ROBUST CONTROL OF SYSTEMS WITH REAL PARAMETER

UNCERTAINTY AND UNMODELLED DYNAMICS

Semi-Annual   Progress   Report

Prepared by:

Principal Investigator:
Bor-Chin Chang
Department of Mechanical Engineering and Mechanics
Drexel University
Philadelphia, PA 19104
(215) 895-1790

and

Co-Principal Investigator:
Robert Fischl
Department of Electrical Engineering
Drexel University
Philadelphia, PA 19104
(215) 895-2254

October 1990

ABSTRACT

Two significant contributions have been made during this research period in the research "Robust Control of Systems with Real Parameter Uncertainty and Unmodelled Dynamics" under NASA Research Grant NAG-1-1102. They are: (1) a fast algorithm for computing the optimal H^∞ norm for the four-block, the two block, or the one-block optimal H^∞ optimization problem, and (2) a construction of an optimal H^∞ controller without numerical difficulty.

In using GD (Glover and Doyle) or DGKF (Doyle, Glover, Khargonekar, and Francis) approach to solve the standard H^∞ optimization problem, the major computation burden is on the computation of the optimal H^∞ norm which required bisection search. In this research period, we developed a very fast iterative algorithm for this computation. Our algorithm was developed based on hyperbolic interpolations which is much faster than any existing algorithm. The lower bound of the parameter, γ, in the H^∞ Riccati equation for solution existence is shown to be the be square root of the supremum over all frequencies of the maximum eigenvalue of a given transfer matrix which can be computed easily. The lower bound of γ such that the H^∞ Riccati equation has positive semidefinite solution can be also obtained by hyperbolic interpolation search. Our another significant result in this research period is the elimination of the numerical difficulties arising in the construction of an optimal H^∞ controller by directly applying the Glover and Doyle's state-space formulas.

With the fast iterative algorithm for the computation of the optimal H^∞ norm and the reliable construction of an optimal H^∞ controller, we are ready to apply these tools in the design of robust controllers for the systems with unmodelled uncertainties. These tools will be also very useful when we consider systems with structured uncertainties.

2

TABLE OF CONTENTS

PROGRESS REPORT

ROBUST CONTROL OF SYSTEMS WITH REAL PARAMETER UNCERTAINTY AND UNMODELLED DYNAMICS

1. INTRODUCTION

This document is the first-period progress report on the NASA supported research, "Robust Control of Systems with Real Parameter Uncertainty and Unmodelled Dynamics", (No. NAG-1-1102). The objective of the proposed research is to develop reliable and efficient algorithms for the computational problems arising in the design of robust optimal controllers for the systems with structured and unmodelled uncertainties and apply them to practical aerospace control systems. We are happy to report that we have obtained some significant results [1,2,3] in the solution of the four-block, the two block, or the one-block optimal H^∞ optimization problem. Meanwhile, we also have some primitive results [4,5] in the mixed H^2/H^∞ optimization problems which may have great impact on our future research. Furthermore, the work to be performed in the succeeding budget period has also been set on the right track.

In the paper, "Iterative Computation of the Optimal H^∞ Norm By Two-Riccati-Equation Method" [1], an iterative algorithm for computing the optimal H^∞ norm by using DGKF two-Riccati-equation approach [6] was proposed. The celebrating two-Riccati-equation solution to a standard H^∞ control problem can be used to characterize all possible stabilizing optimal or suboptimal H^∞ controllers if the optimal or suboptimal H^∞ norm is given. No efficient algorithm for computing the optimal or suboptimal H^∞ norm was available in the literature before this paper was written.

A numerical difficulty caused by the inversion of a singular matrix usually arises in using the DGKF state-space formulas to construct an optimal H^∞ controller. In the paper, "Design of an H^∞ Optimal Controller by DGKF State-Space Formulas" [2], we explained that the numerical difficulty originated from restricting the controller to be strictly proper and showed that the numerical difficulty can be easily eliminated if a proper controller with direct feedthrough is allowed.

In the paper, "Computation of the Optimal H^∞ Norm by H^∞ Riccati Equations and Hyperbolic Interpolations" [3], we proposed an extremely fast iterative algorithm for optimal H^∞ norm computation for a more general H^∞ optimization problem considered by Glover and Doyle [7]. The algorithm was developed based on hyperbolic interpolations

which is faster than any existing methods in the literature. The numerical difficulties arising in the construction of an optimal H^∞ controller by direct applying the Glover and Doyle's state-space formulas were also addressed in the paper.

The papers, "Necessary and Sufficient Conditions for Mixed H^2 and H^∞ Optimal Control" [4] and "Design of a Suboptimal H^∞ Controller with H^2 and Bandwidth Constraints" [5], summarized our primitive results in the mixed H^2/H^∞ optimal control. These results prompt a new research problem which will be briefly described in section 3.

In section 2 of this report, we will show the overall progress in this research period. In section 3, the work for future research will be briefly described.

2. OVERALL PROGRESS

2.1 Introduction

Consider the system

$$
\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} v(s) \\ u(s) \end{bmatrix} := G(s) \begin{bmatrix} v(s) \\ u(s) \end{bmatrix}
\qquad (2.1\text{-}1a)
$$

$$
u(s) = K(s)\, y(s)
\qquad (2.1\text{-}1b)
$$

where $G_{11}(s) \in \mathbb{R}(s)^{p_1 \times m_1}$, $G_{12}(s) \in \mathbb{R}(s)^{p_1 \times m_2}$, $G_{21}(s) \in \mathbb{R}(s)^{p_2 \times m_1}$, and $G_{22}(s) \in \mathbb{R}(s)^{p_2 \times m_2}$. $\mathbb{R}(s)^{p_1 \times m_1}$ is the set of $p_1 \times m_1$ proper rational matrices with real coefficients. Recall that the standard H^∞ optimization problem is the problem of finding a proper controller $K(s)$ such that the closed-loop system is internally stable and $\|\mathcal{F}_\ell(G,K)\|_\infty$ is minimized where

$$
\mathcal{F}_\ell(G,K) := G_{11} + G_{12} K (I - G_{22}K)^{-1} G_{21}
\qquad (2.1\text{-}2)
$$

is the transfer function of the closed-loop system from v to z and

$$
\|\Phi\|_\infty := \sup_\omega \bar{\sigma}\,[\Phi(j\omega)]
\qquad (2.1\text{-}3)
$$

and $\bar{\sigma}(\cdot)$ denotes the maximum singular value.

Let

5

$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \qquad (2.1\text{-}4)$$

be a minimal realization of $G(s)$ and $A \in \mathbb{R}^{n \times n}$. Here

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \{ A, B, C, D \} \qquad (2.1\text{-}5)$$

implies a state-space realization and $G(s) = D + C (sI - A)^{-1} B$.

According to the dimensions of $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$, and $G_{22}(s)$, the standard H^∞ optimization problem has the following situations to consider: (a) $p_1 > m_2$, $p_2 < m_1$; (b) $p_1 \leq m_2$, $p_2 < m_1$; (c) $p_1 > m_2$, $p_2 \geq m_1$; and (d) $p_1 \leq m_2$, $p_2 \geq m_1$. Situation (a) is referred as the four-block H^∞ optimization problem. Situations (b) and (c) are referred as the two-block H^∞ optimization problem. Situation (d) is referred as the one-block H^∞ optimization problem.

Recently, Glover and Doyle [7] presented a celebrating Riccati-equation type solution to the general H^∞ optimization problem (including four-block, two-block, and one-block problems). Glover and Doyle's approach characterizes all possible stabilizing suboptimal H^∞ controllers which order is not higher than that of the plant. In utilizing these Glover and Doyle's formulas to design an optimal (or suboptimal) H^∞ controller, the most computationally demanding work is the computation of the optimal (or suboptimal) H^∞ norm which requires iteration. Up to now in the literature, there is no efficient and systematic search algorithm available for this computation.

In this report we will present a very fast iterative hyperbolic search algorithm for the computation of the optimal (or suboptimal) H^∞ norm. The optimum can occur in three cases. In case (1), the optimum occurs at the smallest γ such that the two H^∞ Riccati equations have stabilizing solutions X and Y, i.e., these X and Y happens to be positive semidefinite and $\rho(XY) < \gamma^2$. Case (2) occurs when $Y=0$ (or $X=0$) for all γ and the optimal H^∞ norm is the smallest γ such that X (or Y) is positive semidefinite. The most likely one to happen most of the time is case (3) in which the optimal H^∞ norm is the γ such that the two H^∞ Riccati equations have positive semidefinite stabilizing solutions X and Y and $\rho(XY) = \gamma^2$ where $\rho(XY)$ is the spectral radius of XY. In subsection 2.3, the optimal H^∞ norm in case (1) is shown to be square root of the supremum over all frequencies of the maximum

eigenvalue of a given transfer matrix which can be computed easily [8,9]. The optimum in either case (2) or case (3) can be obtained by hyperbolic interpolation search which is presented in subsection 2.4.

Except case (1) which does not happen very often, the Glover and Doyle's controller formulas [7] can not be directly used to construct an optimal controller since an inversion of a singular matrix will cause numerical difficulty. As mentioned in [6] and [10], a descriptor (or generalized state-space representation) version of the controller formulas can avoid this numerical difficulty. With slight rearrangement, the Glover and Doyle's controller formulas can be rewritten in a descriptor form. The descriptor form of an optimal controller can then be reduced to an at least one order less state-space representation which is addressed in subsection 2.5.

For convenience, the Glover and Doyle's state-space formulas for suboptimal H^∞ solutions are listed in Subsection 2.2. In Subsections 2.3 and 2.4, algorithms to compute the optimal H^∞ norm in three cases are presented. The numerical difficulty in the construction of an optimal H^∞ controller is addressed in Subsection 2.5. Some illustrative examples are included in Subsection 2.6.

2.2 Glover and Doyle's State-Space Formulas

In [7], Glover and Doyle assume the realization of $G(s)$ is given by (2.1-4) with the following assumptions.

(i) (A, B_2) is stabilizable and (C_2, A) is detectable.

(ii) rank $D_{12} = m_2$, rank $D_{21} = p_2$.

(iii) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$, and D_{11} is partitioned as

$$\begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix} \text{ with } D_{1122} \in \mathbb{R}^{p_2 \times m_2},$$

(iv) $D_{22} = 0$ (this can be removed [7]).

