

TSIEN'S METHOD FOR GENERATING NON-KEPLERIAN TRAJECTORIES  
Part II- The Question of Thrust to Orbit a Sphere  
and the Restricted Three-Body Problem

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ABSTRACT

Tsien's method is extended to treat the orbital motion of a body undergoing accelerations and decelerations. A generalized solution is discussed for the generalized case where a body undergoes azimuthal and radial thrust and the problem is further simplified for azimuthal thrust alone. Judicious selection of thrust could generate either an elliptic or hyperbolic trajectory. This is unexpected especially when the body has only enough energy for a lower state trajectory. The methodology is extended treating the problem of vehicle thrust for orbiting a sphere and vehicle thrust within the classical restricted three-body problem. Results for the latter situation can produce hyperbolic trajectories through eigenvalue decomposition. Since eigenvalues for no-thrust can be imaginary, thrust can generate real eigenvalues to describe hyperbolic trajectories. Keplerian dynamics appears to represent but a small subset of a much larger non-Keplerian domain especially when thrust effects are considered. The need for high thrust long-duration space-based propulsion systems for changing a trajectory's canonical form is clearly demonstrated.

Nomenclature

a	Semi-major axis
A	Areal velocity
e	Eccentricity
E	Energy state
F	Thrust acceleration
g	Gravity
h	Integration constant
p	Semilatus rectum
$\phi$	Spherical coordinate angle
V	Gravity potential
r	Radial distance between mass centers
R	Earth radius
t	Time
$\theta$	azimuthal coordinate angle
x,y,z	Cartesian coordinate variables
$\mu$	Earth's gravitational constant

\* "The views expressed in this paper are solely those of the author and do not reflect the official policy or position of the Defense Intelligence Agency, the Department of Defense, or the U.S. Government."

## Subscripts

o	Initial or reference value
az	Azimuthal
rd	Radial
e	Earth reference value

## I. INTRODUCTION

This paper is a continuation of efforts previously presented in Murad<sup>1</sup>. Some aspects from this reference are included for continuity and the analysis is considerably expanded to treat more problems of general interest to the astrodynamist. The original problem will be briefly addressed followed by a discussion that treats these other situations.

There was a problem of interest concerning a missile event captured on photographic data. The data consisted of two streaks against a star background. Simple evaluations based upon the local sidereal time and the expected distance to the earth day-night terminator indicated that at least one and possibly both streaks were produced in total darkness, possibly by a missile. The problem was to place a trajectory through the streaks to define apogee and velocity which would be used to identify a specific missile system.

Gauss' method<sup>2-4</sup> was used unsuccessfully to place a trajectory through both streaks. The method is adequate for either an elliptic or hyperbolic trajectory, however, it was expected that the missile energy was too low to reach hyperbolic velocities although the software implied that hyperbolic trajectories ought to match the spatial data alleviating any constraint on time. When an elliptic trajectory was considered, adequate spatial matches were obtained, however, the calculated time period was larger than required to support the data.

Clearly a contradiction exists. Assuming that the software was correct, under what conditions could a missile trajectory be defined by a hyperbola when the energy is insufficient to reach hyperbolic velocities? This paper partially examines this concern by evaluating the equations of motion for a vehicle in orbit having azimuthal thrust. As a consequence of treating this problem, significant insights were obtained that have more general applicability to other problems of interest.

### A. Background

To correctly use Gauss' method, several assumptions are implied in the derivation of these orbits. Specifically, the body under investigation is not accelerating or decelerating from forces other than through the attraction of a central force field; bodies undergoing thrust or reentry clearly violate this assumption.

Some words regarding the original data are noteworthy. Several hypotheses were tested concerning what caused the streaks. These hypothesis were used to explain reasons that would have allowed the data to be photographically captured. In the course of trying to match the data, it appeared that the streaks involved thrust creating lateral and axial accelerations or decelerations.

Thus, if these streaks were thrust related, Gauss' method is not applicable.

This problem provided the initial motivation to develop the methodology. This effort's main theme is to present a rationale suggesting that the trajectory canonical form can be altered by thrust.

## B. Current Considerations

There is additional motivation regarding the present paper. During a recent conversation with V. R. Bond<sup>5</sup>, it was suggested that the time required for long space voyages can be reduced significantly by altering thrust to generate specific trajectories based upon suggestions from the author's original paper. This idea generated a different modus operandi. If Tsien's method simplified the problem of altering a spacecraft orbit using thrust, what other problems could be resolved?

The original paper judiciously selected an analytical thrust term to reduce angular momentum simplifying the governing equations of motion. Admittedly biased, the thrust term allows the spacecraft to fly either an elliptical, parabolic or hyperbolic trajectory without any real stipulation on initial velocity. Could this approach treat more complex trajectory problems?

