NASA-CR-192813

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IN-24-CR
156300
p. 16

Final Report

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Performance Period: June 1, 1981 to June 1, 1989

NASA Grant NAG-1-201

(NASA-CR-192813) COMPRESSION STRENGTH FAILURE MECHANISMS IN UNIDIRECTIONAL COMPOSITE LAMINATES CONTAINING A HOLE Final Report, 1 Jun. 1981 - 1 Jun. 1989 (Virginia Polytechnic Inst. and State Univ.) 16 p

N93-23043

Unclas

G3/24 0156300

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April, 1993

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Eric R. Johnson

Introduction

Hole size effect

Experiments on graphite-epoxy laminated plates containing unloaded small holes show that these laminates are notch insensitive. That is, the uniaxial strength of these laminates with small holes exceeds the strength predicted by a point stress criterion using the stress concentration factor for the in-plane stress field. Laminates containing large holes exhibit notch sensitive behavior and consequently their strength is reasonably well predicted by the stress concentration effect. This hole size effect is manifested both in tension and in compression. Apparently some mechanism must cause in-plane stress relief for laminates containing small holes.

A mechanism to explain relief of the in-plane stress concentration in the vicinity of the hole for laminates in compression was suggested by Mikulas (1980). This mechanism is a geometrically nonlinear response that couples the high in-plane stresses with the out-of-plane rotations in the vicinity of the hole. A geometrically nonlinear response is associated with reduced growth of the in-plane stresses as the applied compressive load is increased since, in effect, the in-plane load is directed out-of-plane. Out-of-plane rotations are larger in the vicinity of the hole because of a differential expansion in the thickness of the laminate, or bulging, caused by Poisson's effect.

Objective

The purpose of this research was to study the influence of geometric nonlinearity on the micromechanical response of a filamentary composite material in the presence of a strain gradient caused by a discontinuity such as a hole. A mathematical model was developed at the micromechanical level to investigate this geometrically nonlinear effect.

Mathematical model

The mathematical model was developed for a solid rectangular parallelepiped, or block, of length L, height H, and of a width that is large relative to dimensions L and H. For a cartesian coordinates (x,y,z) with origin at the center of the block, $-L/2 \le x \le L/2$ and $-H/2 \le z \le H/2$. The block is made from continuous fibers parallel to the x-axis that are identical and equally spaced. These fibers are encapsulated by a matrix material. The block is subjected to uniform end shortening $\Delta/2$ at $x = \pm L/2$, traction-free conditions at $z = \pm H/2$, and body forces are neglected. For a very wide block with loads and material properties uniform in the y-direction, plane strain in the x-z plane is valid. Let u, v, and w denote displacements in the x-, y-, and z- directions, respectively. Then, plane strain implies $u = u(x,z)$, $v = 0$, and $w = w(x,z)$. Due to symmetry only a quarter of the x-z plane is modeled as shown in Fig.1. To simulate an in-plane strain gradient, we prescribe the $u(x,0)$ subject to the conditions that $u(x,0)$ is continuous for $0 \le x \le L/2$, $u(0,0) = 0$, and $u(L/2,0) =$

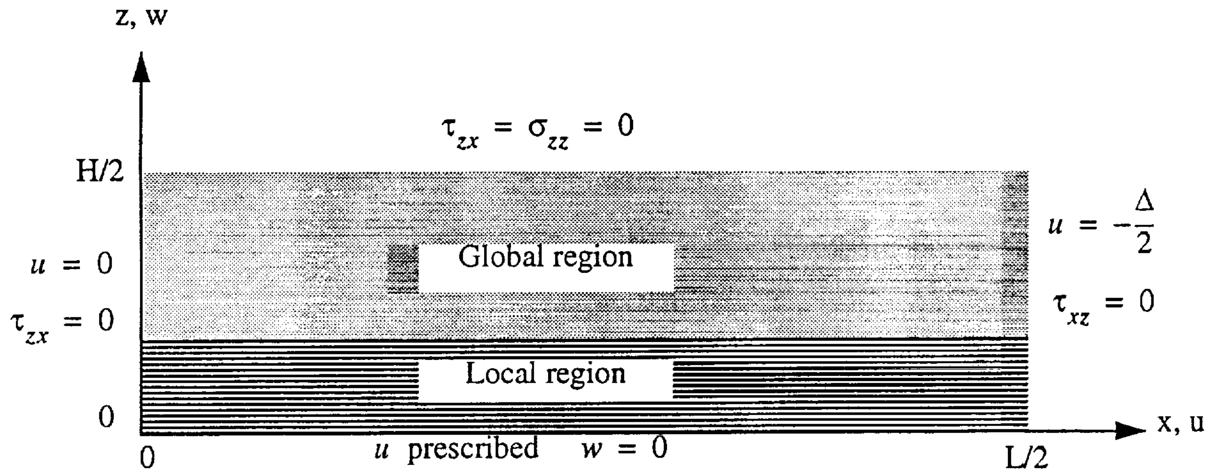

Fig. 1 Solution domain for a fiber reinforced solid block subjected to uniform end shortening, traction free conditions on the remaining external surfaces, and symmetry boundary conditions along the coordinate axes.

$-\Delta/2$. Within this class of functions, $u(x,0)$ is prescribed such that the magnitude of the normal strain $\varepsilon_x(x,0)$ ($= \frac{\partial}{\partial x}[u(x, 0)]$) is a local maximum at $x = 0$, and that this local maximum is larger than the average shortening Δ/L. Prescribing $u(x,0)$ in this manner is the method by which an in-plane strain gradient is introduced into the uniform compressive strain field.