(v) rank $\begin{bmatrix} j\omega I - A & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ $\forall\, \omega \in \mathbb{R}.$

7

$$\text{(vi) rank}\begin{bmatrix} j\omega I - A & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2 \quad \forall\, \omega \in \mathbb{R}.$$

Define two Hamiltonian matrices as follows, ·

$$H_\infty(\gamma) := \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C_1^T D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^T C_1 & B^T \end{bmatrix} \qquad (2.2\text{-}1a)$$

and

$$J_\infty(\gamma) := \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} - \begin{bmatrix} C^T \\ -B_1 D_{\bullet 1}^T \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^T & C \end{bmatrix} \qquad (2.2\text{-}1b)$$

where

$$D_{1\bullet} = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}, \quad D_{\bullet 1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}, \qquad (2.2\text{-}1c)$$

and

$$R = D_{1\bullet}^T D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{R} = D_{\bullet 1} D_{\bullet 1}^T - \begin{bmatrix} \gamma^2 I_{p1} & 0 \\ 0 & 0 \end{bmatrix}. \qquad (2.2\text{-}1d)$$

Then the following theorem shows an easy way to construct a suboptimal stabilizing controller such that $\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma$ where $\mathcal{F}_\ell(G,K)$ is the closed-loop transfer matrix from v to z.

Theorem 2.2-1: [7]

There exists a stabilizing controller such that $\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma$ if and only if the following three conditions hold.

(i) $\gamma > \max \left(\bar{\sigma} \begin{bmatrix} D_{1111} & D_{1112} \end{bmatrix}, \bar{\sigma} \begin{bmatrix} D_{1111}^T & D_{1121}^T \end{bmatrix} \right)$ $\qquad (2.2\text{-}2a)$

(ii) $H_\infty(\gamma) \in \text{dom(Ric)}$ and $X(\gamma) := \text{Ric}[H_\infty(\gamma)] \geq 0$.

$\quad J_\infty(\gamma) \in \text{dom(Ric)}$ and $Y(\gamma) := \text{Ric}[J_\infty(\gamma)] \geq 0$. $\qquad (2.2\text{-}2b)$

(iii) $\rho[X(\gamma)Y(\gamma)] < \gamma^2$. $\qquad (2.2\text{-}2c)$

Moreover, when these conditions hold, one such controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \qquad (2.2\text{-}3a)$$

where

$$\hat{D} = -D_{1121}D_{1111}^T(\gamma^2 I - D_{1111}D_{1111}^T)^{-1}D_{1112} - D_{1122} \qquad (2.2\text{-}3b)$$

$$\hat{C} = \{F_2 - \hat{D}(C_2 + F_{12})\}Z \qquad (2.2\text{-}3c)$$

$$\hat{B} = -H_2 + (B_2 + H_{12})\hat{D} \qquad (2.2\text{-}3d)$$

$$\hat{A} = A + HC + (B_2 + H_{12})\hat{C} \qquad (2.2\text{-}3e)$$

$$Z := (I - \gamma^{-2}YX)^{-1} \qquad (2.2\text{-}3f)$$

$$F^T = \begin{bmatrix} F_{11}^T & F_{12}^T & F_2^T \end{bmatrix} = -(XB + C_1^T D_{1\bullet})R^{-1} \qquad (2.2\text{-}3g)$$

$$H = \begin{bmatrix} H_{11} & H_{12} & H_2 \end{bmatrix} = -(YC^T + B_1 D_{\bullet 1}^T)\tilde{R}^{-1} \qquad (2.2\text{-}3h)$$

and $F_{11} \in \mathbb{R}^{(m_1 - p_2) \times n}$, $F_{12} \in \mathbb{R}^{p_2 \times n}$, $F_2 \in \mathbb{R}^{m_2 \times n}$, $H_{11} \in \mathbb{R}^{n \times (p_1 - m_2)}$, $H_{12} \in \mathbb{R}^{n \times m_2}$, $H_2 \in \mathbb{R}^{n \times p_2}$.

In the above theorem, condition (ii) means that there exist positive semi-definite solutions X and Y to the algebraic Riccati equations corresponding to the Hamiltonians $H_\infty(\gamma)$ and $J_\infty(\gamma)$ respectively. Condition (iii) means that the spectral radius of XY is less than γ^2.

The above theorem shows an easy state-space approach to construct a stabilizing suboptimal controller such that $\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma$. The order of the suboptimal controller can be the same as that of the plant G(s). The major computation involved is the solution of two H^∞ Riccati equations which are easy to solve if solutions exist.

The Glover and Doyle's approach is a great break-through in the solution of H^∞ optimization problem. Theorem 2.2-1 can also be used to compute the optimal H^∞ norm and to construct an optimal H^∞ controller. Algorithms for computing the optimal H^∞ norm will be presented in subsection 2.4 and the construction of an optimal H^∞ controller will be addressed in subsection 2.5. Subsection 2.3 gives the lower bounds of γ for the H^∞ Riccati equations to have solutions and to have positive semi-definite solutions respectively.

9

2.3 Lower Bounds for Solution Existence of H^∞ Riccati Equations

As mentioned in the previous subsection, Theorem 2.2-1 can also be employed to compute the optimal H^∞ norm. The optimal H^∞ norm is the smallest γ such that the three conditions in Theorem 2.2-1 are satisfied. The computation of the optimal H^∞ norm is closely related to the algebraic Riccati equations associated to the Hamiltonian matrices $H_\infty(\gamma)$ and $J_\infty(\gamma)$.

For the algebraic Riccati equation associated to the Hamiltonian matrix $H_\infty(\gamma)$ to have a unique stabilizing solution, $H_\infty(\gamma)$ must have stability and complementarity properties [6]. Stability property means that $H_\infty(\gamma)$ has no eigenvalues on the $j\omega$-axis. With stability property, $H_\infty(\gamma)$ has n eigenvalues in Re s < 0 and n in Re s > 0. Let $\mathfrak{X}_-(H_\infty)$ be an n-dimensional spectral space corresponding to eigenvalues in Re s < 0. Partitioning the matrix constructed from the basis vectors of $\mathfrak{X}_-(H_\infty)$, we have

$$\mathfrak{X}_-(H_\infty) = \mathrm{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad (2.3\text{-}1)$$

where $X_1, X_2 \in \mathbb{R}^{n \times n}$. Complementarity property means that X_1 is nonsingular, or equivalently, the two subspaces

$$\mathfrak{X}_-(H_\infty), \quad \mathrm{Im}\begin{bmatrix} 0 \\ I \end{bmatrix} \qquad (2.3\text{-}2)$$

are complementary. When the Hamiltonian matrix $H_\infty(\gamma)$ has stability and complementarity properties, $X = X_2 X_1^{-1}$ is the unique stabilizing solution of the algebraic Riccati equation associated to $H_\infty(\gamma)$.

Let α_X be the smallest γ such that the Hamiltonian matrix $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues. Then for every $\gamma \geq \alpha_X$ except some isolated γ at which X_1 is singular, the algebraic Riccati equation associated to $H_\infty(\gamma)$ will have a unique stabilizing solution. The smallest γ such that the Hamiltonian $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues is given by the following theorem.

Theorem 2.3-1:

The smallest γ such that the Hamiltonian $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues, denoted by α_X, is the square root of

$$\sup_\omega \lambda_{max}\{G_{11}^*[I - G_{12}(G_{12}^*G_{12})^{-1}G_{12}^*]G_{11} \, (j\omega)\} \qquad (2.3\text{-}3)$$

10

where $G_{11}^*(j\omega)$ ($G_{12}^*(j\omega)$ resp.) is the conjugate transpose of $G_{11}(j\omega)$ ($G_{12}(j\omega)$ resp.) and λ_{max} means the maximum eigenvalue.

Proof:

Let

$$\Gamma(s) = \begin{bmatrix} G_{11}^T(-s)G_{11}(s) - \gamma^2 I_{m1} & G_{11}^T(-s)G_{12}(s) \\ G_{12}^T(-s)G_{11}(s) & G_{12}^T(-s)G_{12}(s) \end{bmatrix} \qquad (2.3\text{-}4)$$

It is straightforward to show that

$$\Gamma^{-1}(s) = \left[\begin{array}{cc|c} A-BR^{-1}D_{1\bullet}^T C_1 & -BR^{-1}B^T & -BR^{-1} \\ -C_1^T C_1 + C_1^T D_{1\bullet} R^{-1} D_{1\bullet}^T C_1 & -(A-BR^{-1}D_{1\bullet}^T C_1)^T & C_1^T D_{1\bullet} R^{-1} \\ \hline R^{-1}D_{1\bullet}^T C_1 & R^{-1}B^T & R^{-1} \end{array}\right] \qquad (2.3\text{-}5)$$

and therefore the Hamiltonian $H_\infty(\gamma)$ of (2.2-1a) is the A-matrix of $\Gamma^{-1}(s)$. The realization has no uncontrollable or unobservable modes on the $j\omega$-axis. Thus, $H_\infty(\gamma)$ has no eigenvalues on the $j\omega$-axis if and only if $\Gamma^{-1}(s)$ has no poles on the $j\omega$-axis. Hence, α_X, the infimum of γ such that $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues, is equivalent to the supremum of γ such that

$$\det[\Gamma(j\omega)] = 0 \qquad \text{for all } \omega. \qquad (2.3\text{-}6)$$

That is, α_X is the supremum of γ such that

$$\begin{bmatrix} G_{11}^*(j\omega)G_{11}(j\omega) & G_{11}^*(j\omega)G_{12}(j\omega) \\ G_{12}^*(j\omega)G_{11}(j\omega) & G_{12}^*(j\omega)G_{12}(j\omega) \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \gamma^2 I_{m1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad \text{for all } \omega \qquad (2.3\text{-}7a)$$

or

$$G_{11}^*[\, I - G_{12}(G_{12}^* G_{12})^{-1}G_{12}^*\,]\, G_{11}\, \xi_1 = \gamma^2\, \xi_1 \qquad \text{for all } \omega \qquad (2.3\text{-}7b)$$

Therefore, α_X is the square root of $\displaystyle\sup_\omega \lambda_{max}\{G_{11}^*[I - G_{12}(G_{12}^* G_{12})^{-1}G_{12}^*]G_{11}\,(j\omega)\}$ which completes the proof.

