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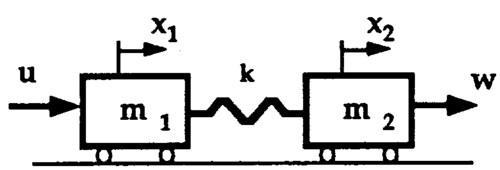
Synthesis of Robust Controllers

At the 1990 American Controls Conference a benchmark problem was issued as a challenge for designing robust compensators. Many compensators were presented in response to the problem. In previous work Stochastic Robustness Analysis (SRA) was used to compare these compensators. In this work SRA metrics are used as guides to synthesize robust compensators, using the benchmark problem as an example.

The benchmark problem consists of 2 masses (m_1 and m_2) connected by a spring (k), with control of the first mass, and output defined as position of the second mass. The state, control, and output vectors have dimensions of 4, 1, and 1. The Benchmark Problem requires an output settling time of 15 sec after an impulsive disturbance on m_2, with limited actuator use and closed-loop robustness in the presence of parameter variations. Plant parameters may vary in the ranges $0.5 < k < 2$, $0.5 < m_1 < 1.5$, and $0.5 < m_2 < 1.5$. In the present study, the settling-time limit is considered to be violated if the displacement of m_2 exceeds a ±0.1-unit envelope, 15 or more seconds after the disturbance. The control-usage limit is violated if control displacement exceeds one unit, and the stability requirement is violated with one or more positive closed-loop roots. Parameters are assumed to have uniform probability distributions within their ranges.

Benchmark Problem



$y = x_2, \quad z = x_2 + n$

$k = 1$ N/m, $m_1 = 1$ kg, $m_2 = 1$ kg

Requirements

- Robust to Variation in Parameters
- Settling Time of 15 secs
- Minimum Actuator Use

By using SRA we can find estimates of the probabilities of instability (P_i), settling time violation (P_{ts}), and actuator usage (P_u). These can be used by a designer to guide the adjustment of design parameters to produce a robust compensator. As a method of trading off different design requirements the probabilities can be combined into a scalar cost. The minimum of this cost function can then be sought. For the benchmark problem the cost function chosen was a weighted quadratic sum of the probabilities of instability, settling time violation, and actuator usage.
The nature of the probabilistic metrics and variations in the functional estimates due to the Monte Carlo Evaluation make the cost function difficult to minimize.

Optimization of Robustness

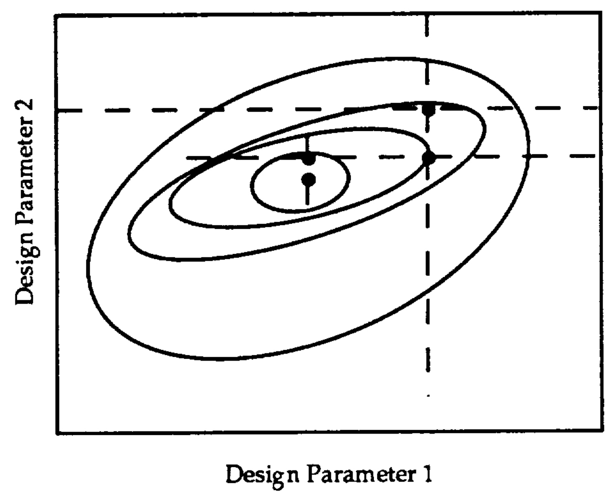
$$J = aP_i^2 + bP_u^2 + cP_t^2$$

Problems with Optimization

- N Dimensional
- Not Continuously Differentiable
- Plateaus where P=0 or 1
- Noise in Evaluation

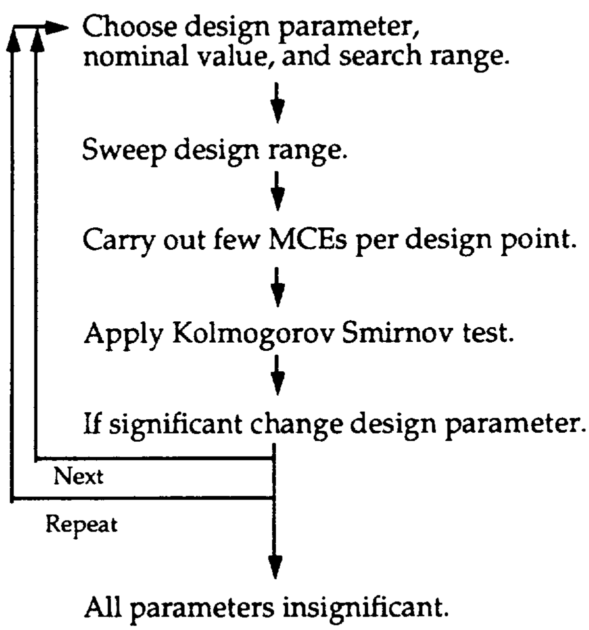
Efficient methods for minimizing this cost are still under research. For the benchmark problem the multidimensional search was reduced to a series of line searches by using the standard univariate search method which involves changing only one design parameter at a time.