A global and local region are defined for the model as shown in Fig. 1. In the local region the response of individual fiber-matrix units is determined by functional degrees of freedom for each unit, such that the total number of degrees of freedom depends on the number of fiber-matrix units. In the global region the response of individual fiber matrix units are determined by only a few functional degrees of freedom defined over the entire global region, and these global degrees of freedom are independent of the number of fiber-matrix units. The global region represents a smeared model of the individual fiber-matrix units remote from the stress concentration at the origin. Details of the fiber matrix units are shown in Fig. 2.

By approximating the displacements in the thickness coordinate z, the dependence of the field equations on the two coordinates x and z is reduced to a set of equations with one-dimensional dependence in x. Details of the mathematical development are given in Appendix A. The resulting equations are summarized in Eqs. (73) to (77) in Appendix A. There are 8N+8 first order, nonlinear, ordinary differential equations subject to 8N+8 boundary conditions, where N denotes the number of fiber-matrix units in the local region. The numerical solution to this nonlinear two-point boundary value problem was attempted by using the computer code called PASVART(Lentini and Pereyra, 1977). Persistent numerical ill-conditioning of the system matrices were encountered using this code, so that no acceptable solution could be found.

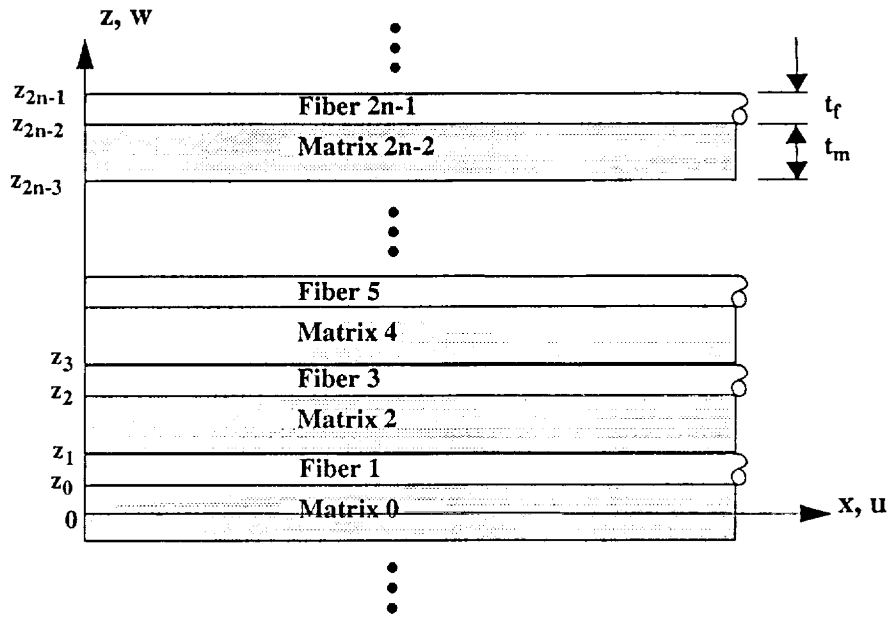

Fig. 2 Configuration and nomenclature of the fiber-matrix units.

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Lentini, M., and Pereyra, V., 1977, "An Adaptive Finite Difference Solver for Nonlinear Two Point Boundary Value Problems with Mild Boundary Layers," **SIAM Journal of Numerical Analysis**, Vol. 14, pp. 91-111.

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Appendix A
Governing Equations

Kinematic assumptions

Local region (n = 1, 2,..., N)

Let t_f and t_m denote the thickness of the fiber and thickness of the matrix in a typical fiber-matrix unit. The total unit thickness $t = t_f + t_m$, and the volume fractions are $V_f = t_f/t$ and $V_m = t_m/t$. The displacements in a typical fiber-matrix unit are assumed to be explicit functions of the thickness coordinate z. It is convenient to use dimensionless thickness coordinates in the fiber and matrix. For the matrix in the nth fiber-matrix unit, $n = 1, 2, \ldots$, N, let

$$\zeta = (z - z_{2n-3})/t_m \qquad z_{2n-3} \le z \le z_{2n-2} \qquad 0 \le \zeta \le 1 \tag{1}$$

and for the fiber let

$$\eta = (z - \bar{z}_{2n-1})/t_f \qquad z_{2n-2} \le z \le z_{2n-1} \qquad -\frac{1}{2} \le \eta \le \frac{1}{2} \tag{2}$$

where $\bar{z}_{2n-1} = (z_{2n-2} + z_{2n-1})/2$. For the nth unit the displacements are assumed to be

$$u(x, z) = u_{2n-3} - \frac{t_f}{2}\theta_{2n-3} + \zeta\left[u_{2n-1} + \frac{t_f}{2}\theta_{2n-1} - u_{2n-3} + \frac{t_f}{2}\theta_{2n-3}\right] + \zeta(1-\zeta)\,u_{2n-2} \tag{3}$$

$$u(x, z) = u_{2n-1} - \eta t_f \theta_{2n-1} \tag{4}$$

$$w(x, z) = w_{2n-3} + \zeta\,[w_{2n-1} - w_{2n-3}] \tag{5}$$

$$w(x, z) = w_{2n-1} \qquad -\frac{1}{2} \le \eta \le \frac{1}{2} \tag{6}$$

There are three functional degrees of freedom for the nth unit. The fiber displacements are $u_{2n-1}(x)$ and $w_{2n-1}(x)$, and the matrix displacement is $u_{2n-2}(x)$. Fiber rotation is $\theta_{2n-1} = \frac{dw_{2n-1}}{dx} = w'_{2n-1}$, where a prime denotes ordinary derivative with respect to x. These displacement assumptions given by Eqs. (3-6) satisfy continuity between adjacent units. The functional forms for the matrix, Eqs. (3) and (5), are the fewest number of terms in a power series in z (or ζ) which model thickness shearing deformation and thickness stretch. The assumption for the fiber displacements in Eqs. (4) and (6) are those of the Euler-Bernoulli beam theory.

