NONLINEAR RESCALING OF CONTROL VALUES SIMPLIFIES FUZZY CONTROL

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Abstract. Traditional control theory is well-developed mainly for linear control situations. In non-linear cases there is no general method of generating a good control, so we have to rely on the ability of the experts (operators) to control them. If we want to automate their control, we must acquire their knowledge and translate it into a precise control strategy.

The experts' knowledge is usually represented in non-numeric terms, namely, in terms of uncertain statements of the type "if the obstacle is straight ahead, the distance to it is small, and the velocity of the car is medium, press the brakes hard". Fuzzy control is a methodology that translates such statements into precise formulas for control. The necessary first step of this strategy consists of assigning membership functions to all the terms that the expert uses in his rules (in our sample phrase these words are "small", "medium", and "hard").

The appropriate choice of a membership function can drastically improve the quality of a fuzzy control. In the simplest cases, we can take the functions whose domains have equally spaced endpoints. Because of that, many software packages for fuzzy control are based on this choice of membership functions. This choice is not very efficient in more complicated cases. Therefore, methods have been developed that use neural networks or genetic algorithms to "tune" membership functions. But this tuning takes lots of time (for example, several thousands iterations are typical for neural networks).

In some cases there are evident physical reasons why equally spaced domains do not work: e.g., if the control variable u is always positive (i.e., if we control temperature in a reactor), then negative values (that are generated by equal spacing) simply make no sense. In this case *it sounds reasonable to choose another scale* $u' = f(u)$ *to represent* u, *so that equal spacing will work fine for* u'.

In the present paper we formulate the problem of finding the best rescaling function, solve this problem, and show (on a real-life example) that after an optimal rescaling, the un-tuned fuzzy control can be as good as the best state-of-art traditional non-linear controls.

1. INTRODUCTION TO THE PROBLEM

Traditional control theory is not always applicable, so we have to use fuzzy control. Traditional control theory is well-developed mainly for linear control situations. In non-linear cases, although for many cases there are good recipes, there is still no general method of generating a good control (see, e.g., [M91]).

Therefore, we have to rely on the ability of the experts (operators) to control these systems. If we want to automate their control, we must acquire transform their knowledge it into a precise control strategy.

The experts' knowledge is usually represented in non-numeric terms, namely, in terms of uncertain statements of the type "if the obstacle is straight ahead, the distance to it is small, and the velocity of the car is medium, press the brakes hard". Fuzzy control is a methodology that

translates such statements into precise formulas for control. Fuzzy control was started by L. Zadeh and E. H. Mamdani [Z71], [CZ72], [Z73], [M74] in the framework of fuzzy set theory [Z65]. For the current state of fuzzy control the reader is referred to the surveys [S85], [L90] and [B91].

Choice of membership functions: an important first step of fuzzy control methodology. The necessary first step of this methodology consists of assigning membership functions to all the terms that the expert uses in his rules (in our sample phrase these words are "small", "medium", and "hard"). The appropriate choice of a membership function can drastically improve the quality of a fuzzy control.

Simplest case: equally spaced functions. In the simplest cases, we can take the functions whose domains have equally spaced endpoints: e.g., we can fix a neutral value N (usually, $N = 0$), and a number Δ, and take "negligible" with the domain $[N - \Delta, N + \Delta]$, "small positive" with the domain $[N, N + 2\Delta]$, "medium positive" with the domain $[N + \Delta, N + 3\Delta]$, etc. Correspondingly, "small negative has the domain $[N - 2\Delta, N]$, "medium negative corresponds to the domain $[N - 3\Delta, N - \Delta]$, etc. If an interval $[a - \Delta, a + \Delta]$ is given, then we can take a membership function $\mu(x)$ that is equal to 0 outside this interval, equal to 1 for $x = a$, and is linear on the intervals $[a - \Delta]$ and $[a, a + \Delta]$. Many software packages for fuzzy control are based on this choice of membership functions.

