A Fuzzy Controller with Nonlinear Control Rules Is the Sum of a Global Nonlinear Controller and a Local Nonlinear PI-like Controller

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Abstract

The fuzzy controllers studied in this paper are the ones that employ N trapezoidal-shaped members for input fuzzy sets, Zadeh fuzzy logic and a centroid defuzzification algorithm for output fuzzy set. The author analytically proves that the structure of the fuzzy controllers is the sum of a global nonlinear controller and a local nonlinear proportional-integral-like controller. If N approaches ∞, the global controller becomes a nonlinear controller while the local controller disappears. If linear control rules are used, the global controller becomes a global two-dimensional multilevel relay which approaches a global linear proportional-integral (PI) controller as N approaches ∞.

1. Introduction

Efforts have been made to clarify the fuzzy controller structures. The structure of a nonlinear fuzzy controller was revealed using a novel method (Ying, 1987; Ying et al., 1990). The work showed that a simplest possible nonlinear fuzzy controller was equivalent to a nonlinear PI controller. In (Ying, 1991), the author analytically proved that the structure of a typical nonlinear fuzzy controllers with linear fuzzy control rules is the sum of a global two-dimensional multilevel relay and a local nonlinear PI controller. The author makes further efforts in this paper to investigate the structure of fuzzy controllers using any type of fuzzy control rules, covering a much broader range of fuzzy controllers.

2. Analytical Analysis of the Structure of the Fuzzy Controllers

2.1 Components of the Fuzzy Controllers

A fuzzy controller usually employs error and rate change of error (rate, for short) about a setpoint as its inputs. That is

$$e^* = GE \cdot e(nT) = GE[y(nT) - \text{setpoint}] \tag{2.1}$$
$$r^* = GR \cdot r(nT) = GR[e(nT) - e(nT-T)] \tag{2.2}$$

where e(nT), r(nT) and y(nT) designate error, rate, and process output at sampling time nT (T is sampling period), respectively. Error at sampling time (n-1)T is specified as e(nT-T). The setpoint is the desired target of the process output and GE and GR are the scalars for the error and rate.

Input fuzzy sets, "error" and "rate," are obtained by fuzzifying e^* and r^*. Assume there are J (J ≥ 1) members for positive "error" ("rate"), J members for negative "error" ("rate") and one member for zero "error" ("rate"). Therefore, there are total

$$N = 2J+1 \qquad\qquad (2.3)$$

members for the fuzzy set "error" ("rate"). Members of "error" ("rate") are represented as E_i (R_i) where -J ≤ i ≤ J. The membership functions corresponding to these members are denoted as $\mu_i(x)$ which has a central value λ_i. Define λ_{-J}=-L, λ_0=0, and λ_J=L. Let the space between the central values of two adjacent members be equal. Then the space, denoted as S, is:

$$S = \frac{L}{J} \qquad\qquad (2.4)$$

and consequently the central value of $\mu_i(x)$ is $\lambda_i = i \cdot S$.

The $\mu_i(x)$ in this study is the commonly-used trapezoidal-shaped membership function. Assume the membership functions for "error" and "rate" are identical, and specifically denote $\mu_i(e^*)$ as the membership function for E_i and $\mu_i(r^*)$ as the membership function for R_i. The trapezoidal-shaped membership function $\mu_i(x)$ satisfies the following two conditions:

(1) For -J+1 ≤ i ≤ J-1, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (2.5)

$$\mu_i(x) = \begin{cases} 0, & x < (i-1)S \\ \dfrac{1}{S-A}[x-(i-1)S], & (i-1)S \le x \le i \cdot S - A \\ 1, & i \cdot S - A \le x \le i \cdot S + A \\ -\dfrac{1}{S-A}[x-(i+1)S], & i \cdot S + A \le x \le (i+1)S \\ 0, & x > (i+1)S \end{cases}$$

