N93-18824

MODEL REDUCTION RESULTS FOR FLEXIBLE SPACE STRUCTURES

Trevor Williams and Masoud Mostarshedi

Department of Aerospace Engineering
University of Cincinnati
Cincinnati, Ohio 45221

Abstract

This paper describes the novel subsystem balancing technique for obtaining reduced-order models of flexible structures, and investigates its properties fully. This method can be regarded as a combination of the best features of modal truncation (efficiency) and internal balancing (accuracy); it is particularly well suited to the typical practical case of structures which possess clusters of close modes. Numerical results are then presented demonstrating the results obtained by applying subsystem balancing to the Air Force Phillips Laboratory ASTREX testbed, the Jet Propulsion Laboratory antenna facility, and the NASA Marshall Space Flight Center ACES structure.

Introduction

Model reduction is a very important practical problem related to the control of flexible space structures (FSS), and a considerable amount of work has been carried out on this topic. Well-known methods include *modal truncation* [1], based either on the natural frequencies of the structure or its modal costs, and *balancing* [2] of the entire structure and then truncation to retain a dominant model for it. An advantage of the balancing approach is that it typically yields a more accurate reduced-order model than does simple modal truncation. This is particularly true when the structure possesses clustered natural frequencies, as is often the case for realistic flexible space structures. However, the disadvantages of balancing are its high computational cost, possible numerical sensitivity problems resulting from the large matrices being operated on, and the difficulty involved in providing a physical interpretation for the resulting balanced "modes".

The purpose of this paper is to investigate the practical performance of the alternative *subsystem balancing* technique when tested on realistic flexible space structures. This method, introduced in [3], retains the desirable properties of standard balancing while overcoming the three difficulties listed above. This is achieved by first decomposing the structural model into subsystems of highly correlated modes, based on the *modal correlation coefficients* derived in [4] from the Grammians of the structure. Each subsystem is approximately uncorrelated from all others, so balancing each separately and concatenating the dominant reduced-order models obtained yields roughly the same result as balancing the entire structure directly. The computational cost reduction produced by this block-by-block technique is considerable: an operation count reduction by a factor of roughly $1/r^2$ if the system decomposes into r equal subsystems. The numerical accuracy of the resulting reduced-order model is also improved considerably, as the matrices being operated on are of reduced dimension, and its modes do now permit a clear physical interpretation. This is a consequence of the fact that each correlated subsystem must necessarily only include modes with close natural frequencies. The balanced modes of each subsystem are, therefore, to first order linear combinations of repeated-frequency modes, and so can themselves be taken as an equally valid set of physical modes. Balancing the entire structure, on the other hand, combines modes of widely differing frequencies, making interpretation difficult.

The numerical results to be presented in this paper are for the Air Force Phillips Laboratory ASTREX structure, the Jet Propulsion Laboratory antenna testbed, and the NASA Marshall Space Flight Center ACES facility. The ACES data to be presented include results both for the *a priori* finite-element model and for a model identified from vibration tests of the structure. Details will also be given of the implementation of the algorithm, in particular, of the method used for determining the dimensions of each subsystem and the number of balanced modes that should be retained from each in the final reduced-order model. Confirmation will also be given of the efficiency advantages of the new method over standard balancing, in terms of floating-point operation counts, and comparisons given of the accuracy properties of the three model reduction procedures.

Problem Formulation

Consider an n-mode model for the structural dynamics of a modally damped, non-gyroscopic, non-circulatory FSS with m actuators and p sensors, not necessarily collocated. This model can be written in modal form [1] as

$$\begin{aligned} \ddot{\boldsymbol{\eta}} + diag(2\zeta_i\omega_i)\dot{\boldsymbol{\eta}} + diag(\omega_i^2)\boldsymbol{\eta} &= \widehat{B}\mathbf{u}, \\ \mathbf{y} &= \widehat{C}_r\dot{\boldsymbol{\eta}} + \widehat{C}_d\boldsymbol{\eta}, \end{aligned} \tag{1}$$

where $\boldsymbol{\eta}$ is the vector of modal coordinates, **u** that of applied actuator inputs and **y** that of sensor outputs, and ω_i and ζ_i are the natural frequency and damping ratio of the i^{th} mode, respectively. For the typical FSS [5], the $\{\zeta_i\}$ are quite low (e.g. 0.5 %), and the $\{\omega_i\}$ occur in clusters of repeated, or nearly repeated, frequencies as a result of structural symmetry. In order to ensure asymptotic stability, as needed in the next section, we shall assume that all natural frequencies and damping ratios are non-zero. (This rigid-body mode restriction can actually be relaxed fairly easily if required.)