What is usually done in more complicated cases. This choice of equally spaced functions is not always very efficient in more complicated cases. Therefore, methods have been developed that use neural networks or genetic algorithms to "tune" membership functions (see, e.g., numerous papers in [RSW92]). But this tuning takes lots of time (for example, several thousands iterations are typical for neural networks).

The idea of a rescaling. In some cases there are evident physical reasons why equally spaced domains do not work. For example, if the control variable u is always positive (i.e., if we control the flow of some substance into a reactor), then negative values (that will be eventually generated by an equal spacing method) simply make no sense.

A natural idea is to choose another scale $u' = f(u)$ to represent the control variable u, so that equal spacing will work fine for u'. This idea is in good accordance with our common-sense description of physical processes. For example, from the physical viewpoint it is quite possible to describe the strength of an earthquake by its energy, but, when we talk about its consequences, it is much more convenient to use a logarithmic scale (called Richter scale). Non-linear scales are used to describe amplifiers and noise (decibels, in electrical engineering), to describe hardness of different minerals in geosciences, etc. (for a general survey of different scales and rescalings see [SKLT71, 89]).

In our case we want to design such a scale that for $f(u)$ the equally spaced endpoints $N - k\Delta$ and $N + k\Delta$ would make sense for all integers k. Therefore, we are looking for a function $f(u)$, whose domain is the set of all positive values, and whose range is all possible real numbers. In mathematical notations, f must map $(0, \infty)$ onto $(-\infty, \infty)$. There are lots of such functions, and evidently not all of them will improve the control. So we arrive at the following problem:

The main problem. *What rescaling to choose?*

What we are planning to do. We formulate the problem of choosing the best rescaling function $f(u)$ as a mathematical optimization problem, and then we solve this problem under some reasonable optimality criteria. As a result, we get an optimal function $f(u)$. We show that its application to non-linear systems really improves fuzzy control.

2. MOTIVATIONS OF THE PROPOSED MATHEMATICAL DEFINITIONS

Why is this problem difficult? We want to find a scaling function $f(u)$ that is the best in some reasonable sense, that is, for which some characteristic I attains the value that corresponds to the best performance of the resulting fuzzy control. As examples of such characteristics, we can take an average running time of an algorithm, or some characteristics of smoothness or stability of the resulting control, etc. The problem is that even for the simplest linear plants (controlled systems), we do not know how to compute any of these possible characteristics. How can we find $f(u)$ for which $I(f(u))$ is optimal if we cannot compute $I(f(u))$ even for a single function $f(u)$? There does not seem to be a likely answer.

However, we will show that this problem is solvable (and give the solution).

The basic idea for solving these kind of problems is described in [K90]; for its application to fuzzy logic see [KK90], to neural networks see [KQ91], to genetic algorithms see [KQF92], and to different problems of fuzzy control see [KQLFLKBR92].

We must choose a family of functions, not a single function. Suppose that for some physical quantity u (e.g., for x coordinate) equal spacing leads to a reasonably good control strategy.

In order to get numerical values of x coordinate, we must fix some starting point and some measuring unit (e.g., a meter). In principle we could as well choose feet to describe length. Then the numerical values of all the coordinates will be different (x meters are equal to λx feet, where λ is the number of feet in 1 meter). However, the intervals that were equally spaced when we used one unit, are still equally spaced, if we use another unit to measure this coordinate.

In a similar way, we could choose a different starting point for the x coordinate. If we take as a starting point a point that had a coordinate x_0 (so that now its coordinate is 0), then all other coordinates will be shifted: $x \to x - x_0$. Again intervals that were equal in the old scale (x) will still be equal if we measure then in the new scale ($x - x_0$).

We can also change both the measuring unit and the starting point. This way we arrive at a transformation $x \to \lambda x + x_0$.

Summarizing: if x is a reasonable scale, in the sense that equally spaced membership functions lead to a reasonably good control, then the same is true for any scale $\lambda x + x_0$, where $\lambda > 0$, and x_0 is any real number. The reason is that if we have a sequence of equally spaced intervals $[N + k\Delta, N + (k+1)\Delta]$, then these intervals will remain equally spaces after these linear rescalings $x \to \lambda x + x_0$: namely, these intervals will turn into intervals $[N' + k\Delta', N' + (k+1)\Delta']$, where $N' = \lambda N + x_0$ and $\Delta' = \lambda\Delta$.