For the first fiber-matrix unit (n =1) symmetry reduces the displacement assumptions in matrix 0 to

$$u(x,z) = u_1 + \frac{t_f}{2}\theta_1 + \zeta(1-\zeta)u_0 \qquad \frac{1}{2} \le \zeta \le 1$$
$$w(x,z) = -w_1 + 2\zeta w_1 \tag{7}$$

Global region (m = N+1, N+2,..., N = M)

The global region consists of M fiber-matrix units. The thickness of the global region is denoted h_G so that $h_G = Mt$. The middle of the global region is at $z = z_G = z_{2N-1} + h_G/2$. We define a dimensionless thickness coordinate ξ in the global region by

$$\xi = (z - z_G)/h_G \qquad -\frac{1}{2} \le \xi \le \frac{1}{2} \tag{8}$$

A separate displacement field is assumed for the global region that is explicit in the thickness coordinate. Introducing the five functional degrees of freedom $u_G(x)$, $\psi_x(x)$, $\Phi(x)$, $w_G(x)$, and $\psi_z(x)$, the displacements in the global region are

$$u(x,z) = u_G + h_G\xi\psi_x + \frac{1}{2}h_G^2\xi^2\Phi$$
$$w(x,z) = w_G + h_G\xi\psi_z \tag{9}$$

Displacement continuity with the top fiber at each x in the local region requires

$$u_{2N-1} - \frac{t_f}{2}\theta_{2N-1} = u_G - \frac{h_G}{2}\psi_x + \frac{h_G^2}{8}\Phi \tag{10}$$

$$w_{2N-1} = w_G - \frac{h_G}{2}\psi_z \tag{11}$$

Point matching the displacements at the top of a typical layer in the global region, using Eqs. (4) and (6) with $n \to m$, to the global field in Eq. (9) gives

$$u_{2m-1} - \frac{t_f}{2}\theta_{2m-1} = u_G + h_G\xi_{2m-1}\psi_x + \frac{1}{2}h_G^2\xi_{2m-1}^2\Phi$$
$$w_{2m-1} = w_G - \frac{h_G}{2}\psi_z \tag{12}$$

where $\xi_{2m-1} = (z_{2m-1} - z_G)/h_G$. Equations (10-12) can be solved for the typical unit fiber displacements in terms of the global displacements to get

$$u_{2m-1} = u_{2N-1} - \frac{t_f}{2}(1+2\xi_{2m-1})\theta_{2N-1} + \frac{t_f}{2}(1+2\xi_{2m-1})\theta_G +$$
$$\frac{h_G}{2}(1+2\xi_{2m-1})\psi_x + \frac{h_G^2}{8}(4\xi_{2m-1}^2 - 1)\Phi \tag{13}$$

$$w_{2m-1} = (1+2\xi_{2m-1})\,w_G - 2\xi_{2m-1}w_{2N-1} \tag{14}$$

$$\theta_{2m-1} = (1+2\xi_{2m-1})\,\theta_G - 2\xi_{2m-1}\theta_{2N-1} \tag{15}$$

where $\theta_G = \frac{dw_G}{dx} = w'_G$. The axial displacement at the center of the matrix in the mth unit in the global region, obtained from Eq. (3) by letting $n \to m$ and $\zeta = 1/2$, is equated to the axial displacement at the center of the mth unit from the global field Eq. (9). Doing this we get

$$\frac{1}{2}\left[u_{2m-1} + \frac{t_f}{2}\theta_{2m-1} + u_{2m-3} - \frac{t_f}{2}\theta_{2m-3}\right] + \frac{1}{4}u_{2m-2} = u_G + \frac{h_G}{2}(\xi_{2m-1} + \xi_{2m-3})\,\psi_x + \frac{h_G^2}{8}(\xi_{2m-1} + \xi_{2m-3})^2\Phi \tag{16}$$

Equation (16) is not a point matching condition between the mth fiber-matrix unit and the global field since $t_m/2 \neq t/2$. Using Eqs. (13-15) the final expression for the matrix axial displacement u_{2m-2} is

$$\frac{1}{4}u_{2m-2} = -\frac{t_f}{2}[\,(1+2\xi_{2m-1})\,\theta_G - 2\xi_{2m-1}\theta_{2N-1}] - \frac{t^2}{8}\Phi \tag{17}$$

Equations (13-15) and (17) relate the fiber and matrix displacements for an individual unit in the global region to the global degrees of freedom.

Strains

Local region (n = 1, 2,..., N)

Assuming small strain and moderate rotations in the solution plane, the strain displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{18}$$

Substituting the displacements in Eqs. (3) and (5) into Eqs. (18) we get the strains in the matrix of the nth unit, $n > 1$, as explicit functions of ζ. Theses matrix strains in the nth unit are

$$\varepsilon_{x_{2n-2}} = \bar{\varepsilon}_{2n-2} + (2\zeta - 1)\,\tilde{\varepsilon}_{2n-2} + (2\zeta - 1)^2\hat{\varepsilon}_{2n-2} \tag{19}$$

$$\varepsilon_{z_{2n-2}} = [w_{2n-1} - w_{2n-3}]/t_m \tag{20}$$

$$\gamma_{xz_{2n-2}} = \bar{\gamma}_{2n-2} + (2\zeta - 1)\,\tilde{\gamma}_{2n-2} \tag{21}$$

in which the generalized strain-displacement measures are defined as

$$\bar{\varepsilon}_{2n-2} = \frac{1}{2}\left[u'_{2n-1} + \frac{t_f}{2}\theta'_{2n-2} + u'_{2n-3} - \frac{t_f}{2}\theta'_{2n-3}\right] + \frac{1}{4}u'_{2n-2} + \frac{1}{8}[\theta_{2n-1} + \theta_{2n-3}]^2 \tag{22}$$