Univariate Minimization



Each line search was carried out by adjusting a
chosen design parameter across a range and assessing the cost
at several points along the range. If the parameter produced
a statistically significant reduction in the cost, then the parameter
was adjusted to the minimizing value. The search then went on
to the next parameter.

Optimization Procedure



For the benchmark problem the compensator structure chosen was the Linear Quadratic Gaussian Regulator. The search first found a robust LQR, a robust filter was added (the method for finding this robust filter is explained later), then the full LQGR was fine tuned using the univariate minimization.

The final compensator was fully analyzed with many Monte Carlo evaluations to validate the design. For the benchmark problem the validating analysis used 8,000 evaluations.

Synthesis for Benchmark Problem

Choose Structure of
LQR with Kalman Filter

↓

Choose Design Parameters

↓

Optimize LQR

↓

Add Estimator

↓

Produce Robust Filter

↓

Optimize Estimator

↓

Carry Out Full Analysis

By using the state propagation equations
we can group the effects of parameter variations with disturbance effects. We can simulate the time response of the system and calculate the disturbance residual, q. This residual includes the effect of parameter variations with the disturbance effects. q can be used to estimate a disturbance covariance matrix. When a filter is designed with this matrix it handles parameter variations as expected disturbances and is more robust in producing estimates of the state.

Robust Kalman Filter

$$x_{k+1} = \Phi x_k + \Gamma u_k + \Lambda w_k$$
$$x_{k+1} = \hat{\Phi} x_k + \hat{\Gamma} u_k + (\Delta\Phi x_k + \Delta\Gamma u_k + \Lambda w_k)$$
$$\hat{x}_{k+1} = \hat{\Phi}\hat{x}_k + \hat{\Gamma} u_k + K(z_k - H\hat{x}_k)$$

$$Q_k = E\left[(\Lambda w_k)(\Lambda w_k)^T\right]$$

$$Q_k =$$
$$E\left[(\Delta\Phi x_k + \Delta\Gamma u_k + \Lambda w_k)(\Delta\Phi x_k + \Delta\Gamma u_k + \Lambda w_k)^T\right]$$

$$\begin{aligned} q_k &= x_{k+1} - \hat{\Phi} x_k - \hat{\Gamma} u_k \\ &= (\Phi - \hat{\Phi}) x_k + (\Gamma - \hat{\Gamma}) u_k + \Lambda w_k \end{aligned}$$

$$\hat{Q} = \frac{1}{N-1} \Sigma_{k=1}^{N} q_k q_k^T$$

Four compensators were designed with different cost function weights to represent different design concerns. The first design (LQG1) has the weight predominantly on the probability of instability; this would be the case when it is very important that even marginal instability should never occur. LQG2 puts weight on the probability of actuator saturation; this will have the effect of allowing a slight decrease in other robustness variables to ensure that actuator limits are rarely violated. For LQG 3 most of the weight is on the performance (settling time) robustness. LQG4 is designed to have a general blend of the robustness properties.

The weights affected the nominal performances in predictable ways e.g., the compensator with high weighting on settling time violation had the lowest nominal settling time.

Characteristics of Compensators Designed for the Benchmark Problem

Weights for Cost Function J

	a	b	c
LQG 1	1	0.01	0.01
LQG 2	0.01	1	0.01
LQG 3	0.01	0.01	1
LQG 4	1	0.02	0.06

3 Zeros, 5 Poles,

1 Non-minimum-phase zero.

Nominal Disturbance Responses

	$Ts_{0.1}$	U_{max}	$X2_{max}$
LQG 1	14.1	0.59	2.12
LQG 2	12.1	0.46	1.59
LQG 3	10.1	0.82	1.09
LQG 4	12.5	0.54	1.43

The robustness of the LQGRs designed using SRA metrics compare very well with the ten compensators (A-J) which had been designed by other methods (these compensators had been analyzed in previous work).