11

Remark 2.3-2: A realization of $I - G_{12}(s)[G_{12}^T(-s)G_{12}(s)]^{-1}G_{12}^T(-s)$ is given by

$$\left[\begin{array}{cc|c} A-B_2D_{12}^TC_1 & -B_2B_2^T & B_2D_{12}^T \\ -C_1^T(I-D_{12}D_{12}^T)C_1 & -(A-B_2D_{12}^TC_1)^T & C_1^T(I-D_{12}D_{12}^T) \\ \hline -(I-D_{12}D_{12}^T)C_1 & D_{12}B_2^T & I-D_{12}D_{12}^T \end{array}\right] \qquad (2.3\text{-}8)$$

which was obtained from a 4n-th order realization and a mathematical pole-zero cancellation. The mathematical pole-zero cancellation was carried out by applying similarity transformation and deleting the uncontrollable and/or uncontrollable states [11].

Computation of α_X, the infimum of γ such that $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues

Let

$$\eta^2(\omega) = \lambda_{max}\{G_{11}^*[I - G_{12}(G_{12}^*G_{12})^{-1}G_{12}^*]G_{11}\ (j\omega)\} \qquad (2.3\text{-}9)$$

Then, α_X, the infimum of γ such that $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues, is the supremum of $\eta(\omega)$. There are several efficient algorithms available for searching for the supremum of $\eta(\omega)$ [8,9]. In [8], a frequency, say ω_1, is chosen based on the pole location of $G_{11}(s)$ and $I - G_{12}(s)[G_{12}^T(-s)G_{12}(s)]^{-1}G_{12}^T(-s)$. Let $\gamma = \eta(\omega_1)$ and then find all the positive real ω's such that $\eta(\omega) = \gamma$. These ω's can be easily obtained from computing the $j\omega$-axis eigenvalues of the Hamiltonian matrix $H_\infty(\gamma)$. Now, we have the frequency intervals in which $\eta(\omega) \geq \gamma$. Evaluate $\eta(\omega)$ for each midpoint of these frequency intervals and set γ to be the maximum of these $\eta(\omega)$'s. Then, find the new frequency intervals in which $\eta(\omega) \geq \gamma$. According to [9], the convergence of this iterative process is quadratic. This process can be repeated until only one frequency interval with $\eta(\omega) \geq \gamma$ is left and the interval length is negligible [9]. In [8], this process is terminated when a frequency interval in which $\eta(\omega)$ is greater than γ and convex is found. A search method called Brent method was used to search for the the supremum of $\eta(\omega)$ in the convex frequency interval.

Let α_Y be the smallest γ such that the Hamiltonian matrix $J_\infty(\gamma)$ has no $j\omega$-axis eigenvalues. The computation of α_Y is similar to that of α_X. Define

$$\alpha = max\ \{\alpha_X, \alpha_Y\} \qquad (2.3\text{-}10)$$

Then α is the infimum of γ such that the two algebraic Riccati equations associated to $H_\infty(\gamma)$

and $J_{\infty}(\gamma)$ have solutions.

Let $\gamma = \alpha$ and denote the solutions of the two algebraic Riccati equations associated to $H_{\infty}(\gamma)$ and $J_{\infty}(\gamma)$ by X and Y respectively. If X and Y happen to be both positive semidefinite and $\rho(XY) < \gamma^2$, then α is the optimal H^{∞} norm according to Theorem 2.2-1. Recall that the case in which the optimum occurs at α, the infimum of γ such that the two Riccati equations have solutions, was referred as case (1).

Computation of β_X, the infimum of γ such that $X(\gamma)$ is positive semidefinite

Let β_X (β_Y resp.) be the infimum of γ such that X (Y resp.) is positive semidefinite. In case (2), Y (or X resp.) is zero for all $\gamma > \alpha$ and hence, the optimum occurs at $\gamma = \beta_X$ (or $\gamma = \beta_Y$ resp.). In the following, a hyperbolic interpolation search algorithm is used to compute β_X.

Denote the eigenvalues of $X(\gamma)$ by $\lambda_i[X(\gamma)]$, i=1,2,...,n and let

$$f[X(\gamma)] = \begin{cases} \max\{\lambda_i[X(\gamma)]\} & \text{if all } \lambda_i[X(\gamma)] \text{ are positive} \\ \min\{\lambda_i[X(\gamma)]\} & \text{if some } \lambda_i[X(\gamma)] \text{ is negative} \end{cases} \qquad (2.3\text{-}11)$$

We observe that $f[X(\gamma)]$ is a hyperbola-like function of γ^2. When γ^2 equals to $\beta_X^2+\epsilon$ (or $\beta_X^2-\epsilon$, resp.) and ϵ approaches to zero, $f[X(\gamma)]$ will approach to infinity (or negative infinity, resp.). As γ^2 increases from $\beta_X^2+\epsilon$ to infinity, $f[X(\gamma)]$ decreases monotonically to a positive number.

To search for β_X, we will use two passes in our algorithm. The first pass is used to find a lower bound x_L such that $X(\sqrt{x_L})$ exists and $x_L < \beta_X^2$ and an upper bound x_U which is greater than β_X^2. The second pass is to search for β_X^2.

Since the shape of the graph $(\gamma^2, f[X(\gamma)])$ resembles that of a hyperbola, a hyperbolic interpolation search algorithm will give fast convergence in computing β_X. For notational simplicity, γ^2 and $f[X(\gamma)]$ are denoted by x and y respectively in the following. In the algorithm, we start from arbitrary three points on the graph (x,y). It is easy to choose three positive numbers x_i, i=1,2,3, with $x_1 < x_2 < x_3$, such that their corresponding Riccati equations associated to $H_{\infty}(\sqrt{x_i})$ have positive semidefinite solutions $X(\sqrt{x_i})$. Computing $y_i = f[X(\sqrt{x_i})]$ from (3-11), we have the trio (x_i, y_i), i=1,2,3. Obviously, β_X^2 is smaller

13

than x_1 and hence x_1 qualifies as an upper bound of β_X^2, denoted by x_U. Let $(x_U, y_U) = (x_1, y_1)$. The abscissa of any point (x_L, y_L) on the graph (x,y) with $y_L < 0$ can serve as a lower bound for β_X^2.

Now, a hyperbola which interpolates these three points, (x_i, y_i), i=1,2,3, can be easily determined. That is, the three parameters a, b, and c in the hyperbolic equation,

$$(x-a)(y-b) = c \tag{2.3-12a}$$

are determined by (x_i, y_i), i=1,2,3. They are,

$$a = \frac{\left[(1 - \frac{x_2}{x_1})(1 - \frac{x_3 y_3}{x_1 y_1}) - (1 - \frac{x_3}{x_1})(1 - \frac{x_2 y_2}{x_1 y_1})\right] x_1}{(1 - \frac{x_2}{x_1})(1 - \frac{y_3}{y_1}) - (1 - \frac{x_3}{x_1})(1 - \frac{y_2}{y_1})} \tag{2.3-12b}$$

$$b = \left[\frac{1 - \frac{x_2 y_2}{x_1 y_1}}{1 - \frac{x_2}{x_1}} - \frac{(1 - \frac{y_2}{y_1}) a}{(1 - \frac{x_2}{x_1}) x_1}\right] y_1 \tag{2.3-12c}$$

$$c = (x_1 - a)(y_1 - b) \tag{2.3-12d}$$

In the graph of the hyperbola, $y \rightarrow \pm\infty$ as $x \rightarrow a$. If the trio are in the neighborhood of $(\beta_X^2, \pm\infty)$, the a of (2.3-12b) will be closer than any x_i to β_X^2. Let $x_a = a$ and evaluate $y_a = f[X(\sqrt{x_a})]$ if $X(\sqrt{x_a})$ exists. If $y_a > 0$, we update (x_U, y_U) and the trio as follows,

$$(x_U, y_U) = (x_a, y_a), \ (x_3, y_3) = (x_2, y_2), \ (x_2, y_2) = (x_1, y_1), \ \text{and} \ (x_1, y_1) = (x_a, y_a).$$

If $y_a < 0$, x_a qualifies as a lower bound for β_X^2 and therefore we have $(x_L, y_L) = (x_a, y_a)$. If $X(\sqrt{x_a})$ does not exist, i.e., $H_\infty(\sqrt{x_a})$ has eigenvalues on the $j\omega$-axis, we suggest to find α_X first. α_X, the infimum of γ such that $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues, can be easily computed by using Theorem 2.3-1 and the algorithm in [8,9]. If $X(\alpha_X)$ is positive semidefinite, then we have case (1) and the optimal H^∞ norm is α_X. Otherwise, we have case (2) and the optimum occurs at β_X which still remains to be found. Let $x_L = \alpha_X^2$ and then x_L qualifies as a lower bound for β_X^2 since $y_L = f[X(\sqrt{x_L})]$ is negative.

14

Once (x_L,y_L) and (x_U,y_U) are available, we are ready to go to the second pass to search for β_X^2. During the first pass, it is possible to have $y_a>0$ all the time and therefore there is no chance to obtain x_L. In this event, we do not need the second pass. Instead, we just keep updating the trio and the point (x_a,y_a) is moving closer and closer to the optimal point $(\beta_X^2,\pm\infty)$. The iterative process can be terminated when $|y_a|$ is large enough.

In the second pass, we start with the new trio (x_L,y_L), (x_U,y_U) and (x_E,y_E). In addition to (x_L,y_L) and (x_U,y_U), the point (x_E,y_E) is the best point which can be chosen from the trio at the end of the first pass. The best point here means the one with the largest $|y_i|$. Let $(x_1,y_1) = (x_L,y_L)$, $(x_2,y_2) = (x_U,y_U)$, and $(x_3,y_3) = (x_E,y_E)$, then a hyperbola which interpolates these three points can be obtained from the formulas in (2.3-12a) - (2.3-12d). If $x_L < a < x_U$, let $x_G=a$. Otherwise, let $x_G=(x_L+x_U)/2$. Evaluate $y_G=f[X(\sqrt{x_G})]$. The point (x_L,y_L) (or (x_U,y_U), resp.) will be updated by (x_G,y_G) if $y_G<0$ (or $y_G>0$, resp.). Before the updating of (x_L,y_L) (or (x_U,y_U), resp.), we like to update (x_E,y_E) by (x_L,y_L) ((x_U,y_U) resp.) if $|y_E| < |y_L|$ (or $|y_E| < |y_U|$, resp.). This iterative process is repeated until $|y_G|$ is sufficiently large or the gap $|x_U-x_L|$ is small enough. In the second pass, we use hybrid search scheme: the hyperbolic interpolation to speed up the convergence and the bisection to guarantee the convergence when the hyperbolic interpolation does not work. Once the trio reaches the neighborhood of the optimum, the convergence rate is quadratic.