Let us now consider a scale u, for which equal spacing does not work. Assume that $u \to f(u)$ is a transformation, after which equal spacing becomes applicable. This means that if we use $f(u)$ as a new scale, then equal spacings work fine. But as we have just shown, for any $\lambda > 0$ and x_0 equal spacing will also work fine for the scale $\lambda f(u) + x_0$.

Therefore, if $f(u)$ is a function that transforms the initial scale into a scale, for which equal spacing works fine, then for every $\lambda > 0$ and x_0 the function $f'(u) = \lambda f(u) + x_0$ has the same desired property.

This means that there is no way to pick one function $f(u)$, because with any function $f(u)$, the whole family of functions $\lambda f(u) + x_0$ has the same property. Therefore, desired functions form

a *family* $\{\lambda f(u) + x_0\}_{\lambda>0, x_0}$. Hence, instead of choosing a single function, we must formulate a problem of choosing a family.

Which family is the best? Among all such families, we want to choose the best one. In formalizing what "the best" means, we follow the general idea outlined in [K90] and applied to neural networks in [KQ91]. The criteria to choose may be computational simplicity, stability or smoothness of the resulting control, etc. In mathematical optimization problems, numeric criteria are most frequently used, where to every family we assign some value expressing its performance, and choose a family for which this value is maximal. However, it is not necessary to restrict ourselves to such numeric criteria only. For example, if we have several different families that lead to the same average stability characteristics T, we can choose between them the one that leads to the maximal smoothness characteristics P. In this case, the actual criterion that we use to compare two families is not numeric, but more complicated: *a family Φ_1 is better than the family Φ_2 if and only if either $T(\Phi_1) < T(\Phi_2)$, or $T(\Phi_1) = T(\Phi_2)$ and $P(\Phi_1) < P(\Phi_2)$*. A criterion can be even more complicated. What a criterion *must* do is to allow us for every pair of families to tell whether the first family is better with respect to this criterion (we'll denote it by $\Phi_2 < \Phi_1$), or the second is better ($\Phi_1 < \Phi_2$) or these families have the same quality in the sense of this criterion (we'll denote it by $\Phi_1 \sim \Phi_2$).

The criterion for choosing the best family must be consistent. Of course, it is necessary to demand that these choices be consistent, e.g., if $\Phi_1 < \Phi_2$ and $\Phi_2 < \Phi_3$ then $\Phi_1 < \Phi_3$.

The criterion must be final. Another natural demand is that this criterion must be *final* in the sense that it must choose a *unique* optimal family (i.e., a family that is better with respect to this criterion than any other family).

The reason for this demand is very simple. If a criterion does not choose any family at all, then it is of no use. If several different families are "the best" according to this criterion, then we still have a problem choosing the absolute "best" family. Therefore, we need some additional criterion for that choice. For example, if several families turn out to have the same stability characteristics, we can choose among them a family with maximal smoothness. So what we actually do in this case is abandon that criterion for which there were several "best" families, and consider a new "composite" criterion instead: Φ_1 is better than Φ_2 according to this new criterion if either it was better according to the old criterion, or according to the old criterion they had the same quality, and Φ_1 is better than Φ_2 according to the additional criterion. In other words, if a criterion does not allow us to choose a unique best family, it means that this criterion is not ultimate; we have to modify it until we come to a final criterion that will have that property.

The criterion must be reasonably invariant. We have already discussed the effect of changing units in a new scale $f(u)$. But it is also possible to change units in the original scale, in which the control u is described. If we use a unit that is c times smaller, then a control whose numeric value in the original scale was u, will now have the numeric value cu. For example, if we initially measured the flux of a substance (e.g., rocket fuel) into the reactor by kg/sec, we can now switch to lb/sec.

