{M}_{mn}^1(K,\mathbf{r})\big[\mathbf{M}_{-mn}^3(K,\mathbf{r}') + a_n\mathbf{M}_{-mn}^1(K,\mathbf{r}')\big] \\ \quad + \mathbf{N}_{mn}^1(K,\mathbf{r})\big[\mathbf{N}_{-mn}^3(K,\mathbf{r}') + b_n\mathbf{N}_{-mn}^1(K,\mathbf{r}')\big]\Big\}, \quad r < r', \\ \sum Q_n^m\Big\{\big[\mathbf{M}_{mn}^3(K,\mathbf{r}) + a_n\mathbf{M}_{mn}^1(K,\mathbf{r}')\big]\mathbf{M}_{-mn}^1(K,\,\mathbf{r}') \\ \quad + \big[\mathbf{N}_{mn}^3(K,\mathbf{r}) + b_n\mathbf{N}_{mn}^1(K,\mathbf{r}')\big]\mathbf{N}_{-mn}^1(K,\mathbf{r}')\Big\}, \quad r' < r < a, \\ \sum Q_n^m\big[c_n\mathbf{M}_{mn}^3(k,\mathbf{r})\mathbf{M}_{-mn}^1(K,\mathbf{r}') + d_n\mathbf{N}_{mn}^3(k,\mathbf{r})\mathbf{N}_{-mn}^1(K,\mathbf{r}')\big], \quad r > a \end{cases} \qquad [10]$$

when the source is inside the sphere ($r' < a$) and

$$\tilde{\mathcal{G}}_e^e = \frac{ik^3}{4\pi\epsilon_o}\begin{cases} \sum Q_n^m\big[\mathfrak{a}_n\mathbf{M}_{mn}^1(K,\mathbf{r})\mathbf{M}_{-mn}^3(k,\mathbf{r}') + \mathfrak{b}_n\mathbf{N}_{mn}^1(K,\mathbf{r})\mathbf{N}_{-mn}^3(k,\mathbf{r}')\big], \quad r < a, \\ \sum Q_n^m\Big\{\big[\mathbf{M}_{mn}^1(k,\mathbf{r}) + \mathfrak{c}_n\mathbf{M}_{mn}^3(k,\mathbf{r})\big]\mathbf{M}_{-mn}^3(k,\mathbf{r}') \\ \quad + \big[\mathbf{N}_{mn}^1(k,\mathbf{r}) + \mathfrak{d}_n\mathbf{N}_{mn}^3(k,\mathbf{r}')\big]\mathbf{N}_{-mn}^3(k,\mathbf{r}')\Big\}, \quad a < r < r', \\ \sum Q_n^m\Big\{\mathbf{M}_{mn}^3(k,\mathbf{r})\big[\mathbf{M}_{-mn}^1(k,\mathbf{r}') + \mathfrak{c}_n\mathbf{M}_{-mn}^3(k,\mathbf{r}')\big] \\ \quad + \mathbf{N}_{mn}^3(k,\mathbf{r})\big[\mathbf{N}_{-mn}^1(k,\mathbf{r}') + \mathfrak{d}_n\mathbf{N}_{-mn}^3(k,\mathbf{r}')\big]\Big\}, \quad r > r' \end{cases} \qquad [11]$$

when the dipole is outside the cloud. The associated magnetic fields are obtained by replacing $\frac{iK^3}{4\pi\epsilon}$ and $\frac{ik^3}{4\pi\epsilon_o}$ by $\frac{K^2\omega}{4\pi}\{1+(\beta-1)H(r-a)\}$ and $\frac{k^2\omega}{4\pi}\left\{1+\left(\frac{1}{\beta}-1\right)H(a-r)\right\}$, $\beta=\sqrt{\frac{\mu\epsilon_o}{\mu_o\epsilon}}$, where H is the Heavyside function, respectively and interchanging those **M**'s and **N**'s which are functions of **r** but not those which are functions of **r**′. Vector fields are recovered by inner multiplication with the source polarization from the right (i.e., $\vec{\mathcal{G}}=\tilde{\mathcal{G}}\cdot\hat{\mathbf{p}}$). If the source is a magnetic dipole, $\tilde{\mathcal{G}}_m^e$ and $\tilde{\mathcal{G}}_m^h$ are obtained from $\tilde{\mathcal{G}}_e^e$ and $\tilde{\mathcal{G}}_e^h$ by replacing $\frac{iK^3}{4\pi\epsilon}$, $\frac{K^2\omega}{4\pi}$, $\frac{ik^3}{4\pi\epsilon_o}$, and $\frac{k^2\omega}{4\pi}$ by $-\frac{K^4}{4\pi\epsilon\omega}$, $\frac{iK^3}{4\pi}$, $-\frac{k^4}{4\pi\epsilon_o\omega}$, and $\frac{ik^3}{4\pi}$ respectively and interchanging only those **M**'s and **N**'s which are functions of **r**′. The constants are given by