Computation of the lower and upper bounds for the optimum of Case (3)

As mentioned, case (3) is the most likely one to happen. In case (3), the optimum occurs at the γ such that $\rho[X(\gamma)Y(\gamma)] = \gamma^2$ where $\rho(XY)$ is the spectral radius of XY. In subsection 2.4, a fast iterative algorithm is proposed to compute the optimum. To be able to use that algorithm, initial lower and upper bounds, γ_L and γ_U, are needed. Any γ with which both $X(\gamma)$ and $Y(\gamma)$ are positive semi-definite and $\rho[X(\gamma)Y(\gamma)] > \gamma^2$ qualifies as a lower bound. Similarly, any γ with which both $X(\gamma)$ and $Y(\gamma)$ are positive semi-definite and $\rho[X(\gamma)Y(\gamma)] < \gamma^2$ can serve as an upper bound. It is easy to have an upper bound which can be chosen arbitrarily large. However, choosing a lower bound is not trivial at all. A small γ may not satisfy the positive semi-definiteness condition and a larger γ may violate the condition $\rho[X(\gamma)Y(\gamma)] > \gamma^2$.

Recall that β_X (β_Y resp.) is the infimum of γ such that X (Y resp.) is positive semi-definite. Let

$$\beta = \max \{\beta_X, \beta_Y\} \qquad (2.3\text{-}13)$$

Then any $\gamma > \beta$ which satisfies $\rho[X(\gamma)Y(\gamma)] > \gamma^2$ qualifies as a lower bound. In order to find a lower bound, it is not necessary to compute β_X and β_Y although they can be obtained from the hyperbolic interpolation search algorithm described above. An easy way to find a lower bound is briefly described in the following.

Denote the eigenvalues of $X(\gamma)Y(\gamma)$ by $\lambda_i[X(\gamma)Y(\gamma)]$, $i=1,2,...,n$ and let

$$g[X(\gamma)Y(\gamma)] = \begin{cases} \max\{\lambda_i[X(\gamma)Y(\gamma)]\} & \text{if all } \lambda_i[X(\gamma)Y(\gamma)] \text{ are positive} \\ \min\{\lambda_i[X(\gamma)Y(\gamma)]\} & \text{if some } \lambda_i[X(\gamma)Y(\gamma)] \text{ is negative} \end{cases} \qquad (2.3\text{-}14)$$

We observe that $g[X(\gamma)Y(\gamma)]$ is also a hyperbola-like function of γ^2. Since the shape of the graph $(\gamma^2, g[X(\gamma)Y(\gamma)])$ resembles that of a hyperbola, a hyperbolic interpolation search algorithm will give fast convergence in finding a lower bound, γ_L, which satisfies $\gamma_L > \beta$ and $g[X(\gamma_L)Y(\gamma_L)] > \gamma_L^2$. In the following, γ^2 and $g[X(\gamma)Y(\gamma)]$ are replaced by x and y respectively to simplify the notation. To find γ_L, we start from arbitrary three points on the graph (x,y). It is easy to choose three positive numbers x_i, $i=1,2,3$, with $x_1<x_2<x_3$, such that their corresponding Riccati equations associated to $H_\infty(\sqrt{x_i})$ $(J_\infty(\sqrt{x_i})$, resp.) have positive semidefinite solutions $X(\sqrt{x_i})$ $(Y(\sqrt{x_i})$, resp.). Computing $y_i = g[X(\sqrt{x_i})Y(\sqrt{x_i})]$ from (2.3-14), we have the trio (x_i,y_i), $i=1,2,3$.

The objective is to search for the optimum point (x_o,y_o) on the graph (x,y) such that $x_o=y_o$. In the proposed algorithm, two passes are used to achieve this objective. The first pass is to find a lower and an upper bound points (x_L,y_L) and (x_U,y_U) such that $y_L>x_L$ and $x_U>y_U>0$, and therefore $x_L<x_o<x_U$ since the graph (x,y) is monotonically decreasing when $y>0$. The second pass is to search for the optimum x_o. If $x_1<y_1$, we have a lower bound point $(x_L,y_L)=(x_1,y_1)$. It is trivial to have an upper bound point (x_U,y_U) from (x_2,y_2), or (x_3,y_3), or some other point on the graph with large abscissa. Most of time we have $x_1>y_1$, and $(x_U,y_U) = (x_1,y_1)$. To find a lower bound point (x_L,y_L), we employ hyperbolic interpolations as follows.

Now, a hyperbola which interpolates these three points, (x_i,y_i), $i=1,2,3$, can be easily determined. That is, the three parameters a, b, and c in the hyperbolic equation of (2.3-12a) can be evaluated by (2.3-12b), (2.3-12c), and (2.3-12d) respectively. Let x_p be the abscissa of the intersection point of the hyperbola and the line y=x and let $x_a=a$. Evaluate $y_a=g[X(\sqrt{x_a})Y(\sqrt{x_a})]$ and $y_p=g[X(\sqrt{x_p})Y(\sqrt{x_p})]$ if they exist. We may have a lower bound point $(x_L,y_L)=(x_p,y_p)$ if $y_p>x_p$ or have $(x_L,y_L)=(x_a,y_a)$ if $y_a>x_a$. If (x_L,y_L) is

16

still not available, we will update (x_U, y_U) and the trio by the information of (x_P, y_P) and (x_a, y_a) and repeat the iterative process until (x_L, y_L) is found. If y_P does not exist (and neither is y_P), we will firstly compute α, the infimum of γ such that $H_\infty(\gamma)$ and $J_\infty(\gamma)$ have no $j\omega$-axis eigenvalues. Recall that $\alpha = \max \{\alpha_X, \alpha_Y\}$ and α_X and α_Y can be easily obtained by using Theorem 2.3-1 and the algorithms in [8,9]. For any $x > \alpha^2$, $y = g[X(\sqrt{x})Y(\sqrt{x})]$ always exists. Denote a lower bound for the lower bound x_L by x_{LL} which is set to be α^2 at the first time y_P does not exist. When the hyperbolic interpolation search does not work (i.e. when $x_P < x_{LL}$), we let $x_B = (x_{LL} + x_U)/2$ and compute $y_B = g[X(\sqrt{x_B})Y(\sqrt{x_B})]$. If $y_B > x_B$, we have $(x_L, y_L) = (x_B, y_B)$. Otherwise, either x_{LL} or x_U will be updated by x_B and the trio can be also updated by using (x_B, y_B).

When a lower and an upper bound points (x_L, y_L) and (x_U, y_U) are available, the second pass of the algorithm can be used to search for the optimum x_o. The second pass of the algorithm will be stated in the following section.

2.4 Computation of the Optimal H^∞ Norm

Recall that the optimal H^∞ Norm is the smallest γ such that the two Riccati equations associated to $H_\infty(\gamma)$ and $J_\infty(\gamma)$ have positive semi-definite solutions $X(\gamma)$ and $Y(\gamma)$ and $\rho[X(\gamma)Y(\gamma)] \leq \gamma^2$. There are three cases to consider. The optimum may occur at $\gamma = \alpha$ which is referred as case (1) and may also occur at either $\gamma = \beta_X$ or $\gamma = \beta_Y$ which is referred as case (2). α, β_X, and β_Y were defined in the previous section. In case (3), the optimal H^∞ Norm is the γ_o such that $\rho[X(\gamma_o)Y(\gamma_o)]$ equals to γ_o^2.

To compute the optimum, we start from any three points on the graph $(\gamma^2, g[X(\gamma)Y(\gamma)])$ where $g[X(\gamma)Y(\gamma)]$ was defined in (2.3-14). If $Y(\gamma)$ or $X(\gamma)$ is zero and the optimum occurs at either $\gamma = \beta_X$ or $\gamma = \beta_Y$, we have case (2). The computation of β_X (or β_Y) has been addressed in the previous section. The optimum can also occur at $\gamma = \alpha_X$ or $\gamma = \alpha_Y$, which was referred as case (1). α_X or α_Y can be easily computed by using Theorem 2.3-1 and the algorithms in [8,9]. For the rest of the section, we assume that $Y(\gamma)$ and $X(\gamma)$ are both nonzero and there are only cases (1) and (3) yet to be considered. In case (1), the optimum occurs at $\gamma = \alpha$ and $\alpha = \max\{\alpha_X, \alpha_Y\}$ which can be easily obtained as mentioned.

If we can find a lower bound γ_L such that $g[X(\gamma_L)Y(\gamma_L)] > \gamma_L^2$, then we have case (3). Unless such a lower bound γ_L does not exist, the computation of α is not necessary.

For notational simplicity, we replace $(\gamma^2, g[X(\gamma)Y(\gamma)])$ by (x, y). The starting three

points on the graph (x, y) are (x_i, y_i), i=1,2,3, with $x_1 < x_2 < x_3$. First of all, we will try to find a lower bound x_L (i.e., γ_L^2) and an upper bound x_U (i.e., γ_U^2). Usually, the x_i's of the initial trio (x_i, y_i), i=1,2,3, are chosen as large numbers to guarantee solution existence for the Riccati equations. An upper bound x_U can be initially assigned as x_1 if y_1 is positive and $y_1 < x_1$. The locating of a lower bound was briefly described at the end of the previous section. Recall that any point (x_L, y_L) on the graph (x,y) which is to the right of β and satisfies $x_L < y_L$ qualifies as a lower bound point. A hyperbola interpolating these three points, (x_i, y_i), i=1,2,3, can be easily obtained from (2.3-12). Let x_P be the abscissa of the intersection point of the hyperbola and the line y=x and let x_a be the abscissa of the singular point of the hyperbola, i.e., $x_a = a$ where a is from (2.3-12b). These two points (x_a, y_a) and (x_P, y_P) are good guesses for (x_L, y_L). If $y_P \geq x_P$, we have $(x_L, y_L) = (x_P, y_P)$. Otherwise, try to see if we have $y_a \geq x_a$ which gives $(x_L, y_L) = (x_a, y_a)$. If the above two inequalities are not satisfied, we will use (x_a, y_a) and (x_P, y_P) to update (x_U, y_U) and the trio and repeat the above iterative process until a lower bound point (x_L, y_L) is procured.

In the above hyperbolic interpolation search process, $y_P = g[X(\sqrt{x_P})Y(\sqrt{x_P})]$ may not exist since the two Riccati equations may not have solutions. If either $H_\infty(\sqrt{x_P})$ or $J_\infty(\sqrt{x_P})$ has eigenvalues on the imaginary axis, we may have case (1). The infimum of γ such that $H_\infty(\gamma)$ and $J_\infty(\gamma)$ have no eigenvalues on the imaginary axis, i.e., α, should be computed by the algorithm proposed in the previous section. If the Riccati solutions $X(\alpha)$ and $Y(\alpha)$ are both positive semi-definite and $\rho[X(\alpha)Y(\alpha)] < \alpha^2$, then α is the optimal H^∞ norm. If either $X(\alpha)$ or $Y(\alpha)$ is indefinite, a bisection search need to be used together with the hyperbolic interpolation to guarantee convergence. The detailed algorithm is listed at the end of this subsection.