Defining the state vector $\mathbf{x} = (\dot{\eta}_1, \omega_1\eta_1, \cdots, \dot{\eta}_n, \omega_n\eta_n)^T$ for this structure yields the state space representation $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \mathbf{y} = C\mathbf{x}$, where $A = blkdiag(A)_i$, $B = (B_1^T, \cdots, B_n^T)^T$ and $C = (C_1, \cdots, C_n)$, with

$$A_i = \begin{pmatrix} -2\zeta_i\omega_i & -\omega_i \\ \omega_i & 0 \end{pmatrix}, B_i = \begin{pmatrix} \mathbf{b}_i \\ 0 \end{pmatrix} \text{and } C_i = \begin{pmatrix} \mathbf{c}_{ri} & \mathbf{c}_{di} / \omega_i \end{pmatrix}; \tag{2}$$

$\mathbf{b}_i$ is the ith row of $\widehat{B}$, and $\mathbf{c}_{ri}$ and $\mathbf{c}_{di}$ are the ith columns of $\widehat{C}_r$ and $\widehat{C}_d$, respectively.

The problem we shall study is that of obtaining a reduced-order model

$$\begin{aligned} \dot{\mathbf{x}}_r &= A_r\mathbf{x}_r + B_r\mathbf{u}, \\ \mathbf{y} &= C_r\mathbf{x}_r \end{aligned} \tag{3}$$

for this structure for which the normalized output error

$$\delta^2 = \frac{\int \|\mathbf{y}(t) - \mathbf{y}_r(t)\|_2^2 \, dt}{\int \|\mathbf{y}(t)\|_2^2 \, dt} \tag{4}$$

is acceptably small. Of course, the size of δ will depend on the order, n_r, chosen for the reduced model. A good model reduction procedure should ideally provide information allowing an intelligent choice for n_r to be made so as to achieve a specified upper bound on δ.

Two techniques for model reduction that have been extensively studied are those of *modal truncation* and *internal balancing* [2]. The purpose of the present paper is to compare the results they produce with those obtained by means of a new method, subsystem balancing, which can be regarded as an intermediate case between the two established techniques. In order to develop this algorithm, it is first necessary to study the *Grammian* matrices which form the basis of balancing. This is the subject of the next section.

Closed-Form Grammians

The controllability and observability Grammians, denoted by W_C and W_O, respectively, of the system described by (2) are the solutions of the algebraic Lyapunov equations

$$AW_c + W_cA^T + BB^T = 0 \tag{5}$$

and

$$A^TW_o + W_oA + C^TC = 0. \tag{6}$$

The block diagonal form of A can be exploited [6][7] to give closed-form solutions for these equations. Taking W_C first and writing it in terms of its (2 x 2) blocks $\{W_{ij}\}$, we have

$$A_i W_{ij} + W_{ij} A_j^T + B_i B_j^T = 0. \tag{7}$$

Applying (2) then yields, after some algebra, the expression

$$W_{ij} = \frac{\beta_{ij}}{d_{ij}} \cdot \begin{pmatrix} 2\omega_i\omega_j(\zeta_j\omega_i + \zeta_i\omega_j) & \omega_j(\omega_j^2 - \omega_i^2) \\ -\omega_i(\omega_j^2 - \omega_i^2) & 2\omega_i\omega_j(\zeta_i\omega_i + \zeta_j\omega_j) \end{pmatrix}, \tag{8}$$

where $\beta_{ij} = \mathbf{b}_i^T\mathbf{b}_j$ and $d_{ij} = 4\omega_i\omega_j(\zeta_i\omega_i + \zeta_j\omega_j)(\zeta_j\omega_i + \zeta_i\omega_j) + (\omega_j^2 - \omega_i^2)^2$. The quantity d_{ij}^{-1} is essentially a measure of how closely correlated modes i and j are; it will be returned to below. Evaluating W_C by this method involves about $7n^2$ floating-point operations (exploiting the symmetry of W_C, i.e. $W_{ji} = W_{ij}^T$); by contrast, the Bartels-Stewart algorithm [8] for general matrices A and B requires order(n^3) operations.

The general expression (8) for W_{ij} simplifies considerably for exactly repeated frequencies, where we obtain

$$W_{ij} = \frac{\beta_{ij}}{2(\zeta_i + \zeta_j)\omega_i} \cdot I_2; \tag{9}$$

in particular, the diagonal blocks are just $W_{ii} = \frac{\beta_{ii}}{4\zeta_i\omega_i} \cdot I_2$. Simplifications also occur for widely separated, lightly-damped modes: in this case,

$$W_{ij} \to \frac{\beta_{ij}}{(\omega_j^2 - \omega_i^2)} \cdot \begin{pmatrix} 0 & \omega_j \\ -\omega_i & 0 \end{pmatrix} \text{as } \zeta_i, \zeta_j \to 0. \tag{10}$$

It is important to note that (9) is inversely proportional to the damping ratios of the structure, while (10) is independent of damping. Thus, the only blocks of W_C which will be of significant magnitude for a structure with light damping are those on the diagonal, and those off-diagonal blocks that correspond to close frequencies. This reflects the well-known result [9]-[11] that the modal model of a flexible structure with widely separated natural frequencies is already approximately balanced. However, balancing a flexible structure with near-repeated frequencies is a much more challenging problem [6], as indeed is determining the controllability properties of its close modes [12].