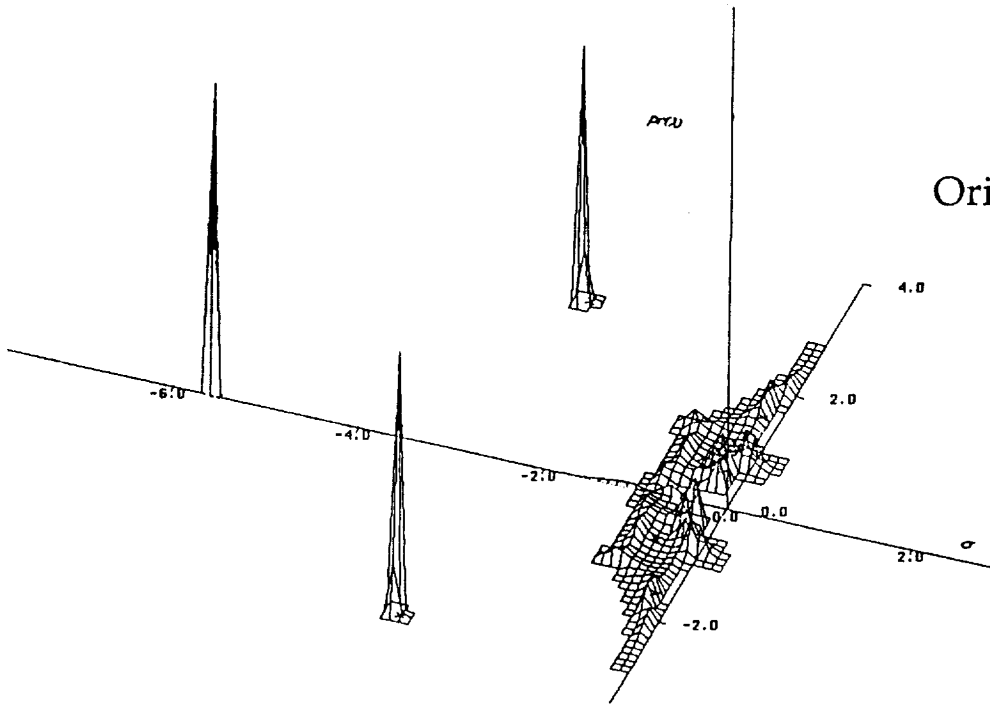
The compensators designed by SRA were better than all the other compensators with respect to stability, actuator saturation, and performance (settling time) robustness.

The only exception was that design D had better settling time robustness than any of the LQGs but this was at the expense of having very high actuator usage.

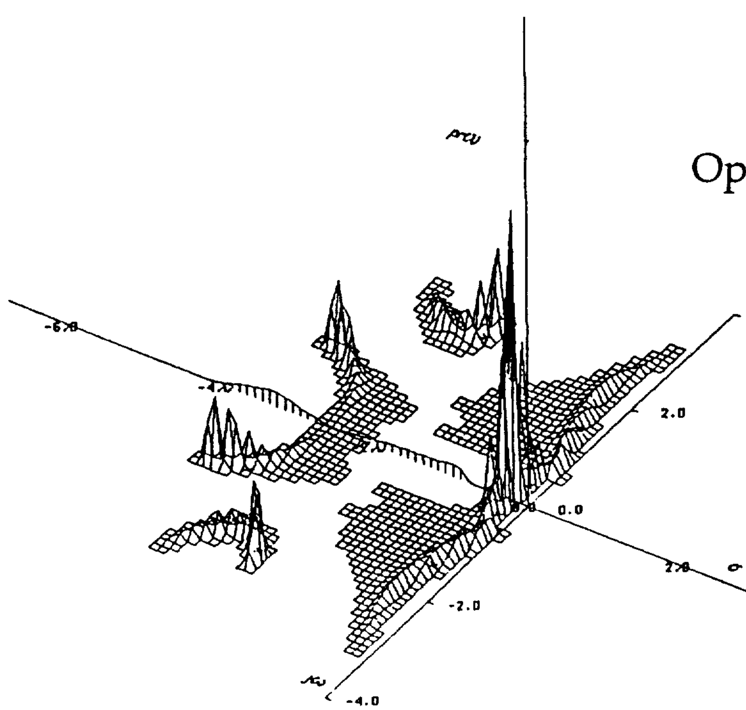
Comparison of Robustness Costs

Design	J1	J2	J3	J4
A	0.03	0.03	0.92	0.08
B	0.01	0.01	0.94	0.06
C	0.01	0.01	0.94	0.06
D	0.01	1.00	0.01	0.02
E	0.02	0.16	1.00	0.07
F	0.03	1.00	0.75	0.08
G	0.06	0.79	1.01	0.12
H	0.01	0.03	0.83	0.05
I	0.008	0.01	0.84	0.05
J	0.07	0.30	1.00	0.12
LQG1	0.006	0.02	0.58	0.04
LQG2	0.004	0.004	0.42	0.03
LQG3	0.09	0.13	0.22	0.10
LQG4	0.006	0.006	0.18	0.02