Now, the lower and upper bound points, (x_L, y_L) and (x_U, y_U) are available. It is well known that in the graph (x,y), y is a monotonically decreasing function of x when $x > \beta^2$. From the observation of numerous examples, we have a conjecture that y is a convex function of x when $x > \beta^2$. Furthermore, the shape of the graph (x,y) resembles a hyperbola and therefore a hyperbolic interpolation search method can be used to speed up the convergence. Although the proposed algorithm is devised by exploiting the conjectural properties of hyperbola resemblance and convexity, it still works even these properties fail to hold. In case that any of these properties does not exist, the algorithm will detect it and switch the search scheme to the bisection and the convergence will still be guaranteed. Even the conjecture is not absolutely correct, the chance for it to fail in practice would be very

rare.

To search for the optimal H^∞ norm, γ_0, which square is the abscissa of the point on the graph (x,y) with x=y, we start from a new trio (x_L,y_L), (x_U,y_U), and (x_E,y_E) which are available at the end of the first pass, i.e., the pass of finding lower and upper bounds x_L and x_U. The point (x_E,y_E) is the available one which has the largest $|y_i|$ excluding (x_L,y_L) and (x_U,y_U). We also initialize $\hat{x}_L=x_L$ and $\hat{x}_U=x_U$. A hyperbola which interpolates these three points can be easily determined. Let $(x_1,y_1) = (x_L,y_L)$, $(x_2,y_2) = (x_U,y_U)$, and $(x_3,y_3) = (x_E,y_E)$, then a hyperbola which interpolates these three points can be obtained from the formulas in (2.3-12a) - (2.3-12d). The abscissa of intersection point of the hyperbola and the straight line y=x, denoted by x_G, will be close to $x_o=\gamma_o^2$. According to our conjecture, x_G is supposed to be between $\hat{x}_L$ and $\hat{x}_U$. If that is not the case, x_G is reassigned as $(\hat{x}_L+\hat{x}_U)/2$. Now, we evaluate $y_G = \rho\left[X(\sqrt{x_G})Y(\sqrt{x_G})\right]$.

In the graph (x,y), if y is a convex function of x, the following theorem can be employed to find a bracket which encloses the optimum x_o.

Theorem 2.4-1:

Given three points (x_L,y_L), (x_U,y_U), and (x_G,y_G) on the graph (x,y). Assume that $y_L > x_L$, $y_U < x_U$, and $x_L < x_G < x_U$. If y is a convex function of x, we have the following:

(i) If $y_G < x_G$, then

$$\theta_6 < x_o < \theta_5 \qquad (2.4\text{-}1a)$$

(ii) If $y_G > x_G$, then

$$\theta_5 < x_o < \theta_6 \qquad (2.4\text{-}1b)$$

where

$$\theta_5 = \frac{y_L - x_L \dfrac{y_L - y_G}{x_L - x_G}}{1 - \dfrac{y_L - y_G}{x_L - x_G}} \qquad (2.4\text{-}2a)$$

and

$$\theta_6 = \frac{y_G - x_G \dfrac{y_U - y_G}{x_U - x_G}}{1 - \dfrac{y_U - y_G}{x_U - x_G}} \qquad (2.4\text{-}2b)$$

Proof: It is straightforward to show these inequalities by geometry.

19

Now we have a bracket which encloses the optimum x_o, i.e.,

$$\hat{x}_L < x_o < \hat{x}_U \qquad\qquad (2.4\text{-}3a)$$

where

$$\hat{x}_L = \theta_6 \quad \text{and} \quad \hat{x}_U = \theta_5 \quad \text{if} \quad y_G < x_G, \qquad\qquad (2.4\text{-}3b)$$

$$\hat{x}_L = \theta_5 \quad \text{and} \quad \hat{x}_U = \theta_6 \quad \text{if} \quad y_G > x_G. \qquad\qquad (2.4\text{-}3c)$$

If the gap between $\hat{x}_L$ and $\hat{x}_U$ is negligible, the search algorithm will be terminated and then we have the optimum $x_o = \hat{x}_U$. Otherwise, the trio (x_L, y_L), (x_U, y_U), and (x_E, y_E) shall be updated and the hyperbolic interpolation search shall be repeated until the gap between $\hat{x}_L$ and $\hat{x}_U$ is small enough. The updating of the trio is executed as follows. When $y_G > x_G$, x_G is a better lower bound than x_L and therefore we update (x_E, y_E) by (x_L, y_L) if $|x_G - x_L| < |x_G - x_E|$ and update (x_L, y_L) by (x_G, y_G). When $y_G < x_G$, x_G is a better upper bound than x_U and therefore we update (x_E, y_E) by (x_U, y_U) if $|x_G - x_U| < |x_G - x_E|$ and update (x_U, y_U) by (x_G, y_G).

In the above proposed search algorithm, we used the conjectured property of convexity. As mentioned above, there is no proof yet for this conjecture. However, it is easy to check if the final $\hat{x}_U$ approximately equals to x_o. Evaluate $\hat{y}_U = \rho\left[X(\sqrt{\hat{x}_U}\,) Y(\sqrt{\hat{x}_U}\,) \right]$, and check if $\hat{y}_U < \hat{x}_U$ and $(\hat{x}_U - \hat{y}_U)$ is negligible. If the condition is satisfied, practically we have the optimum $x_o = \hat{x}_U$, i.e., we have the optimal H^∞ norm $\gamma_o = \sqrt{\hat{x}_U}$. It is almost impossible for the condition to fail. Nevertheless, we can update the lower and upper bounds x_L and x_U and use the bisection search to reduce $x_U - x_L$ in case that the condition fails.

To summarize, an iterative algorithm for computing the optimal H^∞ norm is listed as follows.

Iterative Algorithm for Computing the Optimal H^∞ Norm

Initialization:

A1. Initialize the tolerances ε_1, ε_2, ε_3, which are small positive numbers, and let $x_{LL} = 0$, $x_L = 0$ and $x_U = 0$.

A2. Arbitrarily choose three large x_i, i=1,2,3, such that $x_3 > x_2 > x_1 > 0$ and their corresponding Riccati solutions $X(\sqrt{x_i}\,)$ and $Y(\sqrt{x_i}\,)$ are all positive semi-definite.

A3. If both X and Y are nonzero, go to **D1**.

20

Computation of β_X, the optimum for Case (2): (First Pass)

B1. Either X or Y is zero. Without loss of generality, we assume that Y=0 and let y_i = $f[X(\sqrt{x_i})]$, i=1,2,3, where $f[X(\sqrt{x_i})]$ was defined by (2.3-11). Let (x_U,y_U) = (x_1,y_1).

B2. Let x_a = a where a is evaluated from (2.3-12b). Compute $y_a = f[X(\sqrt{x_a})]$ if $X(\sqrt{x_a})$ exists. If $X(\sqrt{x_a})$ does not exist, go to **C1**.

B3. If $y_a > (1/\varepsilon_3)$, the optimal H^∞ norm is $\sqrt{x_a}$. **Stop.**

B4. If $y_a > 0$, update (x_U,y_U) and the trio as follows:
$(x_U,y_U)=(x_a,y_a)$, $(x_3,y_3)=(x_2,y_2)$, $(x_2,y_2)=(x_1,y_1)$, $(x_1,y_1)=(x_a,y_a)$, Go to **B2**.

B5. $(x_L,y_L)=(x_a,y_a)$, $(x_E,y_E)=(x_2,y_2)$.

Computation of β_X, the optimum for Case (2): (Second Pass)

B6. Let $(x_1,y_1)=(x_L,y_L)$, $(x_2,y_2)=(x_U,y_U)$, $(x_3,y_3)=(x_E,y_E)$, and evaluate the a of (2.3-12b).

B7. If $x_L < a < x_U$, let x_G = a and go to **B9**.

B8. Let $x_G = (x_L+x_U)/2$.

B9. Evaluate $y_G = f[X(\sqrt{x_G})]$.

B10. If $y_G > (1/\varepsilon_3)$, the optimal H^∞ norm is $\sqrt{x_G}$. **Stop.**

B11. If $|x_U - x_L| < \varepsilon_1$, the optimal H^∞ norm is $\sqrt{x_U}$. **Stop.**

B12. If $y_G < 0$, then update (x_E,y_E) by (x_L,y_L) only if $|y_L| > |y_E|$, and update (x_L,y_L) by (x_G,y_G) and go to **B6**.

B13. Update (x_E,y_E) by (x_U,y_U) only if $|y_U| > |y_E|$, and update (x_U,y_U) by (x_G,y_G) and go to **B6**.

Computation of α_X, the optimum for Case (1) with Y=0:

C1. Compute α_X, the infimum of γ such that $H_\infty(\gamma)$ has no $j\omega$-axis eigenvalues, by using Theorem 2.3-1 and the algorithm in [8,9].

C2. If $X(\alpha_X)$ is positive semidefinite, the optimal H^∞ norm is α_X. **Stop.**

C3. Let $x_L = \alpha_X^2$ and and evaluate $y_L = f[X(\sqrt{x_L})]$.
Let $(x_E,y_E) = (x_2,y_2)$ and go to **B6**.

Find upper and lower bounds for Case (3):

D1. Compute $y_i = g[X(\sqrt{x_i})Y(\sqrt{x_i})]$, i=1,2,3, where $g[X(\gamma)Y(\gamma)]$ was defined by (2.3-14).

D2. If $x_1 > y_1$, then let $(x_U, y_U) = (x_1, y_1)$ and go to **D7**.

D3. Let $(x_L, y_L) = (x_1, y_1)$.

D4. If $x_2 > y_2$, then let $(x_U, y_U) = (x_2, y_2)$ and go to **F1**.

D5. If $x_3 > y_3$, then let $(x_U, y_U) = (x_3, y_3)$ and go to **F1**.

D6. Double x_3 and evaluate $y_3 = g[X(\sqrt{x_3})Y(\sqrt{x_3})]$ and go to **D5**.

D7. Evaluate a, b, c from (2.3-12b) - (2.3-12d).

Let $x_a = a$ and $x_P = \frac{1}{2}\left[a+b+\sqrt{(a+b)^2 - 4(ab-c)} \right]$.

If $x_P < x_{LL}$, go to **D18**.