(2) For i = J or i=-J,

$$\mu_J(x) = \begin{cases} 0, & x < (J-1)S \\ \dfrac{1}{S-A}[x-(J-1)S], & (J-1)S \le x \le J \cdot S - A \\ 1, & J \cdot S - A \le x < +\infty \end{cases}$$

$$\mu_{-J}(x) = \begin{cases} 1, & -\infty < x \le -J \cdot S + A \\ -\dfrac{1}{S-A}[x - (-J+1)S], & -J \cdot S + A \le x \le (-J+1)S \\ 0, & x > (-J+1)S. \end{cases}$$

An illustration of the definition is given in Fig. 1.
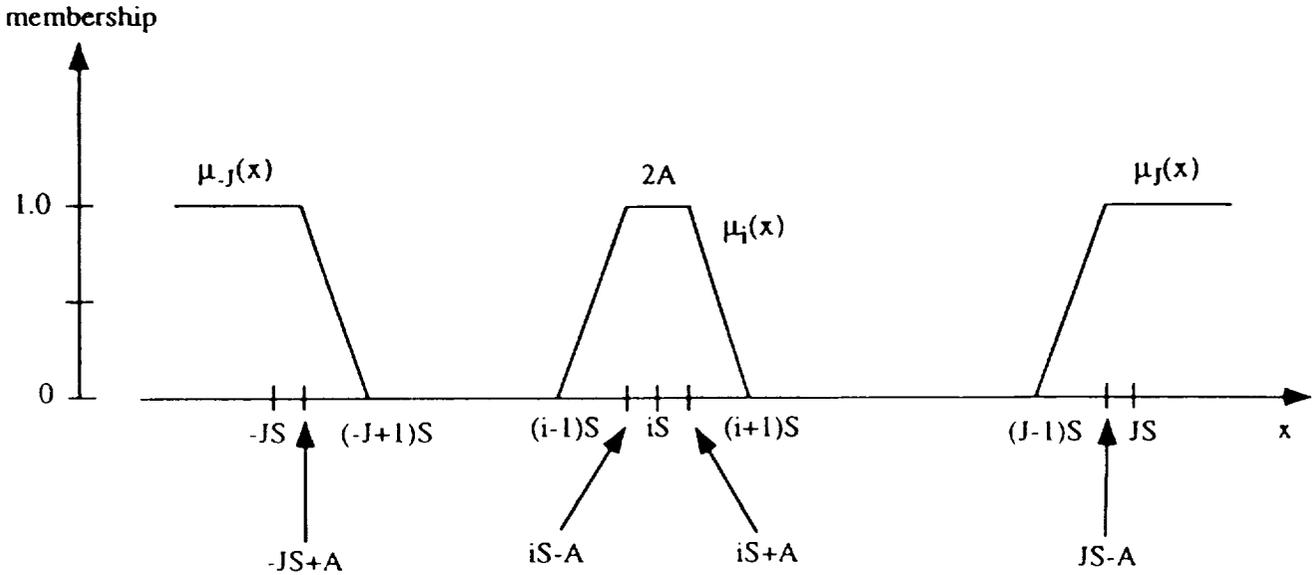
membership

Figure 1. Illustration of the definition of the trapezoidal-shaped membership function.

Denote U_k as a member of the output fuzzy set "incremental output" ("output," for short) and assume there are

$$M = 2K + 1 \tag{2.6}$$

such members where

$$K = \text{Max}\{|f(i, j)|\}. \tag{2.7}$$

f will be described below. The central values of the members of the fuzzy set "output" are designated as γ_k ($-K \le k \le K$) and let $\gamma_{-K}=-H$, $\gamma_0=0$ and $\gamma_K=H$. Further, let the space between the central values of two adjacent members be equal. Consequently, the space, denoted as V, is

$$V = \frac{H}{K} \tag{2.8}$$

and the central value of a member of "output," U_k, can be written as $\gamma_k = k \cdot V$. The membership functions of "output" are required to be regular, unimodal and symmetrical about its central value γ_k. The shape of the membership functions of all the members is identical.