$$\tilde{\varepsilon}_{2n-2} = \frac{1}{2}\left[u'_{2n-1} + \frac{t_f}{2}\theta'_{2n-1} - u'_{2n-3} + \frac{t_f}{2}\theta'_{2n-3}\right] + \frac{1}{4}[\theta^2_{2n-1} - \theta^2_{2n-3}] \tag{23}$$

$$\hat{\varepsilon}_{2n-2} = -\frac{1}{4}u'_{2n-2} + \frac{1}{8}[\theta_{2n-1} - \theta_{2n-3}]^2 \tag{24}$$

$$\bar{\gamma}_{2n-2} = \frac{u_{2n-1} - u_{2n-3}}{t_m} + \frac{t}{2t_m}(\theta_{2n-1} + \theta_{2n-3}) \tag{25}$$

$$\tilde{\gamma}_{2n-2} = -\frac{u_{2n-2}}{t_m} + \frac{1}{2}(\theta_{2n-1} - \theta_{2n-3}) \tag{26}$$

The matrix strains for the first unit ($n = 1$) are determined by substituting the displacements in Eqs. (7) into Eqs. (18). These strains have the same distribution in ζ as shown in Eqs. (19-21) for the nth unit, but the generalized strain-displacement equations are different from those given in Eqs. (22-26). For the $n = 1$ unit the generalized strain-displacement equations are

$$\bar{\varepsilon}_0 = u'_1 + \frac{t_f}{2}\theta'_1 + \frac{1}{4}u'_0 \tag{27}$$

$$\tilde{\varepsilon}_0 = 0 \qquad \hat{\varepsilon}_0 = -\frac{1}{4}u'_0 + \frac{1}{2}\theta^2_1 \tag{28}$$

$$\varepsilon_{z_0} = \frac{2w_1}{t_m} \tag{29}$$

$$\bar{\gamma}_0 = 0 \qquad \tilde{\gamma}_0 = -\frac{u_0}{t_m} + \theta_1 \tag{30}$$

The fiber strains are determined as explicit functions of η by substituting the displacements in Eqs. (4) and (6) into Eqs. (18). For the nth unit the fiber strains are

$$\varepsilon_{x_{2n-1}} = \bar{\varepsilon}_{2n-1} - \eta t_f \kappa_{2n-1} \qquad \varepsilon_{z_{2n-1}} = 0 \qquad \gamma_{xz_{2n-1}} = 0 \tag{31}$$

in which the generalized strain measures for the fiber are defined by

$$\bar{\varepsilon}_{2n-1} = u'_{2n-1} + \frac{1}{2}\theta^2_{2n-1} \qquad \kappa_{2n-1} = \theta'_{2n-1} \tag{32}$$

Strains for the global region (m = N+1, N+2,..., N+M)

The distribution of the strains through the thickness of the mth unit in the global region is assumed to be given by the same expressions for the typical unit in the local region. That is, for the matrix in the mth unit of the global region the strain distributions are given by Eqs. (19-21) with $n \to m$. For the fiber in the mth unit of the global region the strain distributions are given by Eqs. (31) with $n \to m$. The generalized strain measures in the matrix of the mth unit in the global region are then given by Eqs. (22-26) with $n \to m$, and the generalized strains for the fiber in the mth unit are given by Eqs. (32) with $n \to m$. Substituting Eqs. (13), (14), (15), and (17) into these generalized strain measures for the mth unit, we get the generalized strain measures as functions of the variables u'_{2N-1}, θ'_{2N-1}, θ'_G, ψ'_x, Φ', w_{2N-1}, and w_G. Written in matrix form, the generalized strain measures in the mth unit of the global region are

$$\begin{bmatrix} \bar{\varepsilon}_{2m-2} \\ \tilde{\varepsilon}_{2m-2} \\ \hat{\varepsilon}_{2m-2} \\ \bar{\gamma}_{2m-2} \\ \tilde{\gamma}_{2m-2} \\ \varepsilon_{z_{2m-2}} \\ \bar{\varepsilon}_{2m-1} \\ \kappa_{2m-1} \end{bmatrix} = \begin{bmatrix} C^m_{ij} \end{bmatrix} \begin{bmatrix} u'_{2N-1} \\ \theta'_{2N-1} \\ \theta'_G \\ \psi'_x \\ \Phi' \\ w_{2N-1} \\ w_G \end{bmatrix} + \frac{1}{2}\begin{bmatrix} B^m_{ij} \end{bmatrix} \begin{bmatrix} \theta^2_G \\ 2\theta_G\theta_{2N-1} \\ \theta^2_{2N-1} \end{bmatrix} \tag{33}$$

where the 8×7 array of elements C^m_{ij} and the 8×3 array of elements B^m_{ij} for the mth unit in the global region are listed in Appendix B.