D8. If $(x_{LL}=0)$ and (either $X(\sqrt{x_P})$ or $Y(\sqrt{x_P})$ does not exist), then go to **E1**.

D9. Evaluate $y_P = g[X(\sqrt{x_P})Y(\sqrt{x_P})]$.

If $y_P > x_P$, then $(x_L, y_L) = (x_P, y_P)$, and go to **F1**.

D10. If $(x_{LL}=0)$ and $(x_P > y_P > 0)$, then update the following:

$(x_U, y_U) = (x_P, y_P)$, $(x_3, y_3) = (x_2, y_2)$, $(x_2, y_2) = (x_1, y_1)$, $(x_1, y_1) = (x_P, y_P)$,

and go to **D13**.

D11. If $(x_{LL} \neq 0)$ and $(x_P > y_P > 0)$, then update the following:

$(x_U, y_U) = (x_P, y_P)$, $(x_3, y_3) = (x_2, y_2)$, $(x_2, y_2) = (x_P, y_P)$,

and go to **D13**.

D12. $(x_{LL}, y_{LL}) = (x_P, y_P)$, $(x_1, y_1) = (x_P, y_P)$, and go to **D7**.

D13. If $(x_{LL} < x_{LL})$ or (x_a does not exist), go to **D7**.

D14. Evaluate $y_a = g[X(\sqrt{x_a})Y(\sqrt{x_a})]$.

If $y_a > x_a$, then $(x_L, y_L) = (x_a, y_a)$, and go to **F1**.

D15. If $(x_{LL}=0)$ and $(x_a > y_a > 0)$, then update the following:

$(x_U, y_U) = (x_a, y_a)$, $(x_3, y_3) = (x_2, y_2)$, $(x_2, y_2) = (x_1, y_1)$, $(x_1, y_1) = (x_a, y_a)$,

and go to **D7**.

D16. If $(x_{LL} \neq 0)$ and $(x_a > y_a > 0)$, then update the following:

$(x_U, y_U) = (x_a, y_a)$, $(x_3, y_3) = (x_2, y_2)$, $(x_2, y_2) = (x_a, y_a)$,

and go to **D7**.

D17. $(x_{LL}, y_{LL}) = (x_a, y_a)$, $(x_1, y_1) = (x_a, y_a)$, and go to **D7**.

D18. Let $x_B = (x_{LL} + x_U)/2$ and evaluate $y_B = g[X(\sqrt{x_B})Y(\sqrt{x_B})]$.

If $y_B > x_B$, then $(x_L, y_L) = (x_B, y_B)$, and go to **F1**.

D19. If $x_B > y_B > 0$, then update the following:

$(x_U, y_U) = (x_B, y_B)$, $(x_3, y_3) = (x_2, y_2)$, $(x_2, y_2) = (x_B, y_B)$,

and go to **D7**.

D20. $(x_{LL}, y_{LL}) = (x_B, y_B)$, $(x_1, y_1) = (x_B, y_B)$, and go to **D7**.

Computation of α, the optimum for Case (1):

E1. Compute α, the infimum of γ such that $H_\infty(\gamma)$ and $J_\infty(\gamma)$ have no $j\omega$-axis eigenvalues, by using Theorem 2.3-1 and the algorithm in [8,9].

E2. If $X(\alpha)$ and $Y(\alpha)$ are positive semidefinite, and $g[X(\alpha)Y(\alpha)] < \alpha^2$, the optimal H^∞ norm is α. **Stop.**

E3. Let $(x_{LL}, y_{LL}) = (\alpha^2, g[X(\alpha)Y(\alpha)])$ and go to **D18**.

Search for the optimum x_o at which $y_o = x_o$:

F1. Now we have the trio, (x_L, y_L), (x_U, y_U), and (x_E, y_E), where (x_E, y_E) is one of the (x_i, y_i)'s other than (x_L, y_L) and (x_U, y_U) which has the largest $|y_i|$.

F2. Let $\hat{x}_L = x_L$, and $\hat{x}_U = x_U$.

F3. If $|\hat{x}_U - \hat{x}_L| < \varepsilon_1$, the optimum $x_o = \hat{x}_U$. **Stop.**

F4. Let $(x_1, y_1) = (x_L, y_L)$, $(x_2, y_2) = (x_U, y_U)$, and $(x_3, y_3) = (x_E, y_E)$. Compute a, b, and c from (2.3-12b) - (2.3-12d).

F5. $x_G = \frac{1}{2}\left[\, a + b + \sqrt{(a+b)^2 - 4(ab-c)}\,\right]$, $y_G = \lambda_1[X(\sqrt{x_G})Y(\sqrt{x_G})]$.

F6. If $\hat{x}_L \le x_G \le \hat{x}_U$, Go to **F8**.

F7. $x_G = (\hat{x}_L + \hat{x}_U)/2$, $y_G = \lambda_1[X(\sqrt{x_G})Y(\sqrt{x_G})]$.

F8. If $y_G < x_G$, Go to **F10**.

F9. $\hat{x}_L = \max\{\theta_5, \hat{x}_L\}$, $\hat{x}_U = \min\{\theta_6, \hat{x}_U\}$. Go to **F11**.

F10. $\hat{x}_L = \max\{\theta_6, \hat{x}_L\}$, $\hat{x}_U = \min\{\theta_5, \hat{x}_U\}$.

F11. If $|\hat{x}_U - \hat{x}_L| > \varepsilon_1$, go to **F16**.

F12. If $0 < \hat{x}_U - \rho[X(\sqrt{\hat{x}_U})Y(\sqrt{\hat{x}_U})] < \varepsilon_2$, then the optimum $x_o = \hat{x}_U$. **Stop.**

F13. Let $x_G = (x_L + x_U)/2$, $y_G = \lambda_1[X(\sqrt{x_G})Y(\sqrt{x_G})]$.

F14. If $y_G > x_G$, then let $(x_E, y_E) = (x_L, y_L)$, $(x_L, y_L) = (x_G, y_G)$ and go to **F2**.

F15. Let $(x_E, y_E) = (x_U, y_U)$, $(x_U, y_U) = (x_G, y_G)$ and go to **F2**.

F16. If $y_G > x_G$, then let $(x_E, y_E) = (x_L, y_L)$, $(x_L, y_L) = (x_G, y_G)$ and go to **F4**.

F17. Let $(x_E, y_E) = (x_U, y_U)$, $(x_U, y_U) = (x_G, y_G)$ and go to **F4**.

2.5. Construction of Optimal Controllers

From the Glover and Doyle's formulas in Theorem 2.2-1, a suboptimal H^∞ controller can be easily constructed. However, as γ approaches to the optimum we will encounter the inversion of a singular matrix except case (1) which seldom occurs. To eliminate the numerical difficulty, Glover and Limebeer et. al. [10] rederived the optimal controller formulas in a descriptor form (or generalized state-space representation).

Construction of Optimal H^∞ Controllers for Case (3):

The Glover and Doyle's formulas in (2.2-3a) - (2.2-3h) can also be written in a descriptor form after slight rearrangement. First of all, we consider case (3) which occurs much more often than the other two cases. When γ reaches the optimum, γ_o, which satisfies $\gamma_o^2 = \rho[X(\gamma_o)Y(\gamma_o)]$, the matrix Z in (2.2-3f) will become infinity since the matrix $I - \gamma_o^2 Y(\gamma_o)X(\gamma_o)$ is singular. If we try to apply the formulas (2.2-3a) - (2.2-3h) directly to construct an optimal H^∞ controller, a numerical difficulty will arise in the implementation of the $\hat{A}$ and $\hat{C}$ matrices. We will rearrange these formulas such that an optimal H^∞ controller can be constructed without any numerical difficulty.

The dual system of the realization in (2.2-3a) can be easily rewritten in a descriptor form. The state equation (generalized state equation) of the descriptor representation can be split into two set of equations: one involves first derivative of some state variables and the other is just an algebraic equation. The state variables which have no derivative in the equations can be eliminated and then we have a lower order state space representation for the dual system. The dual of the dual system is identical to the original and therefore we have an optimal H^∞ controller as follows:

$$K_{opt}(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array} \right] \tag{2.5-1}$$

where

$$A_c = [\, V_1^T A_D U_1 - V_1^T A_D U_2 (V_2^T A_D U_2)^\dagger V_2^T A_D U_1 \,] \, \Sigma_1^{-1} \tag{2.5-2a}$$

$$B_c = V_1^T B_D - V_1^T A_D U_2 (V_2^T A_D U_2)^\dagger V_2^T B_D \tag{2.5-2b}$$

$$C_c = [\, C_D U_1 - C_D U_2 (V_2^T A_D U_2)^\dagger V_2^T A_D U_1 \,] \, \Sigma_1^{-1} \tag{2.5-2c}$$

$$D_c = \hat{D} - C_D U_2 (V_2^T A_D U_2)^\dagger V_2^T B_D \tag{2.5-2d}$$

and

$$B_D = -H_2 + (B_2 + H_{12})\hat{D} \tag{2.5-3a}$$

$$C_D = F_2 - \hat{D}(C_2 + F_{12}) \tag{2.5-3b}$$

$$A_D = (B_2 + H_{12}) C_D + (A + HC) E_D \tag{2.5-3c}$$

$$E_D = I - \gamma_o^2 X(\gamma_o)Y(\gamma_o) \tag{2.5-3d}$$

Σ_1, U_1, U_2, V_1, and V_2 are obtained from the singular value decomposition of E_D, i.e.,

$$E_D = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{2.5-4}$$

Construction of Optimal H∞ Controllers for Case (2):

In case (2), either Y or X is zero and then the optimum occurs at the smallest γ such that $X(\gamma)$ or $Y(\gamma)$ is positive semidefinite. In the following, with loss of generality we assume $Y = 0$. The optimum occurs at $\gamma = \beta_X$, i.e., the smallest γ such that $X(\gamma)$ is positive semidefinite. If we try to apply the formulas (2.2-3a) - (2.2-3h) directly to construct an optimal H∞ controller, a numerical difficulty will arise since $X(\beta_X)$ is infinity and so are the matrices $\hat{A}$ and $\hat{C}$.

