A COMPARISON OF VISCOELASTIC DAMPING MODELS

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Modern finite element methods (FEMs) enable the precise modeling of mass and stiffness properties in what were in the past overwhelmingly large and complex structures. These models allow the accurate determination of natural frequencies and mode shapes. However, adequate methods for modeling highly damped and highly frequency dependent structures did not exist until recently. The most commonly used method, Modal Strain Energy[1,2], does not correctly predict complex mode shapes since it is based on the assumption that the mode shapes of a structure are real. Recently, many techniques have been developed which allow the modeling of frequency dependent damping properties of materials in a finite element compatible form. Two of these methods, the Golla-Hughes-McTavish[3,4] method and the Lesieutre-Mingori[5,6] method, model the frequency dependent effects by adding coordinates to the existing system thus maintaining the linearity of the model. The third model, proposed by Bagley and Torvik[7], is based on the Fractional Calculus method and requires fewer empirical parameters to model the frequency dependence at the expense of linearity of the governing equations. This work examines the Modal Strain Energy, Golla-Hughes-McTavish and Bagley and Torvik models and compares them to determine the plausibility of using them for modeling viscoelastic damping in large structures.

THE MODAL STRAIN ENERGY MODEL

The most common method used for the modeling of viscoelastic damping in structures presently is the Modal Strain Energy method suggested by Ungar and Kerwin[2]. This method assumes that proportional damping (*Rayleigh Damping*) is an adequate model of the damping mechanisms of a structure. This implies that the modes of the damped structure are the same as that of the undamped structure.

Modal Strain Energy begins with the complex stiffness representation of material damping properties. In this representation, the complex stiffness $K^* = K' + K''j$ where j represents the square root of -1, and K' and K'' are the real and imaginary parts of the complex stiffness, respectively. The ratio K''/K' is the material loss factor. A more detailed description of the complex representation is given by Nashif, Jones and Henderson[8].

The loss factor of any mode i is given by the summation of the strain energy in each element, multiplied by its material loss factor, and divided by the total strain energy of the mode, i.e.,

$$\eta^i = \sum_{j=1}^{n} \frac{V_j^i \eta_j^i}{V^i} \tag{1}$$

The variable η^i is the loss factor of the i^{th} mode, η_j^i is the loss factor of the j^{th} element at the i^{th} natural frequency, V^i is the strain energy of the i^{th} mode at a given amplitude, and V_j^i is the strain energy in the jth element when the structure is deformed in the i^{th} mode shape at the same amplitude. The strain energy V^i in a structure or element with the stiffness matrix defined by K and the deformation defined by x is

$$V = x^T K x \tag{2}$$

Since the imaginary part of the global stiffness matrix is the assembly of the imaginary parts (K'') of the elemental stiffness matrices, equation (1) may be written

$$\eta^i = \frac{\mathbf{x}_i^T K'' \mathbf{x}_i}{\mathbf{x}_i^T K' \mathbf{x}_i} \tag{3}$$

where K' is the real part of the global stiffness matrix and is denoted as K for the undamped and viscously damped systems. Note that this is precisely true in the case of the single degree of freedom system. This is a useful representation of the modal strain energy equation and will be referred to repeatedly.

Although intuitively the concept of using energy ratios weighted by element loss factors is appealing, it has no theoretical basis. In the past there has not been an explanation of why modal strain energy is correct when the imaginary part of the stiffness matrix is proportional to the mass and stiffness matrices. It can be shown that modal strain energy is nothing more than the modal decoupling of a viscoelastic system where it is assumed that the imaginary part of the stiffness matrix must obey $K''/\omega = C$.

The equations of motion for an unforced viscously damped multiple degree of freedom (MDOF) system may be written as

$$M\ddot{y} + C\dot{y} + Ky = 0 \tag{4}$$

Assuming a solution of the form

$$y = u\, e^{i\omega t} \tag{5}$$

then substituting (5) into (4) gives

$$-M\omega^2 u + C\, i\, \omega u + K u = 0 \tag{6}$$

The system equations written in terms of complex modulus corresponding to (4) are then

$$M\ddot{y} + (K' + K''i)\, y = 0 \tag{7}$$

Substituting (5) into (7) similarly gives

$$-M\omega^2 u + (K' + K''i)\, u = 0 \tag{8}$$

Comparing (6) and (8), it is seen that for any given frequency, ω_j,

$$C\,\omega_j = K'' \tag{9}$$

which is the multiple degree of freedom representation of $c=k\eta/\omega$.

Likewise, substituting

$$y = P\,v \tag{10}$$

into (4), where P is the matrix of normalized eigenvectors of $M^{-1}K'$ and premultiplying by P^T, equation (4) becomes

$$P^T M P \ddot{v} + P^T C P \dot{v} + P^T K P v$$

$$= diag\,(m_i)[\ddot{v} + diag(2z_i w_i)\dot{v} + diag\,(w_i^2)v] = 0 \tag{11}$$

where m_i are the modal masses. This is true if and only if C is proportional to M and K. Likewise, for the complex system, substituting (9) and (10) into (7), and premultiplying by P^T gives

$$P^T M P \ddot{v} + P^T K' P v + P^T C\,\omega_j P\,i\,\dot{v}$$

$$= diag(m_i)\,[\,\ddot{v} + (\,diag(\omega_i^2) + diag(2\,\zeta_i\,\omega_i)\,\omega_j\,i\,)\,v\,] = 0 \tag{12}$$

again if and only if C is proportional. This is identical to requiring that K'' be proportional (i.e., $K''/\omega = \alpha M + \beta K$). From (12) it can be seen that

$$(\,P^T K' P\,)^{-1}\,(\,P^T C\,\omega_j P\,) = diag(2\,\zeta_i) = diag(\eta^i) = H \tag{13}$$

when $\omega_j = \omega_i$. However, using (9) gives

$$(\,P^T K' P\,)^{-1}\,(\,P^T K'' P\,) = diag(2\,\zeta_i) = diag(\eta^i) = H \tag{14}$$

Denoting the i^{th} eigenvector as x_i, equation (14) may be written as

$$\eta^i = \frac{\mathbf{x}_i^T K'' \mathbf{x}_i}{\mathbf{x}_i^T K' \mathbf{x}_i} \tag{15}$$

which is identical to equation (3). Therefore, the same rules which apply to decoupling viscously damped systems apply to the proper use of modal strain energy. For non-proportionally damped systems, the matrix H defined by (14) will be non-diagonal and will give some indication of the non-proportionality of the system.

Thus, the modal strain energy technique is nothing more than the modal decoupling of a system with complex modulus damping. Also, the criteria used to define whether or not modal strain energy is a proper method for finding loss factors of a structure described by a complex modulus have been shown to be identical to those for decoupling a viscously damped system.

THE GOLLA-HUGHES-MCTAVISH MODEL

The Golla-Hughes-McTavish (GHM) model is based upon the generalized standard linear model; however, it has been developed for direct incorporation into the finite element method. In the GHM model, the material complex modulus is written in the Laplace domain in the form

$$E^*(s) = E_0(1 + h(s)) = E_0\left(1 + \sum_{n=1}^{k} \hat{a}_n \frac{s^2 + 2\hat{\xi}_n\hat{\omega}_n s}{s^2 + 2\hat{\xi}_n\hat{\omega}_n s + \hat{\omega}_n^2}\right) \tag{16}$$

where the hatted terms are free variables for curve fitting to complex modulus data and s is the Laplace domain operator. From (16) it can be seen that $E^*(\omega) = E_0$ for $j\omega = 0$ which means that no creep is allowed in this model. Also, the number of expansion terms, k, may be modified to represent the high or low frequency dependence of the complex modulus. In general between two and four terms are adequate.

The finite element form of the GHM model for a single modulus and single expansion term is

$$\begin{bmatrix} M & 0 \\ 0 & \dfrac{\hat{\alpha}}{\hat{\omega}^2} E_0 \Lambda_e \end{bmatrix} \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{z}(s) \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{2\hat{\alpha}\hat{\xi}}{\hat{\omega}} E_0 \Lambda_e \end{bmatrix} \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{z}(s) \end{bmatrix} s$$

$$+ \begin{bmatrix} (1+\hat{\alpha})E_0\tilde{K} & -\hat{\alpha}E_0 R_e \Lambda_e \\ -\hat{\alpha}E_0 \Lambda_e R_e^T & \hat{\alpha}E_0 \Lambda_e \end{bmatrix} \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{z}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{F}(s) \\ \mathbf{0} \end{bmatrix} \tag{17}$$

where M is the original mass matrix, $E_0\tilde{K}$ is the original element stiffness matrix, Λ_e is the diagonal matrix of the non zero eigenvalues of $\tilde{K}$, and R_e is the matrix of the eigenvector associated with the eigenvalues of Λ_e. Even for the most complex linear elements, the GHM finite element remains linear and second order. Although the GHM system state

equations are much larger than the original undamped equations, the GHM state matrix is only slightly larger than the state matrix for a viscously damped system. Where the size of the state matrix for a viscously damped system is 2n x 2n, the state matrix of the GHM model is m x m where

$$m = \sum_{i=1}^{n} k_i p_i \tag{18}$$

Here n represents the number of viscoelastic elements, and pi and k_i represent the number of non-zero eigenvalues and number of expansion terms used in the i^{th} element. One drawback of this method which may be overcome simply is the addition of fictitious overdamped modes. These should be recognized as fictitious and discarded.

FRACTIONAL CALCULUS - THE BAGLEY AND TORVIK MODEL

The Bagley and Torvik fractional calculus viscoelastic model has been proposed based on the observations of Nutting[9],Gemant[10,11], Caputo[12,13], Caputo and Minardi[14], and Scott-Blair[15] that the mechanical properties of viscoelastic materials seem to vary as a function of frequency raised to fractional powers. In the time domain, this represents fractional derivatives as defined by

$$\frac{d^a}{dt^a}x(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{x(\tau)}{(t-\tau)^a}dt \qquad 0 < a < 1 \tag{19}$$

where α represents the power of the derivative, Γ is the gamma function, and τ is a dummy variable of integration. This in turn can be represented in the Laplace domain as

$$\pounds\left\{\frac{d^a}{dt^a}x(t)\right\} = s^a x(s) \tag{20}$$

where $\pounds$ represents the Laplace transform operator. The general form of the fractional derivative model is then

$$\sigma + \sum_{n=1}^{\infty} b_n \frac{d^{\beta_n}}{dt^{\beta_n}}\sigma = E\varepsilon + \sum_{m=1}^{\infty} E_m \frac{d^{\beta_m}}{dt^{\beta_m}}\varepsilon \tag{21}$$

The experimental results of Bagley and Torvik demonstrated that, for many materials, the stress-strain relation can be modeled well using only the first expansion term in each series. In the Laplace domain, the Bagley and Torvik viscoelastic model is

$$\sigma(s) = \frac{E_0 + E_1 s^{\alpha}}{1 + b s^{\beta}} \varepsilon(s) = \mu(s) \varepsilon(s) \tag{22}$$

where $\mu(s)$ represents the complex modulus in the Laplace domain. In order to solve the final equations, α and β are restricted to fractional form. In the interest of brevity, no derivation is shown. One may be found in Bagley and Torvik[7].

The final form of the equations of motion are

$$\left\{B_1 s^{1/m} + B_2\right\}\mathbf{y}(s) = \tilde{\mathbf{F}}(s) \tag{23}$$

where m is the smallest common denominator of α and β, B_1 and B_2 are matrices of order $nm(2 + b)$ x $nm(2 + b)$, and $\mathbf{y}(s)$ and $\tilde{\mathbf{F}}(s)$ are the appropriate state vector and forcing function vector respectively. Equation (23) may be posed as an eigenvalue/eigenvector problem (setting $\tilde{\mathbf{F}}(s) = \mathbf{0}$) in order to solve for $\mathbf{y}(s)$ and $s^{1/m}$. The system eigenvalues, s, and the system eigenvectors, $\mathbf{y}(s)$, may then be found using (23). However, notice that the order of the system is dramatically increased. For a second order viscously damped system, the size of the eigenvalue problem is $2n$. The size of the Bagley and Torvik eigenvalue problem is $[m(2+\beta)]\ n$. For the simplest possible Bagley and Torvik viscoelastic model, with $a = b = 1/2$, the order of the system is already $5n$!. This will take 6.25 times more memory and about four times as much time to calculate. Note that in the strictest sense this is not a finite element method, since no viscoelastic element has been developed which could be assembled into an existing finite element model in order to create a global FEM model. Another drawback of this method is the occurrence of unstable eigenvalues as described by Bagley and Torvik[7]. Although these may be disregarded when only interested in mode shapes and loss factors of modes, the forced response of this model would be unstable, which does not agree well with the real behavior of viscoelastic materials.

It should be noted that much work has been done by Morgenthaler[16] of Martin Marietta using the Bagley and Torvik Model on the PACOSS program. The essence of his work is a numerical algorithm incorporating the accelerated subspace iteration technique to the complex modulus problem. Although the initial form of the stiffness matrix is assumed to be the fractional derivative, the algorithm's first step is to evaluate the complex stiffness

matrix at a frequency near the desired natural frequency. Then the desired natural frequency is found, the complex stiffness matrix is evaluated at the new frequency, and the desired natural frequency is found. The procedure iterates until the desired accuracy is reached, although it is mentioned that one iteration may be enough in many cases. The final state matrix is found by recoupling the eigenvalues and eigenvectors.

This method discards the benefit of the Bagley and Torvik model by evaluating the frequency dependent stiffness matrix instead of solving the complete set of frequency dependent equations derived by Bagley and Torvik[7]. This is not bad if all you are interested in are the correct mode shapes and eigenvalues. However, if that is the only goal, then there is no need to use the fractional derivative model. Any other Laplace domain representation which fits the modulus in the frequency domain would work equally as well. There is no reason not to simply use material data sheets directly to evaluate the complex stiffness matrix and avoid the curve fitting altogether. The end result of this method may model the system well, but it does not incorporate all of the frequency dependence of the materials in the final model. Any structural modification requires the complete recalculation of all of the desired eigenvalues and eigenvectors in order to find the new state space matrix, where the GHM method simply requires the assembly of a new element into the existing finite element model.

AN EXAMPLE - THE EVOLUTIONARY VISCO-STRUT

These three models have each been used to model a viscoelastic strut designed for use in the evolutionary model at NASA Langley. The Visco-Strut is a load bearing member capable of supporting tensile and compressive forces in excess of 2000 lbs. In its present configuration, it has a static stiffness on the order of 30,000 lb/in. The viscoelastic material used is G.E. SMRD, manufactured by the General Electric Astro Space Division. Both the GHM and the Bagley and Torvik models have proven to be capable of modeling the frequency dependent complex modulus of the Visco-Strut well, while the Modal Strain Energy method simply uses the raw damping data for any element and therefore is not constrained by the need to curve fit. The results of the curve fitting for the GHM and Bagley and Torvik models are shown below in figures 1 and 2.

For the GHM model, $K_0 = 3.11 \times 10^4$. The remaining parameters are shown in Table 1.

	i=1	i=2	i=3
α_i	2.29	.520	.319
ζ_i	9.83×10^{14}	2.58×10^{23}	7.97×10^{20}
ω_i	1.91×10^{18}	4.44×10^{25}	2.17×10^{22}

Table 1. GHM Parameters for the Visco-Strut.

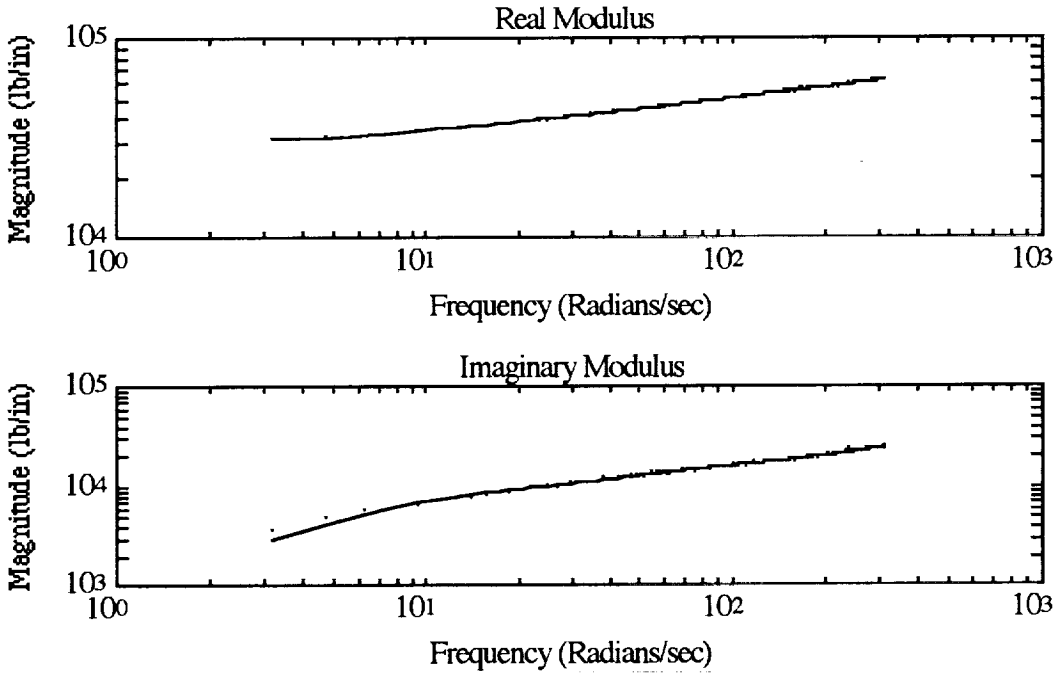

Figure 1. Comparison of the GHM model (solid line) of the complex modulus and test data (dots).

For the Bagley and Torvik model the parameters are K_0=2.86x10^4, K_1=3.6163x10^3, b=1.9028x10^{-2}, and $\alpha=\beta=1/2$.

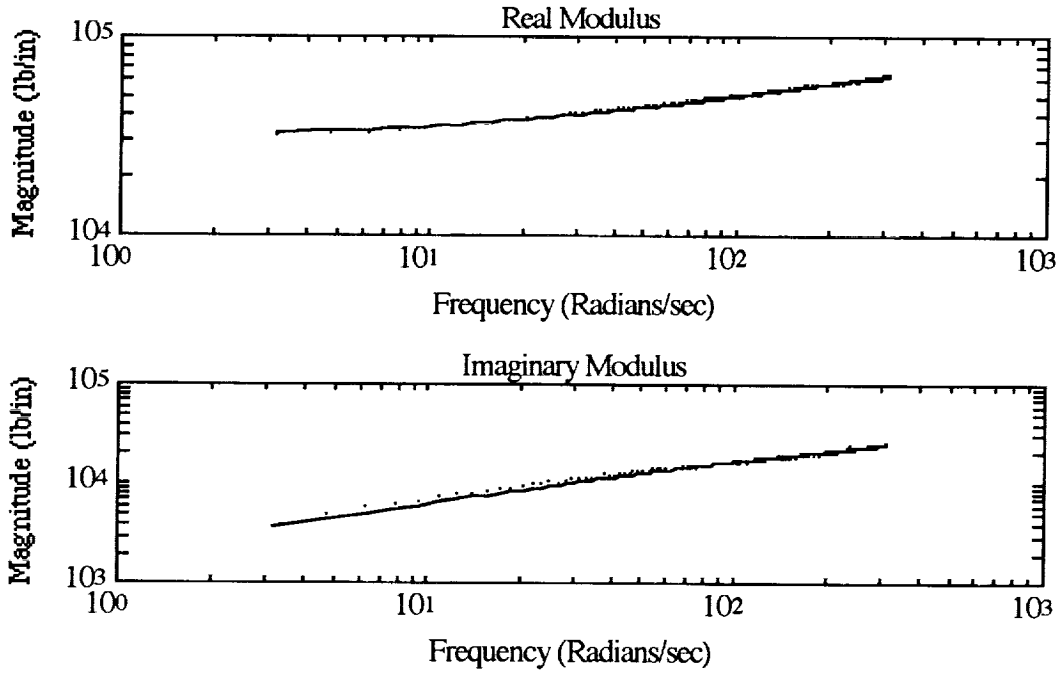

Figure 2. Comparison of the Bagley and Torvik model (solid line) of the complex modulus and test data (dots).

Some first attempts have been taken to model the effects of the Visco-Strut when placed in a small nine bay truss. To date, the only model which has been solvable is the MSE model. The Modal Strain Energy method is simple and quick because it uses the results from the dynamic model of the structure to find modal loss factors using equation (1) or (3). It has been shown by Johnson and Kienholz[1] to correctly find loss factors even when the damping is not proportional. Both the GHM and Bagley and Torvik model solutions have encountered numerical difficulty. The GHM global FEM is ill-conditioned, while the size of the Bagley and Torvik model (1185 x 1185) has caused significant numerical errors. Neither method has yielded useful results for this problem.

CONCLUSIONS

The most robust method for determining modal loss factors is clearly the Modal Strain Energy method. If the system is proportionally damped (as determined by the matrix H in equation (14) being diagonal) then there is no need to use either the GHM or Bagley and Torvik method. Even when the structure is non-proportionally damped, there is little benefit to using either of the higher powered models unless there is a need to predictively model the mode shapes of the damped structure, or model the response of the structure to different excitations. However, if precise modeling of the response is necessary, one must decide whether the GHM modal will become too ill-conditioned for solution, or whether the large increase in the model size using Bagley and Torvik is acceptable.

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