$$a_n=\frac{\beta\frac{h_n}{\mathfrak{x}}(\mathfrak{x}\mathfrak{h}_n)'-\frac{\mathfrak{h}_n}{x}(xh_n)'}{\frac{\mathfrak{h}_n}{x}(xj_n)'-\beta\frac{\mathfrak{j}_n}{\mathfrak{x}}(\mathfrak{x}\mathfrak{h}_n)'},\quad b_n=\frac{\beta\frac{\mathfrak{h}_n}{x}(xh_n)'-\frac{h_n}{\mathfrak{x}}(\mathfrak{x}\mathfrak{h}_n)'}{\frac{\mathfrak{j}_n}{\mathfrak{x}}(\mathfrak{x}\mathfrak{h}_n)'-\beta\frac{\mathfrak{h}_n}{x}(xj_n)'},$$
$$c_n=\frac{\frac{h_n}{x}(xj_n)'-\frac{\mathfrak{j}_n}{x}(xh_n)'}{\frac{\mathfrak{h}_n}{x}(xj_n)'-\beta\frac{\mathfrak{j}_n}{\mathfrak{x}}(\mathfrak{x}\mathfrak{h}_n)'},\quad d_n=\frac{\frac{\mathfrak{j}_n}{x}(xh_n)'-\frac{h_n}{x}(xj_n)'}{\frac{\mathfrak{j}_n}{\mathfrak{x}}(\mathfrak{x}\mathfrak{h}_n)'-\beta\frac{\mathfrak{h}_n}{x}(xj_n)'}, \tag{12}$$
$$x=ka,\ \mathfrak{x}=Ka,\ j_n=j_n(x),\ \mathcal{J}_n=j_n(\mathfrak{x}),\ h_n=h_n^{(1)}(x),\ \mathfrak{h}_n=h_n^{(1)}(\mathfrak{x}).$$

The constants $\mathfrak{a}_n$ and $\mathfrak{b}_n$ are obtained from c_n and d_n respectively by multiplying by β and interchanging h_n with $\mathcal{J}_n$, x with $\mathfrak{x}$, and j_n with $\mathfrak{h}_n$; constants $\mathfrak{c}_n$ and $\mathfrak{d}_n$ are obtained from a_n and b_n respectively by multiplying by -1 and interchanging h_n with $\mathcal{J}_n$ and $\mathfrak{h}_n$ with j_n. The abovementioned interchanges apply only to the respective numerators. Allowing $r\sim\infty$ and using asymptotic forms for the Hansen's functions gives explicit forms for the far-field scattering amplitudes $\tilde{g}$. Obtaining $\tilde{g}$ for external excitation of a water droplet allows calculation of the bulk parameters of the cloud using the methods described in (6).

A dyadic version of Helmholtz's surface integral representation as given in (11) for $\tilde{\psi}=\tilde{\mathcal{E}}^e$ or $\tilde{\mathcal{E}}^h$,

$$\tilde{\psi}(\mathbf{r},\mathbf{r}')=\int_S dS(\hat{\mathbf{r}}_s)\Big\{\Big[\nabla_s\wedge\tilde{G}(\mathbf{r}-\mathbf{r}_s)\Big]^T\cdot\Big[\hat{\mathbf{n}}\wedge\tilde{\psi}(\mathbf{r}_s,\mathbf{r}')\Big]$$
$$-\Big[\hat{\mathbf{n}}\wedge\tilde{G}(\mathbf{r}-\mathbf{r}_s)\Big]^T\cdot\big[\nabla_s\wedge\tilde{\psi}(\mathbf{r}_s,\mathbf{r}')\big]\Big\}\equiv\Big\{\tilde{G},\,\tilde{\psi}\Big\}, \tag{13}$$
$$\tilde{G}(\mathbf{r}-\mathbf{r}_s)=\frac{3ik}{8\pi}\left(\tilde{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)h,\ h=h_o^{(1)}(kR),\ \mathbf{R}=\mathbf{r}-\mathbf{r}_s,$$