N^2 fuzzy control rules are constructed according to the following rule:

IF "error" is E_i and "rate" is R_j THEN "output" is U_k $\hspace{2cm}$ (2.9)

where $k = f(i, j)$. f, determined by the constructors of the fuzzy controllers, may be any function as long as its value is always an integer with respect to the inputs, i and j, because the index k must be an integer.

Zadeh fuzzy logic AND is used to execute the IF side of the fuzzy control rule in (2.9). That is,

$$\mu(i, j) = Min(\mu_i(e^*), \mu_j(r^*)) \hspace{2cm} (2.10)$$

where $\mu(i, j)$ is the membership for the member U_k obtained when E_i and R_j are in the IF side. The center of gravity of defuzzification algorithm is used. The scaled crisp incremental output, denoted as $GU \cdot \Delta u(nT)$, is calculated as

$$GU \cdot \Delta u(nT) = GU \frac{\sum \mu(i, j)\gamma_k}{\sum \mu(i, j)} \hspace{2cm} (2.11)$$

where GU is the scalar for the incremental output.

2.2 Main Results
Theorem 1.

The structure of the fuzzy controllers, constructed by the components defined in the above section, is the sum of a global nonlinear controller (denoted as $\Delta u_G(i, j)$) and a local nonlinear PI-like controller (denoted as $\Delta u_L(i, j)$).

Proof.

Without losing generality, assume that,

$$\begin{aligned} iS \leq e^* \leq (i+1)S \\ jS \leq r^* \leq (j+1)S. \end{aligned} \hspace{2cm} (2.12)$$

$\mu_i(e^*)$, $\mu_{i+1}(e^*)$, $\mu_j(r^*)$ and $\mu_{j+1}(r^*)$, which are the respective memberships for the members E_i, E_{i+1}, R_j and R_{j+1}, are obtained by fuzzifying e^* and r^*. Membership for all other members of "error" and "rate" is zero. Therefore, only the following four fuzzy control rules are executed:

If "error" is E_{i+1} and "rate" is R_{j+1} then "output" is U_{k1} $\hspace{2cm}$ (r1)
If "error" is E_{i+1} and "rate" is R_j $\hspace{0.6cm}$ then "output" is U_{k2} $\hspace{2cm}$ (r2)
If "error" is E_i $\hspace{0.6cm}$ and "rate" is R_{j+1} then "output" is U_{k3} $\hspace{2cm}$ (r3)
If "error" is E_i $\hspace{0.6cm}$ and "rate" is R_j $\hspace{0.6cm}$ then "output" is U_{k4} $\hspace{2cm}$ (r4)

where

$$k_1 = f(i+1, j+1), \quad k_2 = f(i+1, j), \quad k_3 = f(i, j+1) \quad \text{and} \quad k_4 = f(i, j).$$