Stochastic Root Loci
Compensator Optimized for Pi

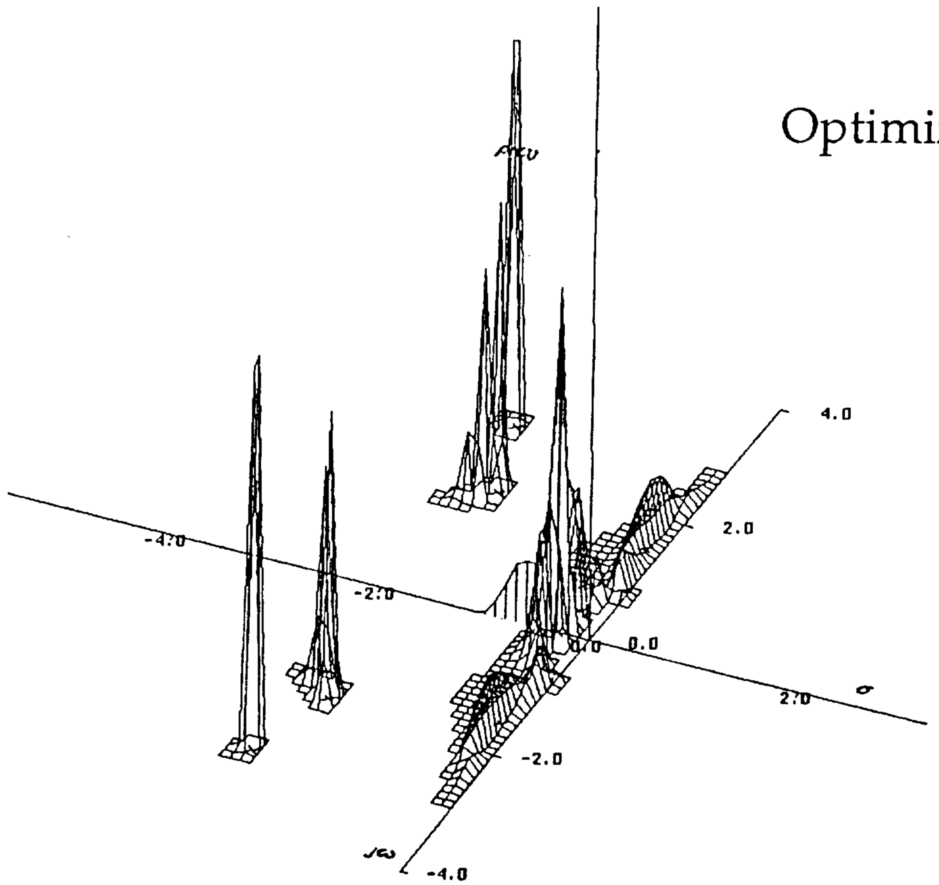


Original LQG



Optimized LQG

Stochastic Root Locus
Compensator Optimized for Pu



Optimized LQG

A careful analysis of the LQGs revealed several interesting lessons for designing robust compensators.

Comparing the stochastic root loci (SRL) of the compensators before and after optimization for stability robustness shows that it is quite possible to have improved robustness but greater sensitivity in the root variation.

There is a large difference in the form of the SRL for the compensator designed for stability robustness and the one designed to minimize actuator saturation. The compensator designed to minimize actuator saturation has used a lot of effort to ensure that there is very little variation in the high speed roots whereas this is unimportant for stability robustness.

Stochastic robustness has been shown to provide a sound basis for designing robust control systems. The design criteria are closely related to design goals and to practical characterization of parametric uncertainty. The method also recognizes that different criteria (e.g., settling time, control usage, and probability of instability) may have greater relative importance in different settings, allowing tradeoffs to be made among competing response requirements. The final designs for the benchmark problem compared very well with designs that had been formulated using other modern synthesis procedures (e.g., H infinity methods).

It appears that stochastic robustness is a powerful design tool. Future work in the near term will be to improve the efficiency of synthesis algorithms.

Conclusions

- Stochastic Robustness Synthesis is flexible and produces practical robust compensators.

Future Work

- More efficient methods of finding the global minimum of the cost function should be found.