This paper will show that an answer is mathematically tractable, however, several issues should be briefly mentioned. Use of control thrust to alter interplanetary trajectories or for station-keeping was limited by technology developed during the sixties and the early seventies. Thrust from reaction control motors or launch boosters used either a single constant setting or several distinct settings; the latter demanded feedback to regulate flowrate of oxidizer or propellant. Inert structural weight of cooling systems, fuel lines, turbines and engines, as well as large amounts of propellants created limitations that stressed launch booster capabilities. Weight and reliability kept propulsion systems to the bare essentials. Thus, altering thrust as a function of orbital parameters or time, was not technically feasible. Furthermore, instrumentation and interpretation of on-board inertial data to identify these parameters also stressed available technology.

The advent of the Shuttle-C<sup>6</sup> and other large boosters such as the Soviet Energiya concepts and its many adaptations<sup>7</sup> (i.e.: Buran-T Space Launch Vehicle, etc.), provides future designers with more flexibility in the design of spacecraft and subsequent payloads. However, chemical propellant mass fraction greatly limits the scope of any extraterrestrial exploration in the near future.

The original paper implies and will be further demonstrated here, large thrust to weight ratios and variable time-dependent long-duration thrust profiles to meet future contingencies are clearly needed. Technology limitations have displaced such ideas only as subliminal thoughts due to the need for finding practical and timely solutions to contemporary problems. Chemical systems have their limitations, although several exciting high risk technology approaches offer promise<sup>8-11</sup>. These potential concepts include: nuclear propulsion, nuclear propulsion with electrical

hybrids, MHD, tachyon beam ejection, and space warp concepts. Gravity gradient or gravity potential drives with their analogues (i.e.: magnetic potential or magnetic gradient concepts) should be included to extend this list.

Admittedly, these are far-reaching propulsion concepts yet to demonstrate technical maturity. Feasibility must parallel long-term serious funding efforts. Without political emphasis, present concepts will keep man bound to both this planet and solar system for a longer period limiting man's imagination and possibilities for growth.

Realizing the thrust-to-weight problem may be unsolvable, there are solutions that are technically feasible that should be examined. Time-dependent thrust appears to offer advantages. Amongst these is the intuitive feeling that expended propellant can be used more efficiently than with constant thrust systems. Time-dependent thrust can be incorporated in liquid rocket chemical systems and hybrid propulsion systems. Hybrid rockets offer the advantage of half the plumbing of a liquid rocket propulsion system with the reliability of a solid propellant rocket motor albeit with a performance degradation. Furthermore, if thrust variation is gradual, a solid core nuclear rocket engine, such as NERVA, could be designed with this built-in feature.

### C. Preliminaries

The equations of motion were examined and cast to account for thrust effects. In the classical derivation, a body in polar coordinates is moving about a much larger body located at the coordinate system origin. The angular momentum equation is simplified, applying Kepler's law, reducing the mathematical complexities. Subsequent substitutions provide an expression for the radius as a function of anomaly. If eccentricity is less than one, the trajectory reduces to an ellipse and if the eccentricity is greater than one, the solution describes a hyperbola. In both cases, foci of the conic represents the location of the larger body central force field.

A brief review of the two-body problem followed by Tsien's approach will be presented as a frame of reference. This is followed by looking at the equations with both axial and azimuthal thrust with the specific example of examining azimuthal thrust and its effects. This problem is extended to a spacecraft with thrust orbiting a large body in two-dimensions to one in three-dimensions. Finally, the problem of a single thrusting spacecraft orbiting two large bodies will be examined by generating different canonical types of trajectories based upon extending further some earlier work by the author.

#### C-1. The Classical Two-Body Problem

The equations of motion in the radial and transverse directions under the influence of a radial inverse gravitational potential are:

$$\ddot{r} - r\dot{\theta}^2 = -g_0 \left(\frac{R}{r}\right)^2 = -\frac{\mu}{r^2} \quad (1a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r} \left[ \frac{d}{dt} (r^2 \dot{\theta}) \right] = 0 \quad (1b)$$

the dot signifies time differentiation,  $r$  is the radial distance to the body measured from the center of the force field and  $\theta$  is the true anomaly.

The integrals for the above ordinary differential equations are:

$$\frac{1}{2} [(\dot{r})^2 + (r\dot{\theta})^2] - \frac{\mu}{r} = E \quad (2a)$$

$$r^2 \dot{\theta} = A \quad (2b)$$

where  $A$ , a constant value, is the areal velocity and the trajectory is Keplerian. By Keplerian, it is implied that the area swept by the radius vector from the central force field to the spacecraft is equal for similar time intervals along the spacecraft's orbit. The quantity  $E$  represents the sum of the spacecraft's kinetic and potential energy which remains constant throughout the trajectory.