The observability Grammian W_O for a system with rate measurements only ($\tilde{C}_d = 0$) can be obtained in a similar fashion to the controllability Grammian, or more simply by noting that $A^T = PAP$ for flexible structures, where $P = diag\{1, -1, \cdots 1, -1\}$. Therefore, pre- and post-multiplying (6) by P gives

$$A[PW_oP] + [PW_oP]A^T + C^TC = 0. \tag{11}$$

Note that this equation makes use of the fact that $CP = P$ for such systems. Thus, W_O is essentially as given in (8), the only alterations being that the signs of the off-diagonal entries are changed and β_{ij} is replaced by $\gamma_{rij} = \mathbf{c}_{ri}^T\mathbf{c}_{rj}$.

If displacement measurements are also allowed, the situation is much less simple; in fact, the analytical expressions that then result for W_O are really too complicated to be useful. The only exception to this is the expression for the i[th] diagonal block of W_O for a lightly-damped structure ($\zeta_i << 1$), where we have the approximation

$$W_{oii} \approx \frac{(\omega_i^2 \gamma_{rii} + \gamma_{dii})}{4\zeta_i \omega_i^3} \cdot I_2 \tag{12}$$

with $\gamma_{dij} = \mathbf{c}_{di}^T \mathbf{c}_{dj}$. Although no general analytical expressions for W_O are now tractable, it is still possible to derive a semi-closed-form method to evaluate the observability Grammian that exploits the special form of the matrix A in (2). This method is nearly as efficient as the true closed-form controllability Grammian results derived previously, and is based on writing the (i,j) block of W_O as

$$W_{oij} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{13}$$

The equation which defines this block (from (6)) is $A_i^T W_{oij} + W_{oij} A_j + C_i^T C_j = 0$, which can be expanded and rewritten as the following system of four simultaneous linear equations.

$$\begin{pmatrix} -2(\zeta_i\omega_i + \zeta_j\omega_j) & \omega_j & \omega_i & 0 \\ -\omega_j & -2\zeta_i\omega_i & 0 & \omega_i \\ -\omega_i & 0 & -2\zeta_j\omega_j & \omega_j \\ 0 & -\omega_i & -\omega_j & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = - \begin{pmatrix} \mathbf{c}_{ri}^T \mathbf{c}_{rj} \\ \mathbf{c}_{ri}^T \mathbf{c}_{dj} / \omega_j \\ \mathbf{c}_{di}^T \mathbf{c}_{rj} / \omega_i \\ \mathbf{c}_{di}^T \mathbf{c}_{dj} / \omega_i \omega_j \end{pmatrix}. \tag{14}$$

Solving this system by means of Gaussian elimination requires approximately 29 floating-point operations, where the special structure of the matrix on the left-hand side has been exploited. It therefore requires a total of about $15n^2$ flops to evaluate the entire symmetric W_O using this approach. It is interesting to note that the determinant of the matrix in (14) is just d_{ij}. This quantity therefore plays a similar rôle in the denominators of both the controllability and observability Grammians. It can also be shown that, just as for W_C, the only blocks of W_O which are large for a lightly-damped structure are those corresponding to two closely-spaced modes.

Finally, if $p \geq m$, as is typical of FSS applications, and there exists a matrix U with orthonormal columns which satisfies $C = UB^T P$, then (2) is said to be *orthogonally symmetric* [13]. A particular class of orthogonally symmetric systems is that of flexible structures with *compatible* (physically collocated and coaxial) actuators and rate sensors: we then have $C = B^T$, i.e. $U = I$. Associated with any orthogonally symmetric system is its *cross-Grammian* W_{CO}, which is defined as the solution of the Lyapunov equation

$$AW_{co} + W_{co}A + BU^T C = 0. \tag{15}$$

The usefulness of W_{co} in balancing applications lies in the fact that it satisfies the relation $W_{co}^2 = W_c W_o$. In fact, as $C^T C = PBU^T UB^T P = BB^T$ and $BU^T C = BU^T UB^T P = BB^T$, (11) and (15) can be seen to reduce to the expressions [13]

$$W_{co} = W_c P = PW_o. \tag{16}$$

Thus, all three Grammians of an orthogonally symmetric system are given directly from (8) with suitable changes of sign, noting, of course, that $\beta_{ij} = \gamma_{rij}$ for such systems. This property will be shown to lead to significant simplifications when balancing models of collocated flexible structures.

Subsystem Balancing

It is always possible [2] to find a state transformation T that takes the model $\{A, B, C\}$ to an *internally balanced* state space representation $\{T^{-1}AT, T^{-1}B, CT\}$, i.e. one with equal and diagonal controllability and observability Grammians

$$\overline{W}_c = \overline{W}_o = \Sigma \equiv diag(\sigma_i), \tag{17}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. These *Hankel singular values* lead to a simple procedure for obtaining a reduced-order approximation to the original system: delete those balanced states corresponding to all singular values below some specified threshold. The resulting dominant reduced-order model will match the full system with an accuracy related to the sizes of those Hankel singular values which were discarded, so giving a guideline for selecting an acceptable reduced model order n_r; see [2] for further details. It should be noted that this model reduction procedure is very straightforward once the balancing transformation T has been found: it merely amounts to discarding trailing rows of the balanced A and B and trailing columns of A and C.