Let $\mathcal{X}_-[H_\infty(\beta_X)]$ be an n-dimensional spectral space corresponding to eigenvalues in Re s < 0. Partitioning the matrix constructed from the basis vectors of $\mathcal{X}_-[H_\infty(\beta_X)]$, we have

$$\mathcal{X}_-[H_\infty(\beta_X)] = \text{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{2.5-5}$$

where X_1, $X_2 \in \mathbb{R}^{n \times n}$. X_1 is singular and therefore $X = X_2 X_1^{-1}$ is infinity. To eliminate the numerical difficulty, we do the singular value decomposition for X_1^T as follows,

$$X_1^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{2.5-6}$$

25

Employing the same techniques, the duality and the elimination of unnecessary state variables, which we just used for case (3), an optimal H^∞ controller can be constructed as follows without any numerical difficulty.

$$K_{opt}(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{2.5-7}$$

where

$$A_c = [\, V_1^T A_D U_1 - V_1^T A_D U_2 (V_2^T A_D U_2)^\dagger V_2^T A_D U_1 \,] \, \Sigma_1^{-1} \tag{2.5-8a}$$

$$B_c = V_1^T B_D - V_1^T A_D U_2 (V_2^T A_D U_2)^\dagger V_2^T B_D \tag{2.5-8b}$$

$$C_c = [\, C_D U_1 - C_D U_2 (V_2^T A_D U_2)^\dagger V_2^T A_D U_1 \,] \, \Sigma_1^{-1} \tag{2.5-8c}$$

$$D_c = \hat{D} - C_D U_2 (V_2^T A_D U_2)^\dagger V_2^T B_D \tag{2.5-8d}$$

and

$$B_D = -H_2 + (B_2 + H_{12})\hat{D} \tag{2.5-9a}$$

$$C_D = \tilde{F}_2 - \hat{D}(C_2 X_1 + \tilde{F}_{12}) \tag{2.5-9b}$$

$$A_D = (B_2 + H_{12})\, C_D + (A + HC)\, X_1 \tag{2.5-9c}$$

Construction of Optimal H^∞ Controllers for Case (1):

For case (1), the formulas (2.2-3a) - (2.2-3h) can be used to construct an optimal H^∞ controller without any numerical difficulty.

2.6. Illustrative Examples

Four illustrative examples will be included in this section. Example 1 is a simple four-block H^∞ optimization problem which belongs to case (3). Example 2 is a two-block H^∞ optimization problem which also belongs to case (3). Example 3 is a four-block H^∞ problem which belongs to case (1). Example 4 is a two-block H^∞ problem which is used to illustrate case (2).

Example 1:

The following is a simple four-block H^∞ optimization problem which is used to illustrate the proposed iterative algorithm of computing the optimal H^∞ norm. A realization of the plant G(s) is given by

$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Two passes are used to compute the optimum. The first pass is to find a lower and an upper bounds which bracket the optimum and the second pass is to search for the optimum. We arbitrarily choose three large numbers $x_1=100$, $x_2=1000$, and $x_3=10000$, and evaluate $y_i = g[X(\sqrt{x_i})Y(\sqrt{x_i})]$, i=1,2,3, by (3-14). That is, we start with the trio

$$(x_1,y_1) = (100, 2.177485978109461e+01)$$
$$(x_2,y_2) = (1000, 2.161627203254472e+01)$$
$$(x_3,y_3) = (10000, 2.160056781642087e+01)$$

Since $x_1>y_1>0$, x_1 qualifies as an upper bound x_U and thus

$$(x_U,y_U) = (100, 2.177485978109461e+01)$$

The hyperbola which interpolates the trio can be easily determined by (3-12). The abscissa of the intersection point of the hyperbola with the line y=x is

$x_G=2.241226047320986e+01$. From (3-14), we evaluate $y_G = g[X(\sqrt{x_G})Y(\sqrt{x_G})]$ and then we have,

$$(x_G,y_G) = (2.241226047320986e+01, 2.241227598351088e+01).$$

After just **one** iteration of the first pass, we have

$$(x_L,y_L) = (x_G,y_G) = (2.241226047320986e+01, 2.241227598351088e+01)$$

since $x_G<y_G$. Now, we have a new trio which include a lower bound point (x_L,y_L), an upper bound point (x_U,y_U), and an additional point (x_E,y_E) which can be chosen from the previous trio as,

$$(x_E,y_E) = (1000, 2.161627203254472e+01).$$

After just **one** iteration of the second pass, we find that the optimum x_o satisfies the following inequality,

$$\hat{x}_L = 2.24122754162167e+01 \leq x_o \leq 2.24122754162171e+01 = \hat{x}_U$$

Thus, the optimum x_o is approximately equal to $\hat{x}_U$ with accuracy up to 14 digits. For double checking, we evaluate

$$\rho[X(\sqrt{\hat{x}_U})Y(\sqrt{\hat{x}_U})] = 2.241227541621705e+01$$

which is a little bit less than $\hat{x}_U$. Therefore, the optimum $x_o = 2.24122754162171e+01$. That is, the optimal H^∞ norm is $\gamma_o = \sqrt{x_o} = 4.734160476390413$.

With $\gamma = 4.7341604768$ which is very close to the optimum, from (2.2-3) and partial fraction expansion we have a suboptimal controller as follows,

$$K(s) = \frac{-8.9043458823e+00}{s+8.7542343140e-01} + \frac{1.0388615552e+11}{s+2.1943944664e+10}$$

The second term of the above expression is a wide band low-pass filter which can be approximated by a direct feed-through term when γ approaches to the optimum. With $\gamma=\gamma_o = 4.734160476390413$, from (2.5-7) we have an optimal controller as follows

$$K_{opt}(s) = \left[\begin{array}{c|c} -0.87541981354051831 & -0.1382887384996682 \\ \hline 4.451128374445501 & -4.734160476390407 \end{array}\right]$$

which is first-order, one order less than that of the plant.

Example 2:

The following is a two-block H^∞ optimization problem which was studied in [12]. A realization of the plant G(s) is given by

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc|c|c} -.1 & 0 & .099 & -.18 \\ 0 & 1 & 0 & 1 \\ \hline 1 & .2 & 0 & .01 \\ 0 & 2.2 & .2 & 0 \\ \hline 0 & 2 & 1 & 0 \end{array}\right]$$

In the first pass, we will find a lower and an upper bounds which bracket the optimum. We arbitrarily choose three large numbers $x_1=10$, $x_2=100$, and $x_3=1000$, and evaluate $y_i = g[X(\sqrt{x_i})Y(\sqrt{x_i})]$, i=1,2,3, by (2.3-14). That is, we start with the trio

$(x_1,y_1) = (10,9.570601437485001e-04)$
$(x_2,y_2) = (100,8.398403602350264e-04)$
$(x_3,y_3) = (1000,8.295556971685659e-04)$

Since $x_1 > y_1 > 0$, x_1 qualifies as an upper bound x_U and thus

$$(x_U,y_U) = (10,9.570601437485001e-04)$$

The hyperbola which interpolates the trio can be easily determined by (2.3-12). The abscissa of the intersection point of the hyperbola with the line $y = x$ is $x_G = 1.237935857327897e+00$ and from (2.3-12b) we have $x_a = a = 1.237024692928173e+00$. From (2.3-14), y_G and y_a can be computed and then we have

$$(x_G,y_G) = (1.237935857327897e+00,4.253228336726583e-02)$$
$$(x_a,y_a) = (1.237024692928173e+00,4.393981296506005e-02)$$

Now, (x_U,y_U) and the trio are updated as follows:

$$(x_U,y_U) = (x_a,y_a) = (1.237024692928173e+00,4.393981296506005e-02)$$
$$(x_3,y_3) = (x_1,y_1) = (10,9.570601437485001e-04)$$
$$(x_2,y_2) = (x_G,y_G) = (1.237935857327897e+00,4.253228336726583e-02)$$
$$(x_1,y_1) = (x_a,y_a) = (1.237024692928173e+00,4.393981296506005e-02)$$

After the **second** iteration of the first pass, we have

$$(x_G,y_G) = (1.210987163764649e+00,1.182214405325450e+00)$$
$$(x_a,y_a) = (1.210025237996426e+00,4.621151922644384e+01)$$

Since $y_a > x_a$ and $x_G > y_G$, we can update (x_U,y_U) and have (x_L,y_L) as follows:

$$(x_U,y_U) = (x_G,y_G) = (1.210987163764649e+00,1.182214405325450e+00)$$
$$(x_L,y_L) = (x_a,y_a) = (1.210025237996426e+00,4.621151922644384e+01)$$

Now, we have a new trio (x_U,y_U), (x_L,y_L), and (x_E,y_E) to start with in the second pass where

$$(x_E,y_E) = (1.237024692928173e+00,4.393981296506005e-02).$$

After just **one** iteration of the second pass, we find that the optimum x_o satisfies the following inequality,

$$\hat{x}_L = 1.210963712471e+00 \leq x_o \leq 1.210963712497e+00 = \hat{x}_U$$

Thus, the optimum x_o is approximately equal to $\hat{x}_U$ with accuracy up to 12 digits.

Example 3:

The following is a four-block H∞ optimization problem which realization is given by

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc|cc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 0 \end{array}\right]$$

This realization is almost the same as that of Example 1. The only difference is in the 22 position of the A matrix. In the first pass, we try to find a lower and an upper bounds which bracket the optimum x_o. We start with the trio

$$(x_1, y_1) = (100, 1.879778879610986e\text{-}01)$$
$$(x_2, y_2) = (1000, 1.873943983436236e\text{-}01)$$
$$(x_3, y_3) = (10000, 1.873362964186965e\text{-}01)$$

The abscissa of the intersection point of the hyperbola interpolating these three points and the line y=x is $x_G = 5.884887925134553e\text{-}01$. $X(x_G)$ does not exist and neither is y_G. By Theorem 2.3-1 and the algorithm in [8,9], we compute α_X, the infimum of γ such that $H_\infty(\gamma)$ has no jω-axis eigenvalues, as

$$\alpha_X = 0.89442719099992.$$

It is easy to check that $X(\alpha_X)$ and $Y(\alpha_X)$ are both positive semidefinite and

$$0.691300276370028 = [\rho[X(\alpha_X)Y(\alpha_X)]]^{1/2} < \alpha_X = 0.89442719099992.$$

Therefore, $\alpha_X = 0.89442719099992$ is the optimal H∞ norm.

Example 4:

The following is a two-block H∞ optimization problem which realization is given by

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc|c|c} -1 & 0 & 1 & 30 \\ 0 & -2 & 0 & 1 \\ \hline 1 & 0 & 0 & 5 \\ 0 & 10 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \end{array}\right]$$

30

The Riccati solution Y is zero for all γ. The optimum will either occur at β_X or α_X. Two passes are used to search for β_X. In the first pass, we will try to use hyperbolic interpolations to find a lower bound. If it does not work, α_X will be computed. It is easy to check if α_X is the optimum. If not, α_X can be used as a lower bound for β_X. Once a lower and an upper bounds are available, we go to the second pass to search for β_X. In the beginning, we arbitrarily choose three large numbers $x_1=100$, $x_2=110$, and $x_3=120$, and evaluate $y_i = f[X(\sqrt{x_i})]$, i=1,2,3, by (2.3-11). That is, we start with the trio

$$(x_1,y_1) = (100, 2.320586064570914e+01)$$
$$(x_2,y_2) = (110, 2.314605879013854e+01)$$
$$(x_3,y_3) = (120, 2.309652871632076e+01)$$

Since $y_1>0$, x_1 qualifies as an upper bound x_U and thus

$$(x_U,y_U) = (100, 2.320586064570914e+01)$$

The hyperbola which interpolates the trio gives

$$x_a = a = 3.560891362913274e+00$$

$X(\sqrt{x_a})$ does not exist and neither is y_a. Now, we compute $\alpha_X = 5.0000000001$. $X(\alpha_X)$ is not positive semidefinite, and so α_X is not the optimum. However, we can use α_X^2 as a lower bound. Let $x_L = \alpha_X^2$ and evaluate $y_L = f[X(\sqrt{x_L})]$ as follows:

$$(x_L,y_L) = (5.000000001, -5.004746766152451e+00)$$

Now, we have a new trio (x_U,y_U), (x_L,y_L), and (x_E,y_E) to start with in the second pass where

$$(x_E,y_E) = (x_2,y_2) = (110, 2.314605879013854e+01).$$

After **eight** iterations of the second pass, we have

$$(x_a,y_a) = (2.633024605035133e+01, 2.530496016219245e+01)$$
$$(x_a,y_a) = (2.509367259661183e+01, 2.551488425827425e+01)$$
$$(x_a,y_a) = (2.500069275875799e+01, -2.360866054339716e+01)$$
$$(x_a,y_a) = (2.500111983723194e+01, -1.353571348053681e+03)$$
$$(x_a,y_a) = (2.500113497987190e+01, 1.906244532943883e+03)$$
$$(x_a,y_a) = (2.500112857428468e+01, -1.238558367307747e+05)$$
$$(x_a,y_a) = (2.500112867114679e+01, 8.504081686717725e+09)$$
$$(x_a,y_a) = (2.500112867114538e+01, 3.624506009602072e+13)$$

We can see that the convergence rate increases rapidly when the trio is in the neighborhood of the optimum. The optimal H^∞ norm is $\beta_X = \sqrt{x_a} = 5.000112865840668$.

3. CONCLUSION AND WORK FOR FUTURE RESEARCH

3.1 Conclusion

Glover and Doyle's Riccati equation method is a breakthrough in the solution of H^∞ optimization problems. However, the algorithms available for the computation of the optimal H^∞ norm are slow and inefficient. In this report, a fast iterative algorithm based on hyperbolic interpolations and a conjecture of convexity was proposed to compute the optimal H^∞ norm for four-block, two-block, and one-block H^∞ optimization problems. This algorithm is complete which can handle all possible cases and its convergence is quadratic when the trio to determine the hyperbola is close to the optimum. The numerical difficulty arising in applying Glove and Doyle's formulas to construct an optimal H^∞ controller was also eliminated.

The H^∞ tools, the efficient algorithm for computing the optimal H^∞ norm and the reliable construction of optimal H^∞ controllers, which we have just developed in the previous research period will be very useful in our future research.

3.2 Work for Further Research

3.2.1 M-Δ Structure and Robust Stability Analysis for Structured Uncertainties

M-Δ structure is a rearrangement of a perturbed system where M(s) is the nominal system and Δ is a block diagonal matrix which consists of all the perturbations. M-Δ structure is essential in the SSV analysis and design techniques. Although it is always possible to pull out all uncertain parts from a perturbed system and form an M-Δ structure, very little about how to obtain this structure has been addressed in the literature. A systematic procedure for constructing M-Δ structure is proposed.

A minimal M-Δ structure means that the dimension of Δ (or M) of the M-Δ structure is minimal. We can construct an M-Δ structure for a given perturbed system, but the dimension of the structure may be unnecessarily large. A nonminimal structure will cause unnecessary complexity in computation and therefore a minimal M-Δ structure is essential in the computation of the SSV. Nevertheless, none about the minimality of M-Δ

structure has been addressed in the literature. A progress in the construction of a minimal M-Δ structure will greatly simplify robust stability analysis for structured uncertainties.

3.2.2 Design of Robust Controllers via H∞ Optimization

The next important issue is how to incorporate robust stability requirement into the picture of the optimal controller design. Doyle et. al. [11] included a robust stability measure into a cost function together with a robust performance measure and formulated a structured singular value minimization problem. The concept of the SSV ball described in the previous subsection can be also used to formulate an SSV optimization problem. It is a realistic and nonconservative formulation. Nevertheless, no easy solution seems to be available for this problem in the near future. Direct incorporating the nonconservative robust stability requirement into the controller design may be difficult at present time. However, we will keep this problem in mind and consider it as one of our long-term research problems. A feasible indirect design approach is proposed.

In the proposed approach, we will use the H∞ norm of the complementary sensitivity function as a measure of robust stability and will formulate an H∞ optimization problem which minimizes the maximum error energy subject to a robust stability constraint. The error reduction and the robust stability can be traded off by choosing the weighting matrices in the cost function. Initially, a weighting matrix is chosen and the corresponding H∞ optimization problem is solved to obtain an optimal controller. The weighting matrices are modified iteratively until the robust stability constraint is just satisfied. The way the weighting matrices affect the trade-off is only partially understood. In addition to the magnitude of the weighting matrix, the structure of the weighting matrix is also an important factor in the trade-off. We will investigate how the weighting matrices affect the trade-off. We will also develop an iterative updating procedure for the weighting matrices by which a robust controller can be designed such that the closed-loop system is robustly stable and the maximum of error energy is minimized.

3.2.3 Design of Robust Controllers via Mixed H²/H∞ Optimization

The optimal H∞ controller which minimizes the H∞ optimization problem usually has wide bandwidth and leads to a poor H² performance. In [5], we found that a little bit sacrifice of the H∞ norm will greatly reduce the bandwidth of the controller and improve the H² performance tremendously. It is practical to formulate a robust control problem as a that of minimizing an H² cost function subject to an H∞ bound. The H² cost function is

33

equivalent to the well known LQG cost function and the H^∞ bound can take care of robust stability and robust error reduction.

The only approach available in the literature for solving the mixed H^2/H^∞ optimization problem was proposed by Bernstein and Haddad [13]. A set of coupled Riccati equations are involved and only in a special case these coupled Riccati equations can be solved. Furthermore, the numerical algorithm for these equations is a homotopy algorithm which is not efficient. A research in finding more efficient approach for the mixed H^2/H^∞ optimization problem is proposed.

3.2.4 Controllability and Observability of Perturbed Plant

Although a number of DOC (Degree of Controllability) and DOO (Degree of Observability) measures have been defined to quantify the controllability and observability of a system, none of them have been used as part of the design of robust controller.

In this study, we shall address this question. Specifically, first we shall determine how sensitivity are the available DOCs and DOOs to the structured uncertainties and unmodelled dynamics. Then we shall find a maximal domain of perturbation $\underline{\delta}$ which guarantees a specified level of Degree of Controllability (DOC) and another domain of perturbation which guarantees a specified level of Degree of Observability (DOO). These domains give the controllability and observability margins which take into account the uncertainties in the system model. Once these margins have been defined, then they will be integrated with the stability margins to develop a methodology for the design of robust controllers.

REFERENCES

[1] B. C. Chang, X. P. Li, H. H. Yeh, and S. S. Banda, "Iterative Computation of the Optimal H∞ Norm By Using Two-Riccati-Equation Method", Proceedings of IEEE Conference on Decision and Control, Dec. 1990.

[2] B. C. Chang, X. P. Li, S. S. Banda, and H. H. Yeh, "Design of an H∞ Optimal Controller By Using DGKF's State-space Formulas", Proceedings of IEEE Conference on Decision and Control, Dec. 1990.

[3] B. C. Chang, X. P. Li, S. S. Banda, and H. H. Yeh, "Computation of the Optimal H^∞ Norm by H^∞ Riccati Equations and Hyperbolic Interpolations", Submitted to IEEE Transactions on Automatic Control.

[4]   H. H. Yeh, S. S. Banda, and B. C. Chang, "Necessary and Sufficient Conditions for Mixed H2 and H∞ Optimal Control", Proceedings of IEEE Conference on Decision and Control, Dec. 1990.

[5]   C. C. Belcastro, B. C. Chang, R. Zong, and R. Fischl, "Design of a Suboptimal H∞ Controller with H² and Bandwidth Constraints", Submitted to the 1991 American Control Conference.

[6]   J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis, "State-space Solutions to Standard H_2 and $H_∞$ Control Problems", IEEE Transactions on Automatic Control, Vol. AC-34, No. 8, pp.831-847, Aug. 1989.

[7]   K. Glover and J.C. Doyle, "State-space Formulae for All Stabilizing Controllers That Satisfy an $H_∞$-norm Bound and Relations to Risk Sensitivity", Systems and Control Letters, Vol. 11, pp.167-172, 1988.

[8]   B.C. Chang, X.P. Li, H.H. Yeh, and S.S. Banda, "Computation of the H∞ Norm of a Transfer Function", Proceedings of the 1990 American Control Conference, May 1990.

[9]   S. Boyd and V. Balakrishnan, "A Regularity Result for the Singular Values of a Transfer Matrix Quadratically Convergent Algorithm for Computing its L∞ Norm", Proceedings of the 28th Conference on Decision and Control, Dec. 1989.

[10] K. Glover, D.J.N. Limebeer, J.C. Doyle, E.M. Kasenally, and M.G. Safonov, "A Characterization of All Solutions to the Four Block General Distance Problem", Preprint, 1989.

[11] Doyle et. al., Lectures Notes, ONR/Honeywell Workshop on Advances in Multivariable Control, Minneapolis, Minnesota, Oct. 1984.

[12] B. C. Chang, S. S. Banda, and T. E. McQuade, "Fast Iterative Computation of Optimal Two-block H∞ Norm", IEEE Transactions on Automatic Control, Vol. AC-34, No.7, pp.738-743, July 1989.

[13] D.S. Bernstein and W.M. Haddad, "LQG Control with an H∞ Performance Bound: A Riccati Equation Approach", IEEE Transactions on Automatic Control, Vol. AC-34, No.3, pp.831-847, Mar. 1989.