Equilibrium equations

The equilibrium equations are obtained from the principle of virtual displacements, or virtual work. The external virtual work is zero for the boundary conditions shown in Fig. 1. Thus, equilibrium equations are determined from vanishing of the internal virtual work per unit width (y-direction); i.e., $\delta W_{int} = 0$ for every admissible variation of the displacement field, which is given by Eqs. (3-6). Let σ_x, σ_z, and τ_{xz} denote the stress components for the solution plane. Then

$$\delta W_{int} = \int_0^{\frac{L}{2}} \int_0^{\frac{H}{2}} (\sigma_x \delta\varepsilon_x + \sigma_z \delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz})\, dz dx = 0 \tag{34}$$

The integral through the thickness is written as a sum of integrals over the individual fiber-matrix units. Then the virtual strains, or variation of the strains, are expressed as explicit functions of the thickness coordinate using Eqs. (19-21) for the matrix and Eqs. (31) for the fiber. After integrating through the thickness of a typical unit the virtual work functional becomes

$$\delta W_{int} = \int_0^{\frac{L}{2}} \sum_{i=1}^{N+M} \{ t_m [\bar{\sigma}\delta\bar{\varepsilon} + \tilde{\sigma}\delta\tilde{\varepsilon} + \hat{\sigma}\delta\hat{\varepsilon} + \bar{\tau}\delta\bar{\gamma} + \tilde{\tau}\delta\tilde{\gamma} + \bar{\sigma}_z\delta\varepsilon_z]_{2i-2} + [N\delta\bar{\varepsilon} + M\delta\kappa]_{2i-1} \} \; dx \tag{35}$$

where we defined the matrix stress resultants

$$[\bar{\sigma}, \tilde{\sigma}, \hat{\sigma}]_{2i-2} = \int_{\frac{1}{2}\delta_{i1}}^{1} [1,\ (2\zeta - 1),\ (2\zeta-1)^2]\, \sigma_{x_{2i-2}} d\zeta$$

$$\bar{\sigma}_z = \int_{\frac{1}{2}\delta i1}^{1} \sigma_{z_{2i-2}} d\zeta \tag{36}$$

$$[\bar{\tau}, \tilde{\tau}]_{2i-2} = \int_{\frac{1}{2}\delta_{i1}}^{1} [1,\ (2\zeta-1)]\, \tau_{xz_{2i-2}} d\zeta$$

$$\delta_{i1} = \begin{pmatrix} 1 & i = 1 \\ 0 & i > 1 \end{pmatrix}$$

and the fiber stress resultants

$$[N, M]_{2i-1} = \int_{-1/2}^{1/2} [1, t_f\eta]\, \sigma_{x_{2i-1}} t_f \; d\eta \tag{37}$$

The variation of the generalized strain-displacement relations, which are obtained from the variation of Eqs. (22-30) for the matrix and Eqs. (32) for the fiber, are substituted into Eq. (35). Integrations by parts with respect to the coordinate x are performed. Vanishing of the functional leads to the following equilibrium equations.

$$\frac{dS_{2n-2}}{dx} + \tilde{\tau}_{2n-1} = 0 \qquad n = 2, 3, \ldots, N \tag{38}$$

$$\frac{d\bar{N}_{2n-1}}{dx} + \bar{\tau}_{2n} - \bar{\tau}_{2n-2} = 0 \qquad n = 1, 2, \ldots N-1 \tag{39}$$

$$-\frac{dV_{2n-1}}{dx} + \sigma_{z_{2n-2}} - \sigma_{z_{2n}} = 0 \qquad n = 1, 2, \ldots, N-1 \tag{40}$$

$$\frac{d\bar{N}_{2N-1}}{dx} - \bar{\tau}_{2N-2} - t_m \sum_{m=N+1}^{N+M} (C_{41}^m \bar{\tau}_{2m-2} + C_{51}^m \tilde{\tau}_{2m-2}) = 0 \tag{41}$$

$$-\frac{dV_{2N-1}}{dx} + \sigma_{z_{2N-2}} + t_m \sum_{m=N+1}^{N+M} C^m_{66}\sigma_{z_{2m-2}} = 0 \tag{42}$$

$$\frac{dS_G}{dx} - t_m \sum_{m=N+1}^{N+M} (C^m_{45}\bar{\tau}_{2m-2} + C^m_{55}\tilde{\tau}_{2m-2}) = 0 \tag{43}$$

$$\frac{dN_G}{dx} - t_m \sum_{m=N+1}^{N+M} (C^m_{44}\bar{\tau}_{2m-2} + C^m_{54}\tilde{\tau}_{2m-2}) = 0 \tag{44}$$

$$-\frac{dV_G}{dx} + t_m \sum_{m=N+1}^{N+M} C^m_{66}\sigma_{z_{2m-2}} = 0 \tag{45}$$

The additional stress resultants appearing in Eqs. (38-45) are defined by

$$S_{2n-2} = \frac{1}{4}t_m(\bar{\sigma} - \hat{\sigma})_{2n-2} \qquad n = 1, 2, \ldots, N \tag{46}$$

$$\bar{N}_{2n-1} = N_{2n-1} + \frac{t_m}{2}[(\bar{\sigma} + \tilde{\sigma})_{2n-2} + (\bar{\sigma} + \hat{\sigma})_{2n}] \qquad n = 1, 2, \ldots, N-1 \tag{47}$$

$$\bar{M}_{2n-1} = M_{2n-1} + \frac{1}{4}t_m t_f[(\bar{\sigma} + \tilde{\sigma})_{2n-2} + (-\bar{\sigma} + \tilde{\sigma})_{2n}] \qquad n = 1, 2, \ldots, N-1 \tag{48}$$

$$V_{2n-1} = \begin{cases} -\dfrac{d\bar{M}_{2n-1}}{dx} + [\bar{N}_{2n-1}\theta_{2n-1} - S_{2n-2}(\theta_{2n-1} - \theta_{2n-3}) + S_{2n}(\theta_{2n+1} - \theta_{2n-1})] + \\ \dfrac{t}{2}(\bar{\tau}_{2n-2} + \bar{\tau}_{2n}) + \dfrac{t_m}{2}(\tilde{\tau}_{2n-2} - \tilde{\tau}_{2n}) \qquad n = 1, 2, \ldots, N-1 \end{cases} \tag{49}$$