where ∇_s operates on the variables associated with $\mathbf{r}_s$ on the surface S of the scatterer and T denotes the transpose, may be used with the asymptotic form of h to obtain

$$\tilde{G}=-\frac{3ik}{8\pi}\left(\tilde{\mathbf{I}}-\hat{\mathbf{r}}\hat{\mathbf{r}}\right)h,\ \tilde{\psi}(\mathbf{r},\mathbf{r}')=h\tilde{g}\,(\hat{\mathbf{r}},\mathbf{r}'),\ \tilde{g}=\frac{3}{8\pi ik}\Big\{\Big(\tilde{\mathbf{I}}-\hat{\mathbf{r}}\hat{\mathbf{r}}\Big)e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}_s},\ \tilde{\psi}(\mathbf{r}_s,\mathbf{r}')\Big\}. \tag{14}$$

Here, **r**, $\mathbf{r}_s$, and **r**′ refer to vectors to the observation point, surface point, and source point respectively. Using Sommerfeld's integral representation (12) for h, we write

$$\left(\tilde{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)h=\frac{1}{2\pi}\int_C d\Omega_c\left(\tilde{\mathbf{I}}-\hat{\mathbf{r}}_c\hat{\mathbf{r}}_c\right)e^{ik\hat{\mathbf{r}}_c\cdot(\mathbf{r}-\mathbf{r}_s)}, \tag{15}$$

in [13], interchange the order of integration, and recognize $\tilde{g}$ from [14] to obtain

$$\tilde{\psi}(\mathbf{r},\mathbf{r}')=\frac{1}{2\pi}\int_C d\Omega_c e^{ik\hat{\mathbf{r}}_c\cdot\mathbf{r}}[\tilde{g}\,(\hat{\mathbf{r}}_c,\mathbf{r}')]. \tag{16}$$

This spectral representation is valid for r greater than the scatterer's projection on $\hat{\mathbf{r}}$.

Substituting the form of $\tilde{g}$ obtained from [10, 11] together with asymptotic representations for Hansen's functions in [16] or, equivalently, using Sommerfeld type complex contour integral representations for the M's and N's (5) which are functions of **r** gives

$$\tilde{\psi}(\mathbf{r},\, \mathbf{r}') = \sum A_n^m(\omega)\vec{\mathbf{R}}_{-mn}(\eta K,\, \mathbf{r}')\frac{1}{2\pi}\int_c d\Omega_c e^{ik\hat{\mathbf{r}}_c\cdot\mathbf{r}}\vec{\mathbf{S}}_{-mn}(\hat{\mathbf{r}}_c), \qquad [17]$$

where $\vec{\mathbf{R}}$ is **M** or **N**, $\vec{\mathbf{S}}$ is **B** or **C**, and $\eta k = K$; A involves the coefficients in [12]. Rotating the coordinate system so that $\hat{\mathbf{r}}$ is parallel to $\hat{\mathbf{z}}$, we obtain $\hat{\mathbf{r}}_c \cdot \mathbf{r} = r\cos\theta_c$. In this form, the time domain solution may be written in a symbolic form by inspection (3). We obtain, for $ct > R$,

$$\tilde{\psi}(\mathbf{r},\, \mathbf{r}',\, t) = -\sum A_n^m\left(\frac{i}{c}\frac{\partial}{\partial t}\right)\vec{\mathbf{R}}_{-mn}\left[\frac{i\eta}{c}\left(\frac{\partial}{\partial t}\right),\, \mathbf{r}'\right]\left\{\frac{\int_0^{2\pi} d\varphi\vec{\mathbf{S}}_{-mn}\left[P_n^m\left(\frac{ct}{R}\right)\right]}{\left[t^2 - \left(\frac{R}{c}\right)^2\right]^{\frac{1}{2}}}\right\}. \qquad [18]$$

When $ct < R$, $\tilde{\psi} = 0$. The sum is to be taken over all relevant combinations and the forms containing derivatives are to be expanded in a series and operate from left to right.

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