Comment. There is no physical sense in changing the starting point for u, because we consider the control variable that takes only positive values, and so 0 is a fixed value, corresponding to the minimal possible control.

We are looking for the universal rescaling method, that will be applicable to any reasonable situation (we do not want it to be adjustable to the situation, because the whole purpose of this rescaling is to avoid time-consuming adjustments). Suppose now that we first used kg/sec,

compared two different scaling functions $f(u)$ and $\tilde{f}(u)$, and it turned out that $f(u)$ is better (or, to be more precise, that the family $\Phi = \{\lambda f(u) + x_0\}$ is better than the family $\tilde{\Phi} = \{\lambda \tilde{f}(u) + x_0\}$). It sounds reasonable to expect that the relative quality of the two scaling functions should not depend on what units we used for u. So we expect that when we apply the same methods, but with the values of control expressed in lb/sec, then the results of applying $f(u)$ will still be better than the results of applying $\tilde{f}(u)$. But the result of applying the function $f(u)$ to the control in lb/sec can be expressed in old units (kg/sec) as $f(cu)$, where c is a ratio of these two units. So the result of applying the rescaling function $f(u)$ to the data in new units (lb/sec) coincides with the result of applying a new scaling function $f_c(u) = f(cu)$ to the control in old units (kg/sec). So we conclude that if $f(u)$ is better than $\tilde{f}(u)$, then $f_c(u)$ must be better than $\tilde{f}_c(u)$, where $f_c(u) = f(cu)$ and $\tilde{f}_c(u) = \tilde{f}(cu)$. This must be true for every c because we could use not only kg/sec or lb/sec, but arbitrary units as well.

Now we are ready for the formal definitions.

3. DEFINITIONS AND THE MAIN RESULT

Definitions. By a *rescaling function* (or a *rescaling* for short), we mean a strictly monotonic function that maps the set of all positive real numbers $(0, \infty)$ onto the set of all real numbers $(-\infty, +\infty)$. We say that two rescalings $f(u)$ and $f'(u)$ are *equivalent* if $f'(u) = Cf(u) + x_0$ for some positive constant C and for some real number x_0.

Comment. As we have already mentioned, if we apply two equivalent rescalings, we will get two scales that are either both leading to a good control, or are both inadequate.

By a *family* we mean the set of functions $\{Cf(u) + x_0\}$, where $f(u)$ is a fixed rescaling, C runs over all positive real numbers, and x_0 runs over all real numbers. The set of all families will be denoted by S.

A pair of relations $(<, \sim)$ is called *consistent* [K90], [KK90], [KQ91] if it satisfies the following conditions:
(1) if $F < G$ and $G < H$ then $F < H$;
(2) $F \sim F$;
(3) if $F \sim G$ then $G \sim F$;
(4) if $F \sim G$ and $G \sim H$ then $F \sim H$;
(5) if $F < G$ and $G \sim H$ then $F < H$;
(6) if $F \sim G$ and $G < H$ then $F < H$;
(7) if $F < G$ then it is not true that $G < F$ or $F \sim G$.

Assume a set A is given. Its elements will be called *alternatives*. By an *optimality criterion* we mean a consistent pair $(<, \sim)$ of relations on the set A of all alternatives. If $G < F$, we say that F is *better* than G; if $F \sim G$, we say that the alternatives F and G are *equivalent* with respect to this criterion. We say that an alternative F is *optimal* (or *best*) with respect to a criterion $(<, \sim)$ if for every other alternative G either $G < F$ or $F \sim G$.

We say that a criterion is *final* if there exists an optimal alternative, and this optimal alternative is unique.

Comment. In the present paper we consider optimality criteria on the set S of all families.

Definitions. By a *result of a unit change* in a function $f(u)$ to a unit that is $c > 0$ times smaller we mean a function $f_c(u) = f(cu)$. By the *result of a unit change* in a family Φ by $c > 0$ we mean the set of all the functions that are obtained by this unit change from $f \in \Phi$. This result will be

denoted by $c\Phi$. We say that an optimality criterion on F is *unit-invariant* if for every two families Φ and $\tilde{\Phi}$ and for every number $c > 0$ the following two conditions are true:

i) if Φ is better than $\tilde{\Phi}$ in the sense of this criterion (i.e., $\tilde{\Phi} < \Phi$), then $c\tilde{\Phi} < c\Phi$.

ii) if Φ is equivalent to $\tilde{\Phi}$ in the sense of this criterion (i.e., $\Phi \sim \tilde{\Phi}$), then $c\Phi \sim c\tilde{\Phi}$.

THEOREM. *If a family Φ is optimal in the sense of some optimality criterion that is final and unit-invariant, then every rescaling $f(u)$ from Φ is equivalent to $f(u) = log(u)$.*

(Proof is given in Section 5).

Comment. This means that the optimal rescalings are of the type $\gamma\, log(u) + \alpha$ for some real numbers $\gamma > 0$ and α.

4. CASE STUDY: APPLICATION OF LOGARITHMIC RESCALING TO FUZZY CONTROL (BRIEF DESCRIPTION)

Description of a plant. We design a control for chemical reaction within a constant volume, non-adiabatic, continuously stirred tank reactor (CSTR). The model that describes the CSTR is [M90]:

$$\dot{x}_1 = -x_1 + Da(1 - x_1)\, exp(x_2/(1 + x_2/\gamma))$$
$$\dot{x}_2 = -x_2 + BDa(1 - x_1)\, exp(x_2/(1 + x_2/\gamma)) - u(x_2 - x_c),$$

where x_1 is the conversion rate, x_2 is the dimensionless temperature, and u is the dimensionless heat transfer coefficient. The objective of the control is to stabilize the system (i.e., bring it closer to the equilibrium point).

What we did. We applied a logarithmic rescaling $x_2 \to X = log\, x_2$, and used membership functions with equal spacing for X. No further adjustment of membership functions was made.

Results. Even without any further adjustment the results of this control were comparable to the results of applying the intelligent "gain scheduled" (non-linear) PID controller ([HK85], [M90]). In other words, we got the control that was as good as the one generated by the state-of-art traditional control theory with respect to stability and controllability of the plant.

With respect to the computational complexity our fuzzy controller is much simpler.

Rescaling is necessary. Without the rescaling, we got a fuzzy control whose quality was much worse than that of a PID controller.

Details. The details of this case study were published in [VT92].

5. PROOF OF THE MAIN RESULT

The idea of this proof is as follows: first we prove that the optimal family is unit-invariant (in part 1), and from that, in part 2, we conclude that any function f from Φ satisfies a functional equation, whose solutions are known.

1. Let us first prove that the optimal family Φ_{opt} exists and is *unit-invariant* in the sense that $\Phi_{opt} = c\Phi_{opt}$ for all $c > 0$. Indeed, we assumed that the optimality criterion is final, therefore there exists a unique optimal family Φ_{opt}. Let's now prove that this optimal family is unit-invariant (this proof is practically the same as in [K90], [KQ91], or [KQF92]). The fact that Φ_{opt} is optimal means that for every other Φ, either $\Phi < \Phi_{opt}$ or $\Phi_{opt} \sim \Phi$. If $\Phi_{opt} \sim \Phi$ for some $\Phi \neq \Phi_{opt}$, then from the definition of the optimality criterion we can easily deduce that Φ is also optimal, which contradicts the fact that there is only one optimal family. So for every Φ either $\Phi < \Phi_{opt}$ or $\Phi_{opt} = \Phi$.

Take an arbitrary c and apply this conclusion to $\Phi = c\Phi_{opt}$. If $c\Phi_{opt} = \Phi < \Phi_{opt}$, then from the invariance of the optimality criterion (condition ii)) we conclude that $\Phi_{opt} < c^{-1}\Phi_{opt}$, and that conclusion contradicts the choice of Φ_{opt} as the optimal family. So $\Phi = c\Phi_{opt} < \Phi_{opt}$ is impossible, and therefore $\Phi_{opt} = \Phi$, i.e., $\Phi_{opt} = c\Phi_{opt}$, and the optimal family is really unit-invariant.

2. Let us now deduce the actual form of the functions $f(u)$ from the optimal family Φ_{opt}. If $f(u)$ is such a function, then the result $f(cu)$ of changing the unit of u to a c times smaller unit belongs to $c\Phi_{opt}$, and so, due to 1., it belongs to Φ_{opt}. But by the definition of a family all its functions can be obtained from each other by a linear transformation $Cf(u) + x_0$, therefore, $f(cu) = Cf(u) + x_0$ for some C and x_0. These values C and x_0 depend on c. So we arrive at the following functional equation for $f(u)$: $f(cu) = C(c)f(u) + x_0(c)$. In the survey on functional equations [A66] the solutions of this equation are not explicitly given, but a for a similar functional equation $f(x + y) = f(x)h(y) + k(y)$ all solutions are enumerated in Corollary 1 to Theorem 1, Section 3.1.2 of [A66]: they are $f(x) = \gamma x + \alpha$ and $f(x) = \gamma exp(cx) + \alpha$, where $\gamma \neq 0$, $c \neq 0$ and α are arbitrary constants. So, let us reduce our equation to the one with known solutions.

The only difference between these two equations is that we have a product, and we need a sum. There is a well known way to reduce product to a sum: turn to logarithms, because $log(ab) = log(a) + log(b)$. For simplicity let us use natural logatithms ln. So let us introduce new variables $X = ln(u)$ and $Y = ln(c)$. In terms of these new variables $x = exp(X)$, $c = exp(C)$. Substituting these values into our functional equation, and taking into consideration that $exp(X)exp(Y) = exp(X + Y)$, we conclude that $F(X + Y) = H(Y)F(X) + K(Y)$, where we denoted $F(X) = f(exp(X))$, $H(Y) = C(exp(Y))$, and $K(Y) = x_0(exp(Y))$. So according to the above-cited result, either $F(X) = \gamma X + \alpha$, or $F(X) = \gamma\, exp(cX) + \alpha$.

From $F(X) = f(exp(X))$, we conclude that $f(u) = F(ln(u))$, therefore either $f(u) = \gamma\, ln(u) + \alpha$, or $f(u) = \gamma\, exp(c\, ln(u)) + \alpha = \gamma u^c + \alpha$. In the second case the function $f(u)$ maps $(0, \infty)$ onto the interval (α, ∞), and we defined a rescaling as a function whose values run over all possible real numbers. So the second case is impossible, and $f(x) = \gamma\, ln(u) + \alpha$, which means that $f(u)$ is equivalent to a logarithm. Q.E.D.

6. CONCLUSIONS

One of the important steps in designing a fuzzy control is the choice of the membership functions for all the terms that the experts use. This choice strongly influences the quality of the resulting control.

For simple controlled systems, it is sufficient to have equally spaced membership functions, i.e., functions that have similar shape (usually triangular or trapezoid), and are located in intervals of equal length $..., [N - \Delta, N + \Delta], [N, N + 2\Delta], [N + \Delta, N + 3\Delta], ...$

For complicated systems this choice does not lead to a good fuzzy control, so it is necessary to tune the membership functions by applying neural networks or genetic algorithms. This is a very time-consuming procedure, and therefore, it is desirable to avoid it as much as possible.

We consider the case, when the equally spaced membership functions are inadequate because the control variable u can take only positive values. Such situations occur, for example, when we control the flux of the substances into a chemical reactor (e.g., the flux of fuel into an engine). Our idea is to "rescale" this variable, i.e., to use a new variable $u' = f(u)$, and to choose a function $f(u)$ in such a way that we can apply membership functions, that are equally spaced in u'.

We give a mathematical proof that the optimal rescaling is logarithmic ($f(u) = a\, log(u) + b$). We also show on a real-life example of a non-linear chemical reactor that the resulting fuzzy control,

without any further tuning of membership functions, can be comparable in quality with the best state-of-art non-linear controls of traditional control theory.

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