$$\begin{aligned} \bar{N}_{2N-1} = N_{2N-1} + \frac{t_m}{2}(\bar{\sigma} + \tilde{\sigma})_{2N-2} + \\ \sum_{m=N+1}^{N+M} [C^m_{71}N_{2m-1} + t_m(C^m_{11}\bar{\sigma}_{2m-2} + C^m_{21}\tilde{\sigma}_{2m-2} + C^m_{31}\hat{\sigma}_{2m-2})] \end{aligned} \tag{50}$$

$$\begin{aligned} \bar{M}_{2N-1} = M_{2N-1} + \frac{1}{4}t_m t_f(\bar{\sigma} + \tilde{\sigma})_{2N-2} + \\ \sum_{m=N+1}^{N+M} [C^m_{82}M_{2m-1} + C^m_{72}N_{2m-1} + t_m(C^m_{12}\bar{\sigma}_{2m-2} + C^m_{22}\tilde{\sigma}_{2m-2} + C^m_{32}\hat{\sigma}_{2m-2})] \end{aligned} \tag{51}$$

$$V_{2N-1} = \begin{pmatrix} -\frac{d\bar{M}_{2N-1}}{dx} + \bar{N}_{2N-1}\theta_{2N-1} - S_{2N-1}(\theta_{2N-1} - \theta_{2N-3}) + \frac{8}{h_G^2}S_G\theta_{2N-1} - \\ \left(\frac{2}{h_G}N_G + \frac{8}{h_G^2}S_G\right)\theta_G + \frac{t}{2}\bar{\tau}_{2N-2} + \frac{t_m}{2}\tilde{\tau}_{2N-2} + t_m \sum_{m=N+1}^{N+M} (C_{42}^m\bar{\tau}_{2m-2} + C_{52}^m\tilde{\tau}_{2m-2}) \end{pmatrix} \tag{52}$$

$$S_G = \sum_{m=N+1}^{N+M} C_{75}^m N_{2m-1} + t_m (C_{15}^m\bar{\sigma}_{2m-2} + C_{25}^m\tilde{\sigma}_{2m-2} + C_{35}^m\hat{\sigma}_{2m-2}) \tag{53}$$

$$N_G = \sum_{m=N+1}^{N+M} C_{74}^m N_{2m-1} + t_m (C_{14}^m\bar{\sigma}_{2m-2} + C_{24}^m\tilde{\sigma}_{2m-2} + C_{34}^m\hat{\sigma}_{2m-2}) \tag{54}$$

$$M_G = \sum_{m=N+1}^{N+M} C_{83}^m M_{2m-1} + C_{73}^m N_{2m-1} + t_m (C_{13}^m\bar{\sigma}_{2m-2} + C_{23}^m\tilde{\sigma}_{2m-2} + C_{33}^m\hat{\sigma}_{2m-2}) \tag{55}$$

$$V_G = -\frac{dM_G}{dx} + \frac{2}{h_G}N_G\theta_G + \left(\frac{2}{h_G}N_G + \frac{8}{h_G^2}S_G\right)(\theta_G - \theta_{2N-1}) + t_m \sum_{m=N+1}^{N+M} (C_{43}^m\bar{\tau}_{2m-2} + C_{53}^m\tilde{\tau}_{2m-2}) \tag{56}$$

The boundary conditions consistent with virtual work and for the particular problem shown in Fig. 1 are

$$u_{2n-2} = 0 \qquad \text{at } x = 0, L/2 \qquad n = 2, 3, \ldots, N \tag{57}$$

$$u_{2n-1}(0) = 0 \qquad u_{2n-1}(L/2) = -\Delta/2 \qquad n = 1, 2, \ldots, N \tag{58}$$

$$\theta_{2n-1} = 0 \qquad \text{at } x = 0, L/2 \qquad n = 1, 2, \ldots, N \tag{59}$$

$$V_{2n-1} = 0 \qquad \text{at } x = 0, L/2 \qquad n = 1, 2, \ldots, N \tag{60}$$

$$\Phi = \psi_x = \theta_G = V_G = 0 \qquad \text{at } x = 0, L/2 \tag{61}$$

At $z = 0$ the axial displacement $u(x,0)$ is prescribed subject to $u(0,0) = 0$ and $u(L/2, 0) = -\Delta/2$. Denote the function $U(x)$ as this prescribed displacement function. From the first of Eqs. (7) with $\zeta = 1/2$, the prescribed displacement relates u_1, θ_1, and u_0 of the first fiber-matrix unit by

$$U(x) = u_1(x) + t_f\theta_1(x) + \frac{1}{4}u_0(x) \tag{62}$$

Equation (62) is used to express the matrix displacement function u_0 in terms of the prescribed function and the displacement and rotation of the first fiber. For u_1 and θ_1 satisfying Eqs. (58) and (59), U properly prescribed, matrix displacement u_0 will vanish at $x = 0, L/2$.

Hooke's law

For plane strain of an isotropic material, Hooke's law for the matrix is

$$\sigma_x = K_m[\varepsilon_x + \bar{\nu}_m\varepsilon_z] \qquad \sigma_z = K_m[\varepsilon_z + \bar{\nu}_m\varepsilon_z] \qquad \tau_{xz} = G_m\gamma_{xz} \tag{63}$$

in which

$$K_m = \frac{E_m(1-\upsilon_m)}{(1+\upsilon_m)(1-2\upsilon_m)} \qquad \bar{\upsilon}_m = \frac{\upsilon_m}{1-\upsilon_m} \qquad G_m = \frac{E_m}{2(1+\upsilon_m)} \tag{64}$$

and E_m and ν_m are Young's modulus and Poisson's ratio, respectively, for the matrix.

For the Euler-Bernoulli kinematics of the fiber, Hooke's law is

$$\sigma_x = \bar{E}_f\varepsilon_x \qquad \text{where } \bar{E}_f = V_fE_f + V_mE_m \tag{65}$$

and E_f is Young's modulus for the fiber, V_f is the fiber volume fraction, and V_m is the matrix volume fraction ($V_f + V_m = 1$). The modulus $\bar{E}_f$ represents the composite modulus of a fiber-matrix sheet parallel to the x-y plane of thickness t_f.

Hooke's law in terms of the stress resultants and generalized strains is obtained as follows. The strain distributions given by Eqs. (19-21) for the matrix are substituted into Eqs. (63), and these results are in turn substituted into the stress resultant definitions for the matrix, Eqs. (36), to get

$$\bar{\sigma}_{2i-2} = (1-\frac{1}{2}\delta_{i1})K_m\left[\bar{\varepsilon} + \frac{1}{3}\hat{\varepsilon} + \bar{\upsilon}_m\varepsilon_z\right]_{2i-2} \qquad i \geq 1 \tag{66}$$

$$\begin{aligned} \tilde{\sigma}_0 &= \frac{1}{4}K_m\left[\bar{\varepsilon}_0 + \frac{1}{2}\hat{\varepsilon}_0 + \bar{\upsilon}_m\varepsilon_{z_0}\right] \\ \tilde{\sigma}_{2i-2} &= \frac{1}{3}K_m\tilde{\varepsilon}_{2i-2} \qquad i > 1 \end{aligned} \tag{67}$$

$$\hat{\sigma}_{2i-2} = (1-\frac{1}{2}\delta_{i1})K_m\left[\frac{1}{3}\bar{\varepsilon} + \frac{1}{5}\hat{\varepsilon} + \frac{1}{3}\bar{\upsilon}_m\varepsilon_z\right]_{2i-2} \qquad i \geq 1 \tag{68}$$

$$\begin{aligned} \bar{\tau}_0 &= \frac{1}{4}G_m\tilde{\gamma}_0 \\ \bar{\tau}_{2i-2} &= G_m\bar{\gamma}_{2i-2} \qquad i > 1 \end{aligned} \tag{69}$$

$$\tilde{\tau}_{2i-2} = (1-\frac{1}{2}\delta_{i1})\frac{1}{3}G_m\tilde{\gamma}_{2i-2} \qquad i \geq 1 \tag{70}$$

$$\bar{\sigma}_{z_{2i-2}} = (1-\frac{1}{2}\delta_{i1})K_m\left[\varepsilon_z + \bar{\upsilon}_m(\bar{\varepsilon} + \frac{1}{3}\hat{\varepsilon})\right]_{2i-2} \qquad i \geq 1 \tag{71}$$

The strains given by Eqs. (31) for the fiber are substituted into Hooke's law given by Eq. (65), and these results in turn are substituted into the resultant definitions in Eqs. (37) to get

$$N_{2i-1} = \bar{E}_f t_f \bar{\varepsilon}_{2i-1} \qquad M_{2i-1} = \frac{1}{12}\bar{E}_f t_f^3 \kappa_{2i-1} \qquad i \geq 1 \tag{72}$$

The task remaining is to substitute the generalized strain-displacement relations for the local region (Eqs. (22-30) and (32)) and the global region (Eqs. (33)) into Eqs. (66-72). These results are then substituted into the stress resultant definitions in Eqs. (46-56), and into the non-differential terms in equilibrium Eqs. (38-45). Although the results of these manipulations are not presented here, the form of the stress resultant-displacement relations are presented in the following section.

Summary of mathematical model

Let the (4N+4) × 1 generalized displacement vector be defined as

$$\vec{q}(x) = \left[u_0\ u_1\ \theta_1\ w_1\ u_2\ u_3\theta_3\ w_3\ \dots\ u_{2N-2}\ u_{2N-1}\ \theta_{2N-1}\ w_{2N-1}\ \Phi\ \psi_x\ \theta_G\ w_G\right]^T \tag{73}$$

and the (4N+1) × 1 generalized force vector as

$$\vec{Q}(x) = \left[S_0\ \bar{N}_1\ \bar{M}_1\ V_1\ S_2\ \bar{N}_3\ \bar{M}_3\ V_3\ \dots\ S_{2N-2}\ \bar{N}_{2N-1}\ \bar{M}_{2N-1}\ V_{2N-1}\ S_G\ N_G\ M_G\ V_G\right]^T \tag{74}$$

The mathematical model can be posed as a system of 8N + 8 nonlinear, first order, ordinary differential equations in the generalized force and displacement vectors subject to boundary conditions at $x = 0$ and $x = L/2$.

Equilibrium equations: There are 3N+2 equilibrium equations given by Eqs. (38-45). After using Hooke's law and the generalized strain-displacement relations, these equilibrium equations are of the form

$$A\frac{d\vec{Q}}{dx} + \vec{F}_1\left[\vec{q}, \frac{d\vec{q}}{dx}\right] = 0 \tag{75}$$

in which A is a (3N+2) × (4N+4) matrix containing ones and zeros, and $\vec{F}_1$ is a (3N+2) × 1 vector functional of the generalized displacements and their derivatives.

Generalized force-displacement equations: The generalized force definitions given by the 4N+4 Eqs. (46-56) can be written in terms of the generalized displacements and their derivatives by using Hookes' law and the generalized strain-displacement relations to eliminate $\bar{\sigma}, \hat{\sigma}, \tilde{\sigma}, \bar{\tau}, \tilde{\tau}, \bar{\sigma}_z, N,$ and M. The result is 4N+4 equations of the type

$$D\frac{d\vec{Q}}{dx} + \vec{Q} = \vec{F}_2\left[\vec{q}, \frac{d\vec{q}}{dx}\right] \tag{76}$$

in which D is a (4N+4) × (4N+4) matrix of constants, and $\vec{F}_2$ is a (4N+4) × 1 vector functional of the displacements and their derivatives. The non-zero rows of matrix D result from the definitions of V_{2n-1} (49), V_{2N-1} (52), and V_G (56).

Kinematic definitions: The rotations are related to the displacements by the equations $\frac{dw_{2n-1}}{dx} = \theta_{2n-1}$, and $\frac{dw_G}{dx} = \theta_G$. These definitions result in N+1 linear equations of the form

$$\bar{J}\,\frac{d\vec{q}}{dx} = \hat{J}\;\vec{q} \tag{77}$$

in which $\bar{J}$ and $\hat{J}$ are (N+1) × (4N+4) matrices containing ones and zeros.

Prescribed displacement condition: The axial displacement $U(x)$ is prescribed in Eq.(62) and this relates displacements in the first fiber-matrix unit. This single equation completes the set of 8N+8 equations for as many unknowns.

Boundary conditions: There 8N+8 boundary conditions given in Eqs. (57-61) for the 8N+8 system of first order differential equations.

Appendix B
Matrix elements in Eq. (33)

The superscript m is dropped for array elements C^m_{ij} and B^m_{ij} in the following formulas for convenience in writing, but it is implied.

$$\begin{gathered} C_{11} = 1 \qquad C_{12} = -t_f/2 \qquad C_{14} = (h_G/2)\,(1+\xi_{2m-1}+\xi_{2m-3}) \\ C_{15} = (h_G^2/8)\,(-1+(\xi_{2m-1}+\xi_{2m-3})^2) \qquad C_{13} = C_{16} = C_{17} = 0 \end{gathered} \tag{B-1}$$

$$\begin{gathered} C_{22} = -t_f\xi_{2m-1} \qquad C_{23} = (t_f(1+2\xi_{2m-1}))/2 \qquad C_{24} = t/2 \\ C_{25} = (h_G/2)^2\,(\xi^2_{2m-1}-\xi^2_{2m-1}) \qquad C_{21} = C_{26} = C_{27} = 0 \end{gathered} \tag{B-2}$$

$$\begin{gathered} C_{32} = -t_f\xi_{2m-1} \qquad C_{33} = (t_f/2)\,(1+2\xi_{2m-1}) \qquad C_{35} = t^2/8 \\ C_{31} = C_{34} = C_{36} = C_{37} = 0 \end{gathered} \tag{B-3}$$

$$\begin{gathered} C_{42} = (-[t_f(\xi_{2m-1}-\xi_{2m-3}) + t(\xi_{2m-1}+\xi_{2m-3})])/t_m \\ C_{43} = [t_f(\xi_{2m-1}-\xi_{2m-3}) + t(1+\xi_{2m-1}+\xi_{2m-3})]/t_m \\ C_{44} = t/t_m \qquad C_{45} = (h_G^2(\xi^2_{2m-1}-\xi^2_{2m-3}))/(2t_m) \\ C_{41} = C_{46} = C_{47} = 0 \end{gathered} \tag{B-4}$$

$$C_{52} = -4\frac{t_f}{t_m}\xi_{2m-1} - \xi_{2m-1} + \xi_{2m-3} \qquad C_{53} = 2\frac{t_f}{t_m}(1 + 2\xi_{2m-1}) + \xi_{2m-1} - \xi_{2m-3}$$

$$C_{55} = t^2/(2t_m) \qquad C_{51} = C_{54} = C_{56} = C_{57} = 0 \tag{B-5}$$

$$-C_{66} = C_{67} = 2\frac{(\xi_{2m-1} - \xi_{2m-3})}{t_m} \qquad C_{61} = C_{62} = C_{63} = C_{64} = C_{65} = 0 \tag{B-6}$$

$$C_{71} = 1 \qquad C_{72} = -\frac{t_f}{2}(1 + 2\xi_{2m-1}) \qquad C_{73} = \frac{t_f}{2}(1 + 2\xi_{2m-1})$$

$$C_{74} = (h_G/2)(1 + 2\xi_{2m-1}) \qquad C_{75} = (h_G^2/8)(4\xi_{2m-1}^2 - 1) \qquad C_{76} = C_{77} = 0 \tag{B-7}$$

$$C_{82} = -2\xi_{2m-1} \qquad C_{83} = 1 + 2\xi_{2m-1} \qquad C_{81} = C_{84} = C_{85} = C_{86} = C_{87} = 0 \tag{B-8}$$

$$B_{11} = (1 + \xi_{2m-1} + \xi_{2m-3})^2 \qquad B_{12} = -(1 + \xi_{2m-1} + \xi_{2m-3})(\xi_{2m-1} + \xi_{2m-3})$$

$$B_{13} = (\xi_{2m-1} + \xi_{2m-3})^2 \tag{B-9}$$

$$B_{21} = 2(\xi_{2m-1} - \xi_{2m-3})(1 + \xi_{2m-1} + \xi_{2m-3})$$

$$B_{22} = -(\xi_{2m-1} - \xi_{2m-3})(1 + 2\xi_{2m-1} + 2\xi_{2m-3}) \qquad B_{23} = 2(\xi_{2m-1}^2 - \xi_{2m-3}^2) \tag{B-10}$$

$$B_{31} = -B_{32} = B_{33} = (\xi_{2m-1} - \xi_{2m-3})^2 \tag{B-11}$$

$$B_{i1} = B_{i2} = B_{i3} = 0 \qquad i = 4, 5, 6, 8 \tag{B-12}$$

$$B_{71} = (1 + 2\xi_{2m-1})^2 \qquad B_{72} = -2\xi_{2m-1}(1 + 2\xi_{2m-1}) \qquad B_{73} = 4\xi_{2m-1}^2 \tag{B-13}$$