Computation of T can be shown to amount to the solution of a standard eigenproblem. This can be formulated in various different ways. The one which follows is not the best numerically (see [14] for a superior alternative), but it makes the significance of the transformation T clearest. Inspection of (5) and (6) reveals that the Grammians of the balanced system are related to those of the original system model as

$$\overline{W}_c = T^{-1} W_c T^{-T} \text{ and } \overline{W}_o = T^T W_o T; \tag{18}$$

multiplying these matrices then gives

$$\Sigma^2 = \overline{W}_c \overline{W}_o = [T^{-1} W_c T^{-T}][T^T W_o T] = T^{-1}[W_c W_o]T. \tag{19}$$

Thus, T is just the matrix of eigenvectors (suitably scaled) of $W_c W_o$, and the Hankel singular values of the system are the corresponding eigenvalues. The usefulness of the cross-Grammian for balancing orthogonally symmetric systems can now also be seen: as T is the eigenvector matrix of $W_c W_o = W_{co}^2$ it is also the eigenvector matrix of W_{co}, and we have $T^{-1} W_{co} T = \Lambda$ with $\Sigma^2 = \Lambda^2$, so $\Lambda = diag(\pm\sigma_i)$. It can be shown

[6] that the appropriate scaling for the eigenvectors making up T for a collocated flexible structure is such that the relation $T^T PT = P$ is satisfied, while the signs of the eigenvalues of W_{co} must alternate in the same way as the diagonal elements of P. This can certainly be seen to be true for the special case of light damping and widely spaced natural frequencies, as (17) and (9) then imply that the $\{\lambda_i\}$ occur in approximate pairs $\{\pm \beta_{ii}/4\zeta_i\omega_i\}$; similarly, the Hankel singular values $\{\sigma_i\}$ of a lightly-damped flexible structure always occur in approximate pairs. The important point about evaluating T in terms of the cross-Grammian directly, rather than using the product $W_c W_o$, is that it is a *square root* method. It therefore possesses the improved accuracy properties typical of these techniques, as exhibited by such applications as least squares estimation by QR decomposition rather than the normal equations [15], Kalman filtering [16], and the FSS problems of on-orbit structural identification [17] and transmission zeros computation [18].

It has already been noted that the Grammians of a lightly-damped flexible structure with widely separated natural frequencies are diagonally dominant, i.e. a modal model of such a structure is already approximately balanced [10][11]. However, consider now the more realistic case of a lightly-damped structure with clusters of close modes, as is typical of flexible spacecraft. The Grammians of such a system will now be **block** diagonally dominant, with a diagonal block corresponding to each cluster of modes. The Grammian eigenvector matrix T obtained from $W_c W_o$ or W_{co} will consequently also be block diagonally dominant. It can therefore be replaced, to first order, by the block diagonal matrix whose (i,i) block is just the eigenvector matrix of the ith dominant diagonal Grammian block. In other words, an approximation to the internally balanced representation of the given FSS can be obtained by balancing each subsystem of close modes independently and then concatenating the results.

This *subsystem balancing* approach, introduced in [3], has several significant advantages over standard balancing. The first is that it is clearly much more efficient to compute the eigenvectors of several small subsystems than it is to evaluate the eigenvector matrix of the entire system. In fact, as eigenstructure evaluation is an order(n^3) operation, this efficiency gain can be quite substantial. Consider for illustrative purposes the case where the structure being studied breaks down into r subsystems of equal dimension. It can then be shown that the standard balancing technique will require on the order of r^2 as many floating-point operations as will subsystem balancing. A second advantage is also a consequence of the fact that we are now operating on matrices of smaller dimension than if the entire system were balanced directly. This tends to reduce the *condition number* [15] of the state transformations being applied, and so limits the effects of rounding errors on the final computed state space model. This therefore helps overcome the numerical problems that have been noted [19][20] when applying classical balancing to systems of high dimension. The final advantage of subsystem balancing relates to the physical interpretation of the resulting balanced state variables $\bar{\mathbf{x}} = T^{-1}\mathbf{x}$. In the new method, the fact that T is taken to be block diagonal implies that each balanced state will be made up of a linear combination of the states corresponding to a single cluster of close modes. This is, to first order, just the repeated eigenvalue case, where any linear combination of eigenvectors (mode shapes) is itself a valid eigenvector. The transformed states produced by subsystem balancing are therefore basically perturbed repeated modes, and so can be visualized quite easily. Standard balancing, by contrast, yields states which are made up of linear combinations of all the modes of the structure, making physical interpretation very difficult.