Substituting the second expression into the first, and changing the independent variable from time to anomaly results in:

$$\frac{d}{d\theta} \left[ \frac{1}{r} \right] \left[ \frac{d^2}{d\theta^2} \left[ \frac{1}{r} \right] + \frac{1}{r} - \frac{\mu}{A^2} \right] = 0 \quad (3)$$

The solution for this initial value problem has the form:

$$r = \frac{p}{1 + e \cos (\theta - \theta_0)} \quad (4)$$

where  $p$  is the semilatus rectum and  $e$  is the eccentricity necessary to satisfy initial conditions. This equation represents an ellipse or a hyperbola depending upon the eccentricity which is based upon parameters such as the kinetic energy,  $E$ , to satisfy this initial value problem.

## C-2. Tsien's Approach

Battin<sup>4</sup> gives an excellent perspective concerning Tsien's contribution to the field of orbital mechanics with regard to non-Keplerian two-body motion. Tsien in several classic papers<sup>12-14</sup> examined two basic problems for predicting orbital change due to constant thrust directed either radially or tangentially along the flight path. Tsien's insights made these difficult problems mathematically tractable and from these initial results, sensitivities resolving problems of practical interest can easily be formulated.

Following Battin's development, Tsien included a constant term in the radial momentum equation signifying radial thrust acceleration. After an integration of the azimuthal momentum equation and substitutions into the radial momentum equation, an integration

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\* The definition of non-Keplerian used in this evaluation is that the areal velocity is no longer a constant.

provided a closed form solution for the velocity as a function of radius and acceleration for an initially circular orbit to reach escape velocity.

$$\frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 + \frac{\mu}{r^2} = a_{rd} \quad (5a)$$

$$\frac{d}{dt} \left( r^2 \frac{d\theta}{dt} \right) = 0 \quad (5b)$$

or

$$\frac{r^2 d\theta}{dt} = \sqrt{\mu r_0} \quad (5c)$$

Various solutions are obtainable. Depending upon definition, the radial thrust problem is Keplerian because of the treatment of the azimuthal equation; the areal velocity is still constant.

For tangential thrust, the case is entirely different. Here, the integration of the azimuthal equation results in an expression for the areal velocity which, even for constant thrust, is now a function of time. In this case, the trajectory should be considered non-Keplerian.

$$\frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 + \frac{\mu}{r^2} = 0 \quad (6a)$$

$$\frac{d}{dt} \left( r^2 \frac{d\theta}{dt} \right) = r a_{az} \quad (6b)$$

which yields various solutions.

Although these examples treat constant thrust acceleration, there are many solutions involving variable thrust which will not be discussed here. Can other more general families of solutions be derived that have practical value to simplify the vehicle trajectory undergoing tangential thrust?

## II. ANALYSIS

### A. The Two-Body Problem

Examining the momentum equations for a vehicle simultaneously having radial and azimuthal thrust yields:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} + a_{rd} \quad (7a)$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = a_{az} \quad (7b)$$

The integral for these equations has the generic form:

$$\frac{1}{2} [\dot{r}^2 + (r\dot{\theta})^2] - \frac{\mu}{r} = \tilde{E}_0 + \int_{t_0}^t \{a_{rd} \dot{r} + a_{az} r\dot{\theta}\} dt \quad (8)$$

In this equation, the vehicle's energy is no longer equal to the integration constant  $E_0$  which includes the kinetic and potential energy at the initial state. The expression for spacecraft energy includes an additional quantity that depends upon the time-

dependent integration of the separate thrust components. As expected, thrust effects alter the vehicle's energy as a function of time or position within the trajectory.

It is feasible to reduce these equations into other simpler forms. For a general class of solutions, let the azimuthal thrust term have the following generic form:

$$a_{az} = \frac{B\dot{r}}{r(r^2\dot{\theta})^n} \quad (9)$$

which, when substituted into the azimuthal momentum equation produces:

$$\frac{1}{(n+1)} \left\{ (r^2\dot{\theta})^{n+1} - (r_0^2\dot{\theta}_0)^{n+1} \right\} = B(r-r_0) \quad (10)$$

The B parameter is selected to eliminate terms defined at the initial state integration.

There are many interesting classes of solutions as well as mathematical problems arising from these expressions. If the exponent n is equal to zero, the term within the integral, using the expression for the rate of change of anomaly, becomes:

$$\int_{t_0}^t a_{az} r \dot{\theta} dt = B^2 \ln \left\{ \frac{r}{r_0} \right\} \quad (11)$$

which represents an embedded logarithmic singularity within the energy integral. Similarly, when n is equal to 1, this term has the same form in the energy expression as the term generated from an inverse-square gravitational force field. If n is larger, the exponent will accordingly increase in the energy forcing function which alters the form of the resulting equation of motion. These higher-order problems require elliptical integral solutions or other more unorthodox approaches.