Applying the equation (2.10) to each of the fuzzy control rules, we get

$$\mu(i+1, j+1) = \text{Min}(\mu_{i+1}(e^*), \mu_{j+1}(r^*)) \qquad (r1^*)$$

$$\mu(i+1, j) \quad = \text{Min}(\mu_{i+1}(e^*), \mu_j(r^*)) \qquad (r2^*)$$

$$\mu(i, j+1) \quad = \text{Min}(\mu_i(e^*), \mu_{j+1}(r^*)) \qquad (r3^*)$$

$$\mu(i, j) \qquad = \text{Min}(\mu_i(e^*), \mu_j(r^*)). \qquad (r4^*)$$

In order to decide the outcomes of the Min operations in ($r1^*$) to ($r4^*$), the author configures a square, which has 16 regions in it as shown in Fig. 2. In different regions, $\mu_i(e^*)$, $\mu_{i+1}(e^*)$, $\mu_j(r^*)$ and $\mu_{j+1}(r^*)$ have different relationships in terms of their magnitudes and consequently the Min operations in ($r1^*$) to ($r4^*$) can be evaluated. For example, in region IC3, the following inequalities can be obtained: $\mu_i(e^*) \geq \mu_j(r^*)$, $\mu_{i+1}(e^*) \leq \mu_{j+1}(r^*)$, $\mu_i(e^*) \leq \mu_{j+1}(r^*)$ and $\mu_j(r^*) \leq \mu_{i+1}(e^*)$. As a result, $\mu(i+1, j+1) = \mu_{i+1}(e^*)$, $\mu(i+1, j) = \mu_j(r^*)$, $\mu(i, j+1) = \mu_i(e^*)$ and $\mu(i, j) = \mu_j(r^*)$, based on ($r1^*$) to ($r4^*$). Similarly, ($r1^*$) to ($r4^*$) for the rest of 15 regions can be evaluated.
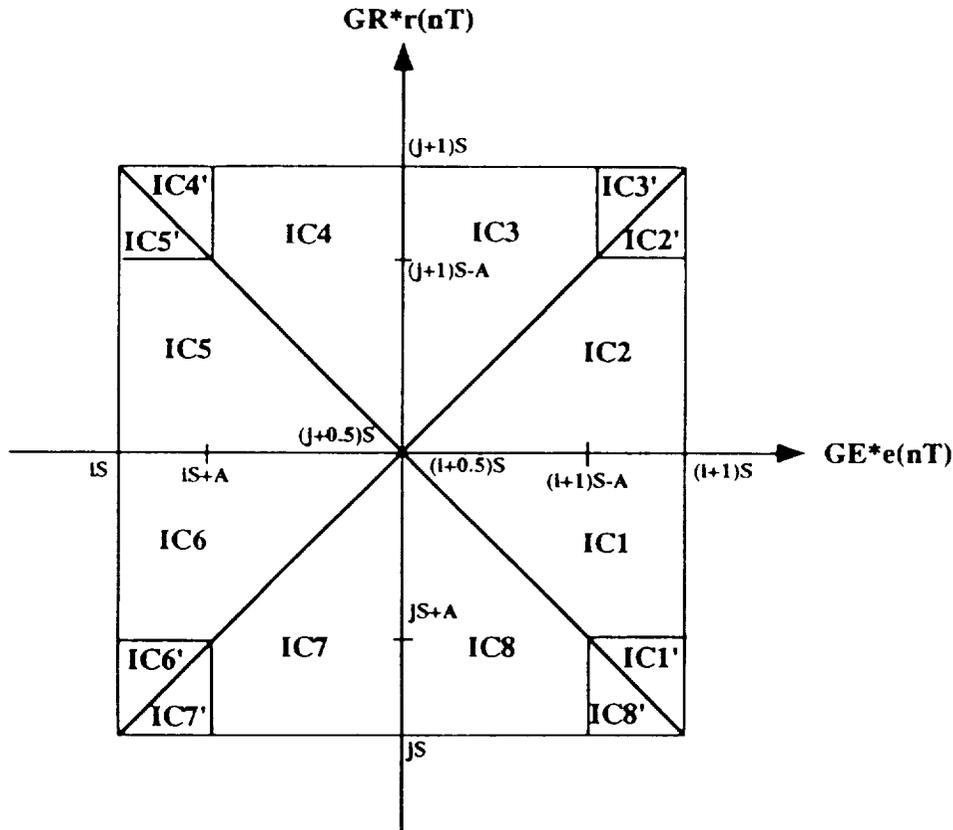
Figure 2. Possible input combinations (IC) of scaled error, e^* (GE·e(nT)), and scaled rate change of error, r^* (GR·r(nT)), of process output when both e^* and r^* are within the interval [-L, L].

Substituting these outcomes into the defuzzification algorithm (2.11) and simplifying the resulting expression, GU·Δu(nT) for the 16 regions can be found. To illustrate this procedure more clearly , let us take region IC3 again as an example.