Model reduction by subsystem balancing therefore proceeds by first dividing the given structure into subsystems of close modes. Each subsystem is then balanced independently, and a reduced-order model for it generated by deleting all balanced states corresponding to Hankel singular values below some specified threshold. (Note that the modified truncation criterion of [21] could be used instead of the Hankel singular values, if desired, without changing the argument in any way.) The resulting reduced-order subsystem models so obtained are then combined to yield a dominant, approximately balanced, reduced-order model for the full system. This method can be applied to any flexible structure, collocated or non-collocated; however, it can be refined somewhat when analyzing collocated structures. In this case, it is possible to define a *modal correlation coefficient* [3][4] between modes i and j, so allowing the interaction between the two modes to be quantified more precisely than in the non-collocated case. This correlation coefficient, defined as

$$\rho_{ij}^2 = \frac{\left\|W_{co_{ij}}\right\|^2}{\left\|W_{co_{ii}}\right\| \cdot \left\|W_{co_{jj}}\right\|}, \tag{20}$$

can be shown to have magnitude lying between 0 and 1. It can also be shown to be small for modes with widely separated natural frequencies, and it may approach unity for close modes. However, it will also be small for modes which are close but have mode shapes which are nearly orthogonal. These correlation coefficients therefore provide a somewhat more precise means of defining the subsystems of structural modes which must be balanced together than does frequency separation by itself. Of course, it must be noted that the cross-Grammian is not defined for non-collocated systems, so (20) cannot be used for such systems. The question of whether a similar correlation coefficient can be defined for such systems is a topic of current research.

In summary, the two algorithms used to compute the state transformations needed for subsystem balancing of flexible structures can be summarized as follows. In both cases, approximate operation counts are given for each step for the illustrative case of a system of order n which breaks down into r equal subsystems.

Non-Collocated:

Define subsystems (by modal frequency separation)
For each subsystem:

Construct closed-form W_c and W_o:	$22(n/r)^2$ flops
Find Cholesky factorization $W_o = LL^T$:	$\frac{1}{6}\cdot(n/r)^3$ flops
Construct $X = L^T[W_c W_o]L^{-T} = L^T W_c L$:	$(n/r)^3$ flops
Find eigenstructure of symmetric X:	$5(n/r)^3$ flops
Transform by L to give eigenvectors of $W_c W_o$:	$\frac{1}{2}\cdot(n/r)^3$ flops
Total (all subsystems):	$\frac{20}{3}\cdot(n^3/r^2) + 22(n^2/r)$ flops

Collocated:

Define subsystems (by modal correlation coefficients)
For each subsystem:

Construct closed-form W_{co}:	$7(n/r)^2$ flops
Find eigenstructure of unsymmetric W_{co}:	$15(n/r)^3$ flops
Total (all subsystems):	$15(n^3/r^2) + 7(n^2/r)$ flops

These operation counts compare very favorably with the total of about $21n^3$ needed for standard balancing; they exhibit a reduction by a factor of approximately $1/r^2$. It is also interesting to note that the collocated method has a higher count than the non-collocated algorithm, which uses the method described by Laub [14]. It may therefore be supposed that there is no advantage to treating collocated structures as a special case, as we have done. However, this ignores two factors. Firstly, use of the modal correlations (20) may permit smaller subsystems to be defined, without any loss of accuracy, than if frequency differences are used as the separation criterion. Secondly, the collocated method is a matrix square root method, and so should be expected to have superior numerical conditioning properties.

Results

Numerical results will now be provided which illustrate the behavior of the subsystem balancing technique when applied to realistic structures. The three structural models studied are the Air Force Phillips Laboratory ASTREX article, the Jet Propulsion Laboratory flexible antenna testbed, and the NASA Marshall Space Flight Center ACES facility. These three structures all possess light damping and a large number of closely-spaced vibration modes. Furthermore, they allow the algorithm to be tested in both the collocated and non-collocated cases. Results will also be given for an identified model for ACES obtained from experimental vibration test data. (The interested reader is referred to [22] for further details.)

1. ASTREX

This graphite-epoxy truss structure [23] provides a good illustration of the application of subsystem balancing to a non-collocated flexible structure. The structural model considered has 22 modes with frequencies below 50 Hz: these are given in Table I. It can be noted that this system does indeed possess modes with close frequencies; for instance, modes 5 and 6 and 14 and 15 differ by only about 0.1 Hz. Each mode has an assumed damping ratio of 0.1 %.

Model reduction for this structure is actually quite challenging, as it is fitted with 8 actuators and 39 sensors. Any reduced-order model will therefore have to be able to approximately match the response of all 39 outputs of the true system to any of the 8 control inputs. Despite this difficulty, the subsystem balancing method was found to give good results when applied to ASTREX. The first step in the procedure is to break the complete model down into subsystems of close modes, based on their relative frequency separation. A separation threshold of 7 % was found to lead to a good balance between having excessively large subsystems (threshold too high) and obtaining inaccurate results as a result of separating modes which actually interact significantly (threshold too low). The subsystem modal groupings found for the chosen threshold are given in Table II. Note that modes 14 and 22, for instance, are included in the same subsystem even though they are separated by over 10 Hz and therefore do not interact directly. The reason for this is that they actually interact indirectly through the other modes in the subsystem: mode 14 is within 7% of modes 15 to 18; mode 18 interacts with mode 19, which in turn interacts with modes 20 and 21, which in turn interacts with 22. This is a common occurrence when defining subsystems of closely coupled modes.

The next step in the procedure is to balance each subsystem independently, making use of the closed-form expressions (8) and (14) to compute the relevant Grammians, and then truncate to give a dominant reduced-order subsystem model. These are then concatenated to obtain a dominant reduced-order model for the entire system. The last column of Table II shows the number of balanced "modes" that were retained from each subsystem when a Hankel singular value threshold of 0.2 was used. It can be seen that several groups of modes at both low and high frequencies do not contribute at all to the final reduced-order model, others are retained in their entirety, and still others are approximated to by a truncated balanced model. The composite reduced-order model so obtained has 11 modes, as opposed to 22 in the original model. Despite this substantial reduction in model order, the difference between the outputs of the full and reduced models, as measured by the normalized impulse response output error δ in (4), is a quite acceptable 6.65 %. As a final point, note that the subsystem balancing technique does actually produce the claimed efficiency gains when applied to this practical system. In fact, the operation count required to balance the subsystems obtained above for ASTREX is only about 7.4 % as large as that required to balance the entire system directly.

2. JPL Antenna

This structure, designed to be representative of a flexible dish antenna, possesses 12 ribs symmetrically distributed about a central pivoted hub. The model provided by JPL for this structure has 84 modes, with the lowest natural frequency at 0.09 Hz; as a result of the symmetry of the system, many of these frequencies are essentially repeated. In the work presented here, a uniform damping ratio of 0.5 % has been assumed for all modes.

The extensive sensor/actuator distribution provided for this structure allows it to be studied in both a collocated and non-collocated configuration. Taking the non-collocated case first, 4 outer levitator sensors (LO1, LO4, LO7 and LO10, in the notation of the JPL model) and 4 actuators (rib root actuators RA1 and RA10; hub actuators HA1 and HA10) were considered to be in use, and all other sensors and actuators disabled. Applying the subsystem balancing technique to this system with a relative frequency separation threshold of 25 %, the 9 subsystems listed in Table III are obtained. (Note that the mode numbers of this system do not increase monotonically with frequency.) If each subsystem is then balanced independently and truncated with a Hankel singular value threshold of 0.0009, the number of balanced modes retained from each is given in the last row of Table III. It can be seen that subsystems 4 and 5 and the large, high-frequency subsystem 9 do not contribute at all to the final reduced model for the structure, whereas subsystems 1 and 3 are retained in their entirety. This 20-mode reduced model matches the output response of the 84-mode full system quite accurately, giving a normalized impulse response error of δ = 11.1 %. By contrast, a reduced model of the same order obtained by modal truncation gave a δ value of 18 %, considerably degraded as a result of ignoring significant interactions (*spillover*) between close modes. The results obtained by balancing the entire system and then truncating were also significantly worse than those obtained by subsystem balancing; in this case, a 20-mode model gave δ = 53.8 %. The reason for this is appears to be numerical conditioning problems that arise when balancing the large (168 states) full system model. Such difficulties are limited in the subsystem balancing approach, as no more than 40 states need ever be balanced at any one time. It should also be noted that the operations count required for subsystem balancing of this structure is only about 2.7 % of that used for standard balancing, a very considerable savings.

To illustrate the application of the collocated version of subsystem balancing, based on the modal correlation coefficients $\{\rho_{ij}\}$ defined by (20) from the cross-Grammian W_{CO}, we shall now restrict the sensors and actuators used to the 6 collocated pairs which exist in the JPL model. These consist of the 4 rib root sensors and actuators RA1/RS1, RA4/RS4, RA7/RS7 and RA10/RS10, as well as the two hub pairs HA1/HS1 and HA10/HS10. Applying the subsystem definition procedure described previously, based on a correlation threshold of $\rho_{th} = 0.03$, yielded the 13 subsystems given in Table IV. It should be noted that there is a degree of correspondence between these subsystems of modes and those obtained by means of the frequency separation criterion (Table III). The main difference is that certain of the subsystems given in Table III have now been broken down into two non-interacting collections of modes. This agrees with the fact that all highly-interacting modes must have close natural frequencies, but all close modes do not necessarily interact strongly. Taking a Hankel singular value threshold of 0.039, a 32-mode reduced model was then obtained for the overall system; the number of balanced modes retained from each subsystem are given in the last row of Table IV. The resulting normalized impulse response error between the full and reduced-order models is $\delta = 4.5$ %; by contrast, a 32-mode model obtained by modal truncation gave an error of 11.2 %, and standard balancing led to $\delta = 15$ %. The new method can thus be seen to give very acceptable results, avoiding the spillover and/or numerical conditioning accuracy problems that affect the other two techniques.

3. ACES

The final system considered is the Astromast-based ACES structure. A 50-mode model for this system has natural frequencies as listed in Table V; as in the previous two examples, the presence of close modes can clearly be observed. ACES is outfitted with a total of 22 sensors and 9 actuators. However, the present model reduction work was carried out in conjunction with the positivity-based controller design discussed in [24], which requires the use of collocated actuators and rate sensors. The model considered here will therefore make use only of the 3 x-, y- and z-axis Advanced Gimbal System (AGS) torquers and their collocated Apollo Telescope Mount (ATM) rate gyros.

Applying the collocated version of subsystem balancing with a modal correlation coefficient threshold of 0.034 leads to the subsystem modal groupings given in Table VI. It is interesting to note that the modes in subsystems 11 and 12 are intermingled; for instance, modes 46 and 47 are extremely close in frequency, yet they are placed in different subsystems. This is another illustration of the fact that two modes can be close and yet nearly orthogonal, and so not highly interacting; the modal correlation coefficients reflect this. Each subsystem was now truncated, based on a singular value threshold of 0.0025, and a 15-mode reduced-order model obtained; the number of modes taken from each subsystem is given as the last column of Table VI. It can be seen that the high-frequency groups 9 through 12 do not contribute at all to the reduced model. For this beam-like structure, it was found that the results obtained by standard balancing and modal truncation were not actually significantly different from those obtained by subsystem balancing, in contrast what was found for ASTREX and the JPL structure.

As a final point, subsystem balancing was also applied to a 15-mode model of the x-axis dynamics of ACES which was identified from vibrational test data. This model was reduced in this way to a 7-mode dominant approximation which matched the observed response well. The fact that the identified modes had quite considerable damping variations did not lead to any difficulties when computing the modal correlation coefficients. Subsystem balancing is therefore certainly not limited to structural models which possess uniform damping ratios.

Conclusions

This paper has described the novel subsystem balancing technique for obtaining reduced-order models of flexible structures, and investigated its properties fully. It was shown that this method can be regarded as a combination of the best features of modal truncation (efficiency) and internal balancing (accuracy); it is particularly well suited to the typical practical case of structures which possess clusters of close modes. Numerical results were then presented demonstrating the results obtained by applying subsystem balancing to the Air Force Phillips Laboratory ASTREX facility, the Jet Propulsion Laboratory antenna testbed, and the NASA Marshall Space Flight Center ACES structure.

References

1. R.R. Craig, *Structural Dynamics*, New York: Wiley, 1981.

2. B.C. Moore, 'Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction', *IEEE Transactions on Automatic Control*, Vol. 26, Feb. 1981, pp. 17-32.

3. T.W.C. Williams and W.K Gawronski, 'Model Reduction for Flexible Spacecraft with Clustered Natural Frequencies', invited paper, 3rd NASA/NSF/DoD Workshop on Aerospace Computational Control, Oxnard, CA, Aug. 1989.

4. W.K Gawronski and T.W.C. Williams, 'Model Reduction for Flexible Space Structures', *Journal of Guidance, Control, and Dynamics*, Vol. 14, Jan.-Feb. 1991, pp. 68-76.

5. M.J. Balas, 'Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams', *IEEE Transactions on Automatic Control*, Vol. 27, 1982, pp. 522-535.

6. T.W.C. Williams, 'Closed-Form Grammians and Model Reduction for Flexible Space Structures', *IEEE Transactions on Automatic Control*, Vol. 35, Mar. 1990, pp. 379-382.

7. R.E. Skelton, R. Singh and J. Ramakrishnan, 'Component Model Reduction by Component Cost Analysis', *Proc. AIAA Guidance, Navigation and Control Conference*, Aug. 1988, pp. 264-274.

8. R.H. Bartels and G.W. Stewart, 'Solution of the Matrix Equation AX + XB = C', *Communications ACM*, 1972, pp. 820-826.

9. E.A. Jonckheere and L.M. Silverman, 'Singular Value Analysis of Deformable Systems', *Proc. 20th IEEE Conference on Decision and Control*, Dec. 1981, pp. 660-668.

10. C.Z. Gregory, 'Reduction of Large Flexible Spacecraft Models using Internal Balancing Theory', *Journal of Guidance, Control, and Dynamics*, Vol. 7, Nov.-Dec. 1984, pp. 725-732.

11. E.A. Jonckheere, 'Principal Component Analysis of Flexible Systems - Open-Loop Case', *IEEE Transactions on Automatic Control*, Vol. 29, Dec. 1984, pp. 1095-1097.

12. T.W.C. Williams, 'Degree of Controllability for Close Modes of Flexible Space Structures', *Proc. 30th IEEE Conference on Decision and Control*, Dec. 1991, pp. 1627-1628.

13. J.A. de Abreu-Garcia and F.W. Fairman, 'A Note on Cross-Grammians for Orthogonally Symmetric Realizations', *IEEE Transactions on Automatic Control*, Vol. 31, Sept. 1986, pp. 866-868.

14. A.J. Laub, 'Computation of "Balancing" Transformations', *Proc. Joint Automatic Control Conference*, Aug. 1980.

15. G.H. Golub and C.F. Van Loan, *Matrix Computations*, Baltimore: Johns Hopkins, 1983.

16. G.J. Bierman, *Factorization Methods for Discrete Sequential Estimation*, New York: Academic, 1977.

17. T.W.C. Williams, 'Identification of Large Space Structures: A Factorization Approach', *Journal of Guidance, Control, and Dynamics*, Vol. 10, July-Aug. 1987, pp. 466-473.