Let us return to the more restrictive case for treating azimuthal thrust alone. The equations of motion are as follows:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \quad (12a)$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = a_{az} \quad (12b)$$

Let us examine the situation for azimuthal thrust and assume a form that allows closure to reduce the azimuthal equation of motion to a quadrature:

$$a_{az} = \frac{B\dot{r}}{r(r^2\dot{\theta})} \quad (13a)$$

$$\frac{d}{dt} \left\{ \frac{1}{2} (r^2\dot{\theta})^2 \right\} = B\dot{r} \quad (13b)$$

Clearly orbits described by this expression are non-Keplerian. The thrust term is non-conservative and alters the nature of the solution. Here, the expression is simplified by judiciously selecting

the following integration factors:

$$B = \frac{1}{2} r_0^3 \dot{\theta}_0^2 \quad (14)$$

resulting in:

$$\dot{\theta} = \frac{1}{r^2} [2Br]^{\frac{1}{2}} \quad (15)$$

There is a need to explain the selection of the acceleration profile and how it satisfies the overall problem regarding the initial streak data. For the case when a missile accelerates toward the apogee (i.e.: boost) and decelerates moving away from apogee (i.e.: reentry/retro thrust), B is positive. The terms involving radius and the rate of change in anomaly are positive valued; they only change in overall magnitude but not in sign. The inclusion of the rate of change of radius with time, however, does change sign when the vehicle passes through apogee. The positive sense of this term represents positive thrust where a negative sign implies retro or reentry decelerations. It is assumed the accelerating/decelerating forces on the body act tangential to the flight path represented by the azimuthal term.

By non-Keplerian, the implication is that areal velocity is not constant and the body governing the central force field may not be collocated with the geometric foci for either an ellipse or hyperbola. This is important in the analysis for the latter situation; the apogee must be the closest point to the foci while for an ellipse the apogee is the furthest from the foci at the center of the Earth for a surface-to-surface missile trajectory.

When used with the radial equation of motion and integrated, the constant  $E_0$  term representing initial energy is not directly removed from the formalism as in the classic sense but remains throughout the derivation. This becomes:

$$\frac{d^2}{d\theta^2} \left\{ \frac{1}{r} \right\} + \lambda^2 \left\{ \frac{1}{r} \right\} = \gamma \quad (16)$$

with a solution that takes either of the following forms depending upon whether lambda is real or imaginary:

$$r = \begin{cases} \frac{\alpha}{1 + \beta \cosh \lambda [\theta - \theta_0]} & \text{for } \lambda^2 < 0 \\ \left[ \frac{E}{B} \theta^2 + \delta_1 \theta + \delta_2 \right]^{-1} & \text{for } \lambda^2 = 0 \\ \frac{\alpha}{1 + \beta \cos \lambda [\theta - \theta_0]} & \text{for } \lambda^2 > 0 \end{cases} \quad (17)$$



where:

$$\gamma = \frac{1}{2B} \left[ \tilde{E}_0 + \frac{B}{r_0} \right]$$

$$\lambda^2 = \left[ 2 - \frac{\mu}{B} \right] \quad (18)$$

$$\alpha = \lambda^2/\gamma \text{ and } \beta = \gamma/\lambda^2$$

Baxter<sup>15</sup> derives a similar expression for the case of force field perturbations in the radial direction. Baxter suggests that the fundamental problem of Keplerian representations of real orbits is the failure to correctly account for the energy of the orbiting body. This could lead to in-track errors in Keplerian mean motion. Baxter compensates by using perturbation terms in the gravitational potential to remove in-track drift. Furthermore, the method can produce Keplerian trajectories in a non-Keplerian environment by inclusion of these radial terms where orbital elements are changed to include perturbative quantities. For example, energy is directly included in these expressions and is not treated as a secondary term through the definition of eccentricity.

The change in the form of the trajectory relies principally upon the nature of whether lambda is real or imaginary. Values for B depend upon location along the trajectory where thrust is applied and as the value of B increases, the sense of lambda becomes more negative. When the magnitude of this term is equal to 2.0, the equation is parabolic. When larger than 2.0, the equation is hyperbolic. This is independent of energy considerations which enters the problem only through eccentricity.

If the coefficients are altered to reflect when this expression is identical to the classically derived equation, an interesting analogy develops. For specific initial conditions defining B and the azimuthal thrust profile, a thrusting trajectory could be derