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Physical Insight into the Simultaneous Optimization of Structure and Control

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Abstract

Recent trends in spacecraft design which yield larger structures with more stringent performance requirements place many flexible modes of the structure within the bandwidth of active controllers. The resulting complications to the spacecraft design make it highly desirable to understand the impact of structural changes on an optimally controlled structure. This work uses low order structural models with optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers to develop some basic insight into this problem. This insight concentrates on several basic approaches to improving controlled performance and how these approaches interact in determining the optimal designs. A numerical example is presented to demonstrate how this insight can be generalized to more complex problems.

1 Introduction

Traditionally, control and structure subsystems in spacecraft have been designed separately. This was an efficient approach when the required bandwidth of rigid body controllers was well below the frequencies of the flexible modes of the structure. Recently however, increasing size in spacecraft structural designs has resulted in ever decreasing frequencies for flexible modes, while more stringent pointing and alignment requirements have resulted in control designs of increasing bandwidth. The net result is that several to many flexible modes of a spacecraft structure can lie within the bandwidth of onboard controllers. The strong interaction of structure and control that arises from this makes simultaneous design of these two subsystems highly desirable.

One approach to this problem can be called numerical control/structure optimization [1]. In this method, one first selects a basic structural and control design (*e.g.* a ten bay truss with full state feedback). Several structural parameters (*e.g.* truss member thicknesses) and control gains are designated as design variables and a dynamic performance metric is formulated. A numerical algorithm is then employed to search over the space of allowable designs for a particular one which optimizes a dynamic performance metric with a suitable constraint on the overall mass or size of the structure.

The Achilles Heel of this approach lies in the lack of physical insight yielded by the numerical solution. This insight is crucial in designing any controlled structure, including one which will ultimately be designed numerically. A good understanding of how changes in the structure influence controlled performance is essential in formulating the optimization problem to be solved numerically. Physical insight will hopefully lead to a wise, rather than arbitrary, selection of the design variables. Otherwise selection of design variables can place design objectives at odds and thereby yield a needlessly compromised solution.

One can envision four basic ways that a change to the structure of a spacecraft can alter its controlled performance. First, it can alter the way that disturbances influence the dynamics of the structure (disturbability). Second, it can affect the influence of control actuators (controllability). Third, it can change the way in which the dynamics of the structure appear in the performance metric (observability). And finally, it can change the frequencies and damping ratios of the structure. These different qualities of the structure; controllability, disturbability, observability, frequency and damping; are defined here as the structure's *modal properties.*

It should be noted that there is also a fifth way that one can alter the performance of a controlled structure, and that concerns changes which affect the robustness of the controllers. However, this is beyond the scope of this work. The next section describes the basic mathematical controlled structure problem and formulates a low order problem (typical section) useful in studying how a controlled structure's modal properties interact to produce dynamic performance.

2 Problem Description and Typical Section

A general linear structure can be described by the equation of motion:

$$\begin{aligned} M(\alpha)\ddot{r} + D(\alpha)\dot{r} + K(\alpha)r &= F(\alpha)u + G(\alpha)v \\ y &= N(\alpha)r \end{aligned} \tag{1}$$

where r is a vector of physical displacements on the structure, y is a vector of displacements (either physical or modal) to be controlled, u is a vector of control forces, and v is a vector of disturbance forces. The vector α is an array of real values which represent quantities in the structure which can be varied by the engineer in the design process. For example, the elements of α could represent the diameters of members in a truss structure. The goal of control/structure optimization is to find a suitable combination of structural parameters and control force which minimizes the performance metric.

$$(\alpha^\star, u^\star(t)) = \arg \min_{u(t),\alpha\in\mathcal{D}} J(M(\alpha), D(\alpha), K(\alpha), F(\alpha), G(\alpha), N(\alpha), u(t)) \tag{2}$$

The set of allowable designs, $\mathcal{D}$, is usually constrained to contain only designs that are below some maximum value of size or mass either directly, or by including a component in the cost which penalizes these values. Furthermore, $\mathcal{D}$ is usually restricted to include only those designs which have physical meaning. For example, one might constrain design variables representing member thicknesses to lie above zero.

Equation 1 can be transformed into a modal state space representation where the state vector is modal displacement, q, and frequency normalized modal velocity, q':

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} q \\ q' \end{bmatrix} &= \begin{bmatrix} 0 & \omega \\ -\omega & -2\zeta\omega \end{bmatrix}\begin{bmatrix} q \\ q' \end{bmatrix} + \begin{bmatrix} 0 \\ \omega^{-1}\Phi^T F \end{bmatrix} u + \begin{bmatrix} 0 \\ \omega^{-1}\Phi^T G \end{bmatrix} v \\ y &= \begin{bmatrix} N\Phi & 0 \end{bmatrix}\begin{bmatrix} q \\ q' \end{bmatrix} \end{aligned} \tag{3}$$

The matrices ω and ζ are diagonal matrices containing the natural frequency and damping ratio of each mode and Φ is the modal transformation matrix:

$$r = \Phi q \qquad q' = \omega^{-1}\dot{q} \qquad \Phi^T M \Phi = I \qquad \Phi^T K \Phi = \omega^2 \qquad \Phi^T D \Phi \approx 2\zeta\omega \tag{4}$$

Altering certain matrices in Equation 3 corresponds exactly to altering the individual modal properties mentioned above. It is useful to make the following definitions.

$$\begin{array}{rll} \text{Frequency Matrix} & \omega & \\ \text{Damping Matrix} & \zeta & \\ \text{Controllability Matrix} & \mathcal{F} = \omega^{-1}\Phi^T F & \\ \text{Disturbability Matrix} & \mathcal{G}_v = \omega^{-1}\Phi^T G \quad \text{or} & \mathcal{G}_d = \omega^{-2}\Phi^T G \\ \text{Observability Matrix} & \mathcal{N} = N\Phi & \end{array} \tag{5}$$

Note the appearance of inverse frequency in the expression for controllability. It reflects the inherent resistance of higher frequency modes to impulsive control forces. Similarly, there are frequency terms

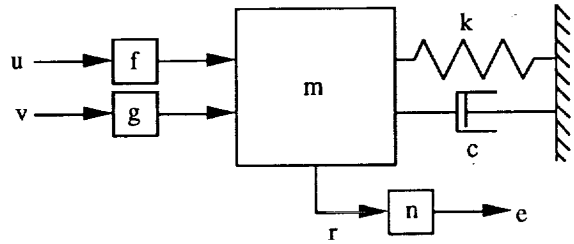

Figure 1: Typical section

in the disturbability expressions, except that there are two forms. The first corresponds to impulsive (velocity) disturbance forces, the second corresponds to static (displacement) disturbance forces. The choice of exponents in these expressions is clarified below.

Any alteration to the structure can be perceived as having two stages of effects. First, a change to the structure alters its modal properties. Second, the changes in these quantities alter the controlled performance of the system. This view of the problem is useful because it is relatively easy to understand how a change to the structure will influence its modal properties. If one could then understand the relative importance of these quantities in determining controlled performance, then one would have a good understanding of the entire problem. To develop this understanding, it is useful to study the system shown in Figure 1. This system is typical of a single mode of a flexible structure. For this reason this model and its associated controller are called a controlled structure typical section. The frequency, damping, controllability, disturbability, and observability matrices for the typical section are simply scalars:

$$\omega = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2m\omega} \qquad \mathcal{F} = \frac{f}{m\omega} \qquad \mathcal{G}_v = \frac{g}{m\omega} \qquad \mathcal{G}_d = \frac{g}{m\omega^2} \qquad \mathcal{N} = nm \tag{6}$$

and the equation of motion in state space form is:

$$\begin{aligned} \dot{x} &= Ax + Bu + Lv \\ y &= Cx \qquad C = \begin{bmatrix} \mathcal{N} & 0 \end{bmatrix} \\ x &= \begin{bmatrix} q \\ q' \end{bmatrix} \quad A = \begin{bmatrix} 0 & \omega \\ -\omega & -2\zeta\omega \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \mathcal{F} \end{bmatrix} \quad L = \begin{bmatrix} 0 \\ \mathcal{G} \end{bmatrix} \end{aligned} \tag{7}$$

The next sections show how the above parameters influence the controlled performance for the optimally controlled system when two different performance metrics are used.

3 $\mathcal{H}_2$ Problem

One of the most common dynamic performance metrics is the infinite horizon, $\mathcal{H}_2$ performance metric:

$$J = E\left[\int_0^\infty \left(y^T(t)y(t) + u^T(t)Ru(t)\right) dt\right] \tag{8}$$

where $E[\cdot]$ is the expectation operator and R is a symmetric, positive-definite control weighting matrix. The disturbance is specified as an expected value of the outer product of the initial state. In this case, it is assumed that the initial state comes about due to either static or impulsive forces, v, applied to the disturbance inputs of the system:

$$
\begin{aligned}
S = E\left[x(0)x^T(0)\right] &= \begin{bmatrix} E[q(0)q^T(0)] & 0 \\ 0 & 0 \end{bmatrix} && \text{displacement disturbance} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & E[q'(0)q'^T(0)] \end{bmatrix} && \text{velocity disturbance} \\
E\left[q(0)q^T(0)\right] &= \mathcal{G}_d V \mathcal{G}_d^T \quad E\left[q'(0)q'^T(0)\right] = \mathcal{G}_v V \mathcal{G}_v^T \quad E\left[vv^T\right] = V && (9)
\end{aligned}
$$

where V is the expected value of the outer product of the disturbance force v. The expectation operator appears in these expressions in order to allow for a statistical instead of a deterministic description of these forces. It should be noted that there is also a stochastic formulation of this performance metric. However, if V is taken to be the intensity of a Gaussian White Noise disturbance, then the resulting analysis would be identical to that for the velocity disturbance.

Notice that frequency does not appear in the above disturbance expressions. This was because any dependence of the disturbance properties of the system on frequency were absorbed into the definitions of the variables $\mathcal{G}_v$ and $\mathcal{G}_d$. This was the chief reason for including frequency in these definitions for the disturbance matrices, as it reflects the differing resistance of stiffer modes to static and impulsive forces.

A well known result of optimal control theory, is that for an optimally controlled system described by Equations 7 and 9 the $\mathcal{H}_2$ cost is [2]:

$$
J_{\text{opt}} = \text{tr}\{PS\} \tag{10}
$$

where P is the symmetric, positive-definite solution of:

$$
PA + A^TP + C^TC - PBR^{-1}B^TP = 0 \tag{11}
$$

For the typical section problem, these equations can be solved in closed form with the control penalty defined as $R = \rho^2$ [3]:

$$
\begin{aligned}
J_{\text{opt}} &= \frac{V\mathcal{G}_d^2\mathcal{N}^2}{\omega}\frac{1}{\beta^2}\left(\sqrt{\beta^2+1}\sqrt{4\zeta^2+2\sqrt{\beta^2+1}-2}-2\zeta\right) && \text{for a displacement disturbance} \\
J_{\text{opt}} &= \frac{V\mathcal{G}_v^2\mathcal{N}^2}{\omega}\frac{1}{\beta^2}\left(\sqrt{4\zeta^2+2\sqrt{\beta^2+1}-2}-2\zeta\right) && \text{for a velocity disturbance} \\
\text{where} \quad & \beta \equiv \left|\frac{\mathcal{N}\mathcal{F}}{\omega\rho}\right| && (12)
\end{aligned}
$$

Notice that the optimal costs for both types of disturbances have two parts. The first part (containing V, $\mathcal{G}$, $\mathcal{N}$, and ω) represents the transmissibility of the disturbance to the performance outputs. The second part, containing only non-dimensional terms for damping, ζ, and control influence, β, represents the improvement in performance gained through the application of passive damping and control. These parts of Equation 12 are too complicated to make any easy inferences about the relationship between modal properties and performance. However, the two non-dimensional values, ζ (damping) and β (control influence) completely determine the character of the equations, and it is illustrative to consider the asymptotic behavior of the performance with respect to these values.

Table 1 shows how Equation 12 behaves for limiting values of control influence, β and damping ζ. The top row of this table shows the behavior of the cost as the control forces become ineffective compared to the internal forces of the damping. In that case, the control terms ($\mathcal{F}$ and ρ) drop out,

Table 1: $\mathcal{H}_2$ Performance Costs

Control Type	Disturbance Type	
	Velocity	*Displacement*
Open Loop or Heavy Damping	$J_{\text{opt}} = \frac{V\mathcal{G}_v^2\mathcal{N}^2}{\omega}\frac{1}{4\zeta}$	$J_{\text{opt}} = \frac{V\mathcal{G}_d^2\mathcal{N}^2}{\omega}\left(\frac{1}{4\zeta} + \zeta\right)$
Expensive Control, Light Damping	$J_{\text{opt}} = \frac{V\mathcal{G}_v^2\mathcal{N}^2}{\omega}\beta^{-1} = \frac{V\mathcal{G}_v^2\rho\lvert\mathcal{N}\rvert}{\lvert\mathcal{F}\rvert}$	$J_{\text{opt}} = \frac{V\mathcal{G}_d^2\mathcal{N}^2}{\omega}\beta^{-1} = \frac{V\mathcal{G}_d^2\rho\lvert\mathcal{N}\rvert}{\lvert\mathcal{F}\rvert}$
Cheap Control, Light Damping	$J_{\text{opt}} = \frac{V\mathcal{G}_v^2\mathcal{N}^2}{\omega}\sqrt{2}\beta^{-3/2} = \frac{V\mathcal{G}_v^2\sqrt{2\lvert\mathcal{N}\rvert\rho^3\omega}}{\sqrt{\lvert\mathcal{F}\rvert^3}}$	$J_{\text{opt}} = \frac{V\mathcal{G}_d^2\mathcal{N}^2}{\omega}\sqrt{2}\beta^{-1/2} = \frac{V\mathcal{G}_d^2\sqrt{2\lvert\mathcal{N}\rvert^3\rho}}{\sqrt{\lvert\mathcal{F}\rvert\omega}}$

leaving an expression that represents the performance of the open loop system. The expressions for both disturbance cases are very similar with the exception of how damping influences performance. For damping levels less than 50%, increased damping improves performance for both disturbance types; however, for larger damping levels, increased damping actually inhibits performance for the displacement disturbance. This reflects the tendency of heavily damped systems to recover slowly from initial displacements.

The second and third rows of the table show the behavior of the performance when the damping becomes small relative to the control influence. In these cases, terms related to passive damping become insignificant and drop out of the cost. Physically, this means that the available control forces are greater than the internal damping forces and are therefore dominant in reducing cost. This will always be true in a system where it is necessary to use control to achieve a significant gain in performance. The implication is that damping should not be used to improve the performance of controlled modes directly. Instead, the damping design should concentrate on improving the performance of uncontrolled modes (modes which are either uncontrollable or outside the bandwidth of the controller) and adding robustness to the controlled modes.

The system shows two different types of behavior for low damping depending on the level of control effort. For low (or expensive) control effort, the costs for both disturbance cases are identical. This is because the low control is providing a small amount of active damping and the response of the system takes several cycles to attenuate (Figure 2). The cost is determined almost entirely by the response envelope and the phase difference in the responses imposed by the disturbance type has little effect. For higher control levels (expensive control), the response of the system attenuates in only one or two cycles and the phase difference becomes more important. Figure 3 depicts the response of a heavily controlled system for both types of disturbances. It is plain in the figure that the height of the peak response is a major factor in determining performance. However, the peak response for the system with the displacement disturbance is unaffected by control, while the peak response for the system with the velocity disturbance can be dramatically affected by control. This gives the costs fundamentally different behaviors at higher control levels for the different disturbance types. This effect appears in the last row of Table 1. In particular, the control influence parameter plays a large role in the cost for the velocity disturbance than it does in that of the displacement disturbance.

Using the information contained in this Table 1, and information about the sensitivity of frequency, damping, controllability, observability, and disturbability to the design parameters, α, one can infer which of these quantities should be adjusted and which should be ignored in designing a good controlled structure.

As an example, consider a case worked out by Milman *et.al.* [1] (Figure 4) This system consists of a three element, cantilevered, Bernoulli-Euler beam with a tip actuator and an impulsive disturbance also at the tip. The control in this case is optimal full state feedback and the design variables are the

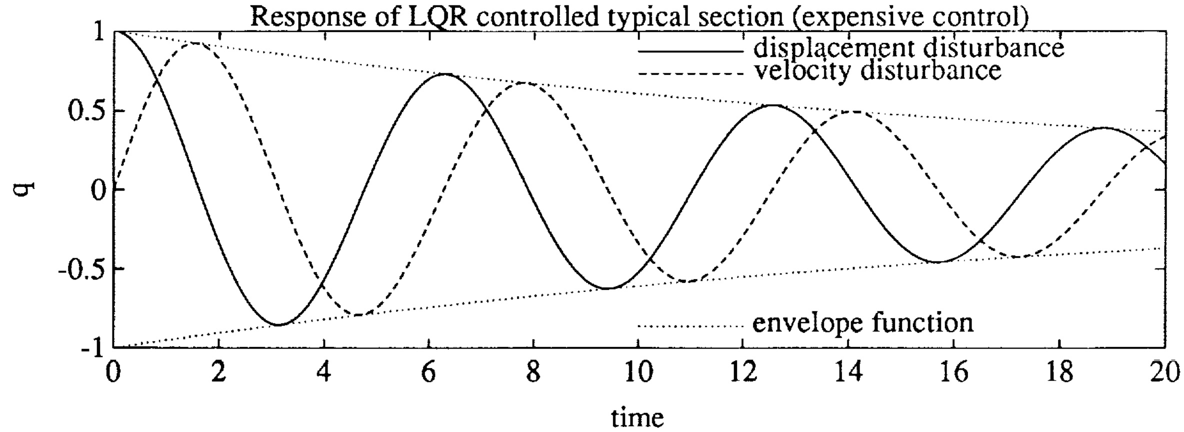

Figure 2: Response of single mass typical section for displacement and velocity disturbances: $\omega = 1$, $\zeta = 0$, $\mathcal{G}_d = \mathcal{G}_v = 1$, $\mathcal{N} = 1$, $\rho = 10$.

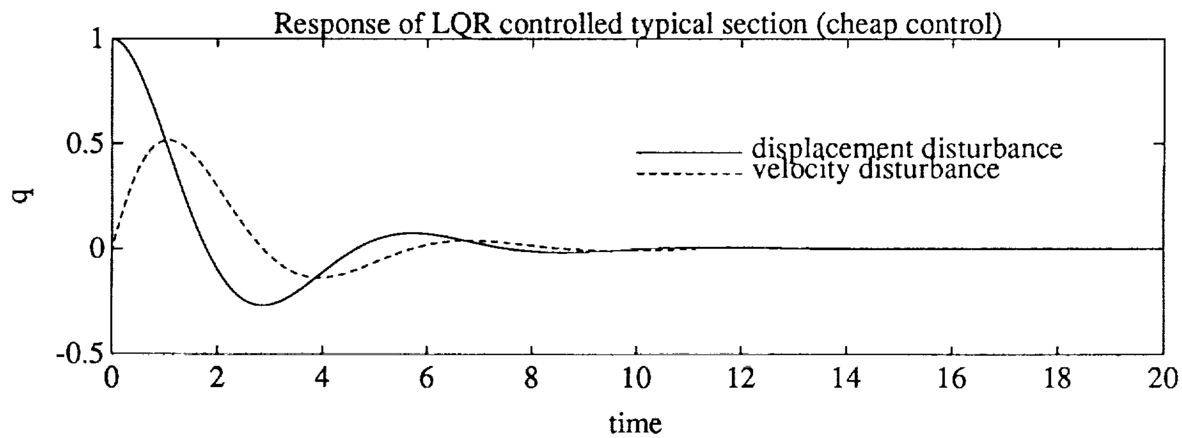

Figure 3: Response of single mass typical section for displacement and velocity disturbances: $\omega = 1$, $\zeta = 0$, $\mathcal{G}_d = \mathcal{G}_v = 1$, $\mathcal{N} = 1$, $\rho = 1$.

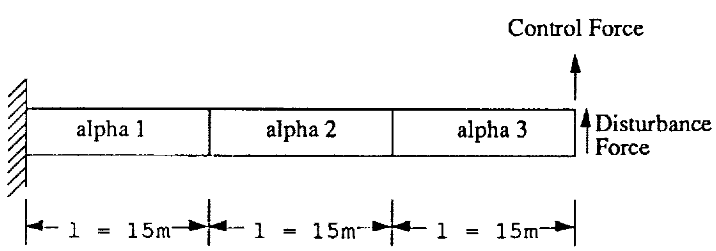

Figure 4: Beam example of Milman *et.al.*

Table 2: Results

λ	Optimal Design α_1	α_2	α_3
0.0000	.10	.10	.10
0.0001	.10	.10	1.59
0.0010	.10	.31	5.00
0.0100	0.38	1.18	15.72
0.1000	2.50	5.48	53.04
0.2000	4.57	8.93	81.48
0.3000	6.58	12.02	108.66
0.4000	8.65	15.10	137.97
0.5000	10.95	18.56	171.79
0.6000	13.77	22.66	213.96
0.7000	17.59	28.22	271.95
0.8000	24.06	37.32	364.60
0.9000	40.21	60.07	565.21
0.9200	47.49	70.68	644.24
0.9400	58.71	87.88	756.79
0.9600	79.40	122.91	937.62
0.9800	136.80	233.95	1307.86
0.9900	230.35	402.91	1779.25

thicknesses of the three elements. The metric optimized was a combination of a dynamic performance metric penalizing strain and kinetic energy and a metric penalizing total mass.

$$J = \lambda W(\alpha) + (1-\lambda)\int_0^\infty \left(r^T K r + \dot{r}^T M \dot{r} + \rho u^2\right) dt \tag{13}$$

The parameter, λ, is adjusted to obtain different levels of tradeoff between weight and performance. It can be shown that for any optimal design obtained for a given value of λ, the same design can be obtained by constraining $W(\alpha)$ and removing it from the performance metric. Hence Equation 13 is equivalent to Equation 2 with J set to the dynamic term and the set $\mathcal{D}$ constrained to include only designs which satisfy a maximum mass constraint.

An important feature of this problem is that the actuator and the disturbance are collocated. Examining the first column of Table 1 reveals that reducing the disturbability and increasing the controllability will both have favorable effects on the performance. However, the table also shows that when the sensitivity of the disturbability, $\mathcal{G}$, and controllability, $\mathcal{F}$, to the design variables is equal (as in this case), then greater gains can be attained by reducing disturbability at the expense of controllability for all levels of control. Table 2 confirms this suspicion. For almost all values of λ, the design obtained through optimization placed the bulk of the mass at the tip of the beam, where its inertia could help resist any forces applied to the tip.

This multimode problem was fairly easy to analyze due to its structure (disturbance and control collocated). However, it is important to understand the implications of the interaction of several modes. The next section explores this problem.

4 $\mathcal{H}_2$ Problem for a multi-mode system

In a system consisting of a single mode, the expressions in Table 1 will be exact; however, it should be expected that the interaction of several modes with a controller will produce somewhat different results. To facilitate this discussion, it is useful to look at the form of the gradient for an optimally controlled system. This gradient can be computed by first combining the cost in Equation 10 and the constraint in Equation 11 into a Lagrangian.

$$L = \text{tr}\{PS\} + \text{tr}\left\{H\left(PA + A^T P + C^T C - PBR^{-1}B^T P\right)\right\} \tag{14}$$

Setting derivatives of this expression with respect to the matrices P and H to zero recovers the constraint equation and an additional equation:

$$\frac{\partial L}{\partial P} = H\left(A - BR^{-1}B^T P\right)^T + \left(A - BR^{-1}B^T P\right) H + S = 0 \tag{15}$$

It can be seen that the Lagrange Multiplier Matrix, H, is the covariance of the state of the closed loop system.

The derivative of the cost with respect to a particular parameter, α_i, is the derivative of the Lagrangian with respect to that parameter;

$$\frac{\partial J}{\partial \alpha_i} = \frac{\partial L}{\partial \alpha_i} = \text{tr}\left\{P\frac{\partial S}{\partial \alpha_i} + H\left(P\frac{\partial A}{\partial \alpha_i} + \frac{\partial A^T}{\partial \alpha_i}P + \frac{\partial}{\partial \alpha_i}\left(C^T C\right) - P\frac{\partial}{\partial \alpha_i}\left(BR^{-1}B^T\right)P\right)\right\} \tag{16}$$

when P and H are given by Equations 11 and 15.

When A, B, C, and S are defined for a modal system description (Equation 3) then the different terms of the above equation describe changes in cost due to changes in individual modal properties. It is therefore useful to make the following definitions:

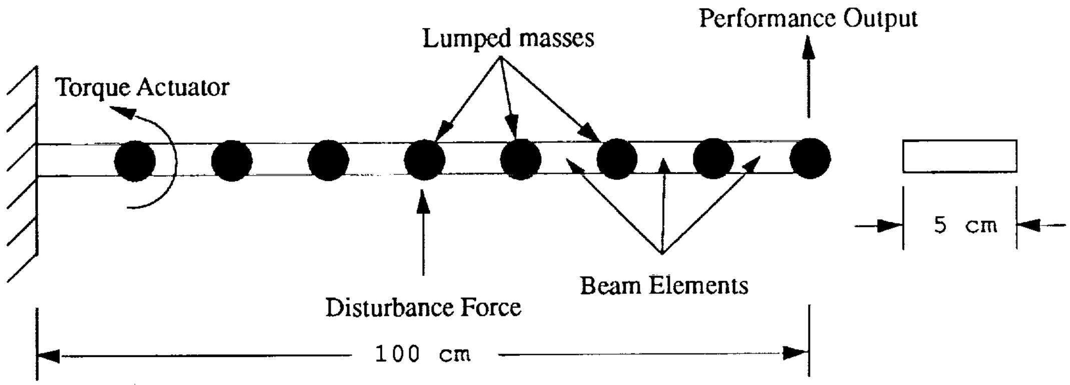

Figure 5: Cantilevered aluminum beam model. $E = 73GPa$, $\rho = 2700kg/m^3$

$$\begin{array}{ll} \delta J_{fre} = \mathrm{tr}\left\{H\left(P\frac{\partial A}{\partial \alpha} + \frac{\partial A^T}{\partial \alpha}P\right)\right\} & \text{Frequency Subgradient} \\ \delta J_{dis} = \mathrm{tr}\left\{P\frac{\partial S}{\partial \alpha}\right\} & \text{Disturbability Subgradient} \\ \delta J_{obs} = \mathrm{tr}\left\{H\frac{\partial}{\partial \alpha}\left(C^TC\right)\right\} & \text{Observability Subgradient} \\ \delta J_{con} = -\mathrm{tr}\left\{HP\frac{\partial}{\partial \alpha}\left(BR^{-1}B^T\right)P\right\} & \text{Controllability Subgradient} \end{array} \quad (17)$$

It is illustrative to consider the application of these definitions to a simple problem. Figure 5 depicts a cantilevered beam model consisting of eight finite elements with lumped masses included at the nodes. The performance output is the tip displacement of the beam, the actuator is a torque applied near the root, and the disturbance is a transverse force applied at the midspan of the beam. The structural design variables in this problem are the thicknesses of the beam elements and the size of the lumped masses. These variables have been scaled so that an equal change in element thickness or lumped mass represents an equal change in mass.

$$\begin{array}{ll} \text{Beam element design variables} & t_i^* = \frac{t_i}{1cm} \\ \text{Lumped mass design variables} & m_i^* = \frac{m_i}{(1cm)w_b h \rho} \end{array} \quad (18)$$

where w_b and h are the element width and length and ρ is the material density. The total mass of the beam and lumped masses is constrained to be less than or equal to that of a 1 cm uniform beam with no lumped masses.

Figures 6-9 depict the magnitudes of the subgradients, normalized by the cost magnitude and projected onto the constant mass constraint, for a 1 cm uniform beam with no lumped mass. Figures 6 and 7 show these values for a displacement disturbance and model orders of one and four modes, respectively. Figures 8 and 9 show these same values for a velocity disturbance.

The horizontal axis in these figures represents the level of control penalty used in the LQR performance metric. The left side of each figure corresponds to cheap control or high control effort. The right side corresponds to expensive control or low control effort. The vertical lines are the different levels of control penalty at which the control influence parameter:

$$\beta = \left|\frac{\mathcal{NF}}{\omega\rho}\right| = \left|\frac{N\phi_i\phi_i^T F}{\omega_i^2\rho}\right| \quad (19)$$

is equal to unity for different modes. (Recall from Equation 4 that Φ is the modal transformation matrix and ω_i is the modal frequency.)

For the single mode models, these figures agree exactly with the results predicted in Table 1. The sensitivity of the cost to disturbability is unaffected by control effort, frequency rises from zero to

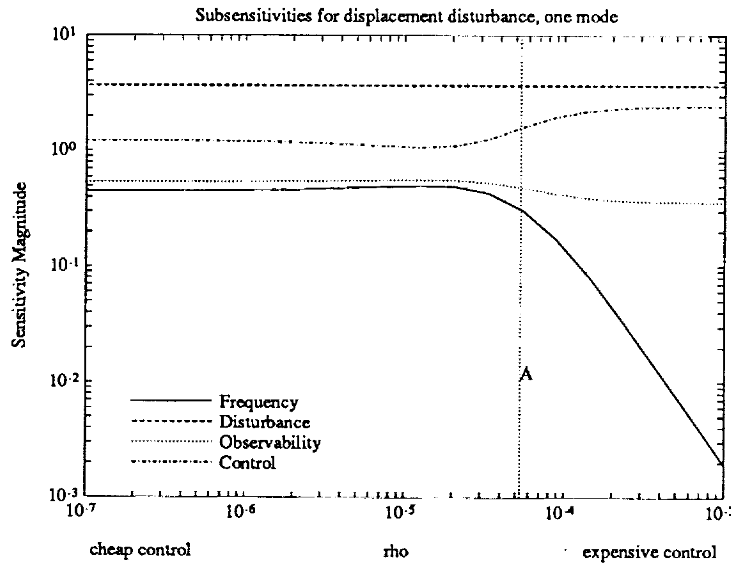

Figure 6: Magnitudes of subgradients normalized by cost for uniform beam with one mode and displacement disturbance versus control weighting; A: β for mode 1 = 1.

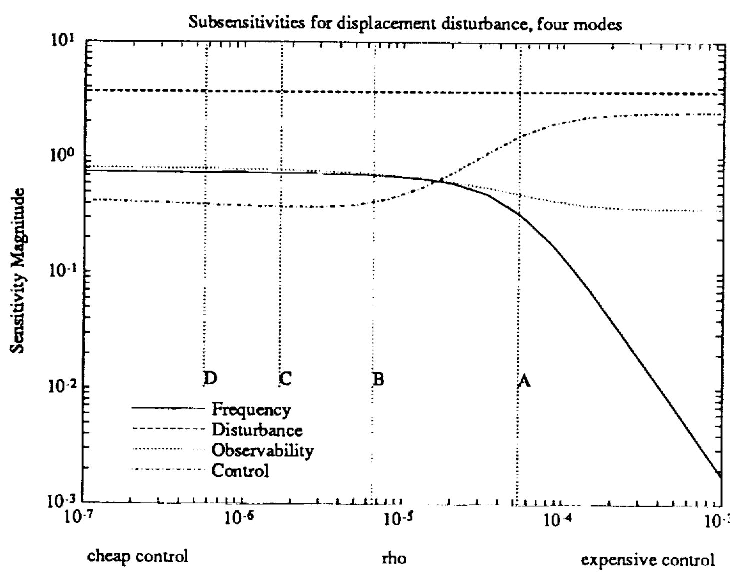

Figure 7: Magnitudes of subgradients normalized by cost for uniform beam with four modes and displacement disturbance versus control weighting; A: β for mode 1 = 1, B: β for mode 2 = 1, C: β for mode 3 = 1, D: β for mode 4 = 1.

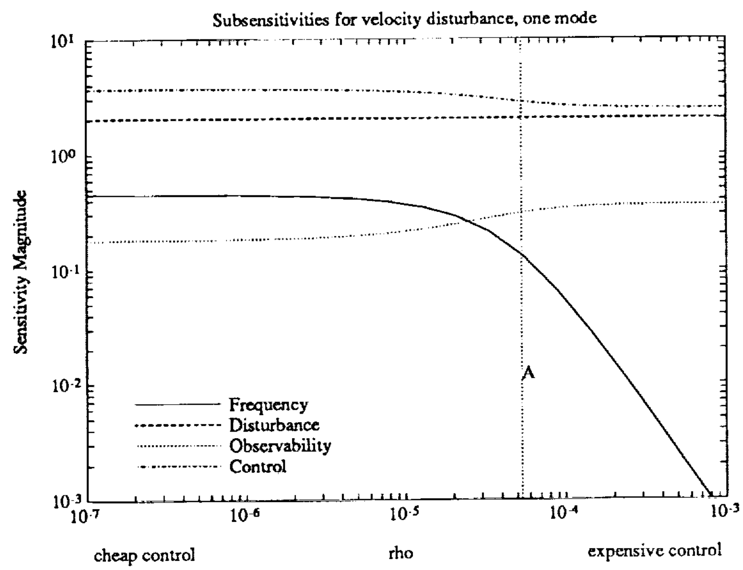

Figure 8: Magnitudes of subgradients normalized by cost for uniform beam with one mode and velocity disturbance versus control weighting; A: β for mode $1 = 1$.

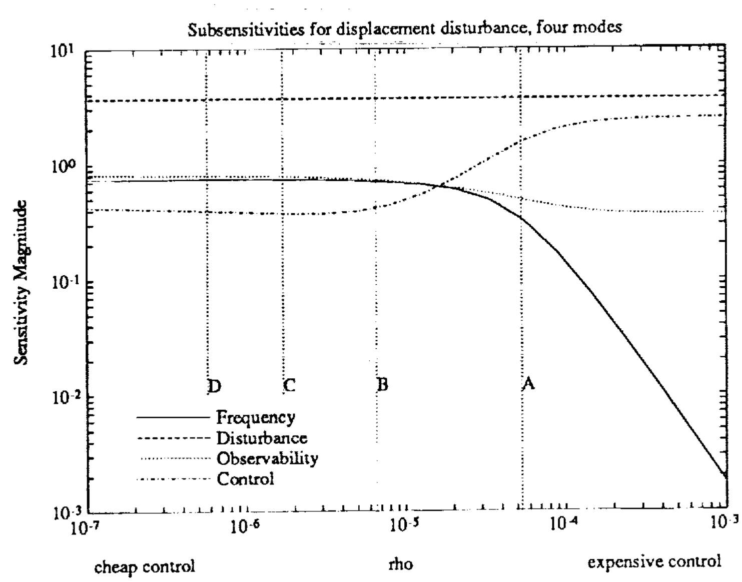

Figure 9: Magnitudes of subgradients normalized by cost for uniform beam with four modes and velocity disturbance versus control weighting; A: β for mode $1 = 1$, B: β for mode $2 = 1$, C: β for mode $3 = 1$, D: β for mode $4 = 1$.

finite importance with decreasing control cost, and for the displacement disturbance, increased control effort increases the importance of observability and decreases the significance of controllability while for the velocity disturbance, the opposite is true.

In the four mode model, there are several differences in the behavior of the cost. In the cheap control cases, there is a marked increase in the sensitivity of the cost to observability and a decrease in the sensitivity to controllability. Also, for the velocity disturbance, there is a slight increase in the sensitivity to disturbances. Yet, for expensive control, the sensitivities are almost identical to those for the single mode model. The reason for this discrepancy can be traced to the use of a single actuator to control several modes. At low levels of control, the actuator will need to provide only active damping for each mode. Velocity feedback will provide this active damping to all modes. At higher levels of control, however, each mode will require more shape control. Unfortunately the inputs required by each mode for shape control can be very different and the optimal control must represent a compromise. In other words, a shortage of actuators makes the control less effective at high levels. The effect this has on the cost is to reduce the role of the control influence term β in the expressions in the third row of Table 1. Terms which can reduce transmissibility of the disturbance to the performance output directly (*i.e.* disturbability and observability) will be enhanced in significance while controllability will be diminished. This is exactly the behavior in the results noted above.

5 $\mathcal{H}_\infty$ Problem

Another performance metric in common use is the $\mathcal{H}_\infty$ metric:

$$J = \sup_{\omega} \left(y(\imath\omega)^T y(-\imath\omega) + u(\imath\omega)^T R u(-\imath\omega) \right) \tag{20}$$

where the disturbance in Equation 7, v, is Gaussian White Noise of unit intensity. It can be shown that when there is a minimum positive value, γ, such that there exists a symmetric, positive-definite solution to the equation

$$PA + A^T P + C^T C - P \left(BR^{-1}B^T - \frac{1}{\gamma^2} LL^T \right) = 0 \tag{21}$$

then for optimal full state feedback control, the value of the performance metric is the square of this limiting value of γ [4].

For the typical section, it is possible to solve for this γ in closed form:

$$J_{\text{opt}} = \gamma_{\min}^2 = \begin{cases} \mathcal{G}^2 \left(\frac{\mathcal{F}^2}{\rho^2} + \frac{\omega^2}{\mathcal{N}^2} 4\zeta^2(1-\zeta^2) \right)^{-1} & \zeta \leq \frac{1}{\sqrt{2}} \\ \mathcal{G}^2 \left(\frac{\mathcal{F}^2}{\rho^2} + \frac{\omega^2}{\mathcal{N}^2} \right)^{-1} & \zeta \geq \frac{1}{\sqrt{2}} \end{cases} \tag{22}$$

The same proportionality to $\mathcal{G}^2$ that was present in the $\mathcal{H}_2$ case is also present in the $\mathcal{H}_\infty$ case. However, the dependence on the remaining terms has an interesting form. For very expensive control ($\rho \to \infty$), the term containing the controllability drops out and the cost reverts to the open loop cost. This solution implies that for problems in which the disturbability is relatively insensitive to parameter changes ($\frac{\partial \mathcal{G}}{\partial \alpha} = 0$), then a structure which was optimized for open loop response will also have optimal closed loop response. In other words, sequential design of the structure and control will achieve the same result as simultaneous design. The insensitivity of a system to disturbability can occur frequently in controlled structure design. This is most likely to happen when the disturbance is not well known, or widely distributed and uncorrelated. Hence, there is a possibility that it may actually be easier to design many $\mathcal{H}_\infty$ controlled structures than their $\mathcal{H}_2$ counterparts. However, the complexity of the $\mathcal{H}_\infty$ problem makes it infeasible to show here that it applies to higher order systems.

6 Conclusions

The typical section is a useful tool for understanding the implications of structural changes under different types of control. The ability to formulate closed form solutions makes it very easy to understand the functionality of the performance on modal properties. As long as the coupling between modes in a structure remains light, the typical section results, which exactly capture the behavior of a single controlled mode, should be reliable in higher order systems. However at high control levels, the modal coupling becomes more severe and the typical section insights become less reliable. In particular, the importance of controllability is reduced when fewer actuators than modes are available.

There exists the possibility that for systems in which the disturbability is insensitive to design parameters, sequential design (open loop structural optimization followed by control optimization) can actually yield an optimal design. However, it remains to be shown that this conclusion, which is valid for the typical section, generalizes to more complicated systems.

References

[1] M. Milman, M.Salama, R. Scheid, R. Bruno, and J.S.Gibson, JPL Technical Report D-6767, October, 1989.

[2] R. Kwakernaak and R.Sivan, *Linear Optimal Control Systems.* John Wiley and Sons, Inc., 1972.

[3] R.N. Jacques and D.W. Miller. MIT SERC Report 1-91, February 1991.

[4] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A.Francis. *IEEE Transactions on Automatic Control*, 34(8), August 1989.