18. T.W.C. Williams, 'Computing the Transmission Zeros of Large Space Structures', *IEEE Transactions on Automatic Control*, Vol. 34, Jan. 1989, pp. 92-94.

19. M.S. Tombs and I. Postlethwaite, 'Truncated Balanced Realization of a Stable Non-Minimal State-Space System', *International Journal of Control*, Vol. 46, 1987, pp. 1319-1330.

20. M.G. Safanov and R.Y. Chiang, 'A Schur Method for Balanced Model Reduction', *Proc. American Control Conference*, 1988, pp. 1036-1040.

21. R.E. Skelton and P. Kabamba, 'Comments on "Balanced Gains and Their Significance for L^2 Model Reduction"', *IEEE Transactions on Automatic Control*, Vol. 31, Aug. 1986, pp. 796-797.

22. M. Mostarshedi, *Model Reduction of Flexible Structures using Subsystem Balancing and the Closed Form of the Grammians*, Master of Science Thesis, University of Cincinnati, Dec. 1991.

23. N.S. Abhyankar and J.L. Berg, 'ASTREX Model Information: Supplement to astabcd.mat File', Phillips Laboratory Internal Report, Aug. 15, 1991.

24. A.B. Bosse and G.L. Slater, 'Digital Implementation of Vibration Suppression Controllers for Large Space Structures', *Proc. 8th VPI & SU Symposium on Dynamics and Control of Large Structures*, May 1991.

Table I. Natural Frequencies (Hz) of the ASTREX Structure

Mode	Frequency
1	3.71
2	5.45
3	14.94
4	15.09
5	19.79
6	19.91
7	21.73
8	25.41
9	29.31
10	30.68
11	33.07
12	33.76
13	35.19
14	38.40
15	38.50
16	38.74
17	38.99
18	40.37
19	42.36
20	43.66
21	45.28
22	48.57

Table II. Subsystems Defined for the ASTREX Structure

Subsystem	Modes	Number in ROM
1	1	0
2	2	0
3	3, 4	2
4	5, 6	1
5	7	1
6	8	1
7	9, 10	0
8	11, 12, 13	0
9	14, ..., 22	6

Table III. Subsystems Defined for the Non-Collocated JPL Antenna

Subsystem	1	2	3	4	5	6	7	8	9
Modes included	8	1	9	2	11	3	4	5	6
	15	22	16	10	18	12	13	14	7
		29		17		19	20	21	27
		36		23		24	25	26	28
		43		30		31	32	33	34
		50		37		38	39	40	35
		57		44		45	46	47	41
		64		51		52	53	54	42
		71		58		59	60	61	48
		78		65		66	67	68	49
				72		73	74	75	55
				79		80	81	82	56
									62
									63
									69
									70
									76
									77
									83
									84
Number kept	2	7	2	0	0	4	3	2	0

Table IV. Subsystems Defined for the Collocated JPL Antenna

Subsystem	1	2	3	4	5	6	7	8	9	10	11	12	13
Modes included	1	2	3	4	5	6	8	9	11	15	16	18	22
		10	12	13	14	7							29
		17	19	20	21	27							36
		23	24	25	26	28							43
		30	31	32	33	34							50
		37	38	39	40	35							57
		44	45	46	47	41							64
		51	52	53	54	42							71
		58	59	60	61	48							78
		65	66	67	68	49							
		72	73	74	75	55							
		79	80	81	82	56							
						62							
						63							
						69							
						70							
						76							
						77							
						83							
						84							
Number kept	0	0	2	6	6	12	1	1	1	1	1	1	0

Table V. Natural Frequencies (Hz) of the ACES Structure

Mode	Frequency	Mode	Frequency	Mode	Frequency
1	0.0102	18	7.4870	35	28.5100
2	0.0268	19	7.5907	36	29.5787
3	0.1569	20	7.6027	37	29.5806
4	0.5051	21	7.8395	38	29.5806
5	0.9118	22	8.4980	39	33.6301
6	0.9292	23	9.6258	40	36.4142
7	3.4540	24	10.5690	41	43.3590
8	3.7229	25	11.4674	42	55.0998
9	3.7323	26	12.0870	43	55.3988
10	3.7855	27	12.0958	44	64.4592
11	4.4967	28	13.7005	45	68.0280
12	5.3601	29	13.9286	46	86.0042
13	5.5579	30	15.6527	47	86.8839
14	5.9523	31	16.8346	48	104.5961
15	5.9523	32	20.6836	49	109.1766
16	7.1019	33	20.7823	50	112.2931
17	7.3312	34	20.7917		

Table VI. Subsystems Defined for the ACES Structure

Subsystem	Modes	Number in ROM
1	1	0
2	2	1
3	3	0
4	4	1
5	5	1
6	6	0
7	7, ..., 15	4
8	16, ..., 34	8
9	35, ..., 38	0
10	39, 40, 41	0
11	42,43,44,46,49,50	0
12	45, 47, 48	0