Substituting $\mu(i+1, j+1)$, $\mu(i+1, j)$, $\mu(i, j+1)$ and $\mu(i, j)$ for the IC3 region into (2.11),

$$GU \cdot \Delta u(nT) = \frac{k_1 \mu_{i+1}(e^*) + k_2 \mu_j(r^*) + k_3 \mu_i(e^*) + k_4 \mu_j(r^*)}{\mu_{i+1}(e^*) + \mu_j(r^*) + \mu_i(e^*) + \mu_j(r^*)} V \cdot GU \qquad (2.13)$$

$$= k_3 \cdot V \cdot GU + \frac{(k_1 - k_3)\mu_{i+1}(e^*) + (k_2 + k_4 - 2k_3)\mu_j(r^*)}{\mu_{i+1}(e^*) + \mu_i(e^*) + 2\mu_j(r^*)} V \cdot GU$$

$$= f(i, j+1)V \cdot GU + \{K_i[e(nT) - \frac{(i+0.5)S}{GE}] + K_p[r(nT) - \frac{(j+0.5)S}{GR}] + \varepsilon\}$$

where

$$K_p = \frac{(2f(i, j+1) - f(i+1, j) - f(i, j))V \cdot GR \cdot GU \cdot S}{2S - 2[GR \cdot r(nT) - (j+0.5)S]}$$

$$K_i = \frac{(f(i+1, j+1) - f(i, j+1))V \cdot GE \cdot GU \cdot S}{2S - 2[GR \cdot r(nT) - (j+0.5)S]}$$

$\varepsilon = 0$.

Denote $f(i, j+1)V \cdot GU$ as $\Delta u_G(i, j)$ and the rest of the expression as $\Delta u_L(i, j)$. $\Delta u_G(i, j)$ is a global nonlinear controller because it calculates control action with respect to i and j. $\Delta u_L(i, j)$ is a local nonlinear PI-like controller because it calculates control action according to the relative position of the current input state (e(nT), r(nT)) with respect to a dynamically changing point, ((i+0.5)S/GE, (j+0.5)S/GR). K_p and K_i are the proportional-gain and integral-gain. ε is nonzero in some IC regions.

Similar proof can be conducted for the rest of 15 regions.

Theorem 2 (General Limit Theorem for Control Rules).

When N approaches ∞,

$$(2.14)$$

(1) $\Delta u_L(i, j) = 0$

and

(2) $\Delta u_G(i, j)$ becomes

$$\underset{i,j,J \to \infty}{\text{Lim}} \frac{f(i, j) \cdot H \cdot GU}{K} \qquad (2.15)$$

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Proof.

Proof is trivial.

■

Theorem 3.

If linear control rules are used, i.e., if $f(i, j) = -(i + j)$, then

(1)   The global nonlinear controller becomes a global two-dimensional multilevel relay

$$\Delta u_G(i, j) = -(i + j + 1)\frac{H \cdot GU}{N - 1}.$$ (2.16)

(2)   As N approaches ∞, the global two-dimensional multilevel relay becomes a global linear PI controller:

$$GU \cdot \Delta u(nT) = -(K_i \cdot e(nT) + \overset{\bullet}{K}_p \cdot r(nT))$$ (2.17)

where

$$K_p = \frac{GR \cdot GU \cdot H}{2L}$$ (2.18)

$$K_i = \frac{GE \cdot GU \cdot H}{2L}.$$

Proof.
(1) $K = \text{Max}\{|f(i, j)|\} = 2J = N - 1$. $f(i+1, j) = f(i, j+1) = -(i + j + 1)$. Therefore,

$$\Delta u_G(i, j) = -(i + j + 1)\frac{H \cdot GU}{N - 1}.$$ (2.19)

■

(2)   See (Ying, 1991) for proof.

3.   Conclusions

With fuzzy control rules being expressed by a function f, the author has been able to analytically reveal the structure of the fuzzy controllers. The structure is the sum of a global nonlinear controller and a local nonlinear PI-like controller.

The work accomplished in this paper furthers understanding on the nature of fuzzy controllers. Fuzzy controllers generally are nonfuzzy nonlinear controllers. Therefore, nonlinear control theory can be utilized to solve fuzzy control problems, such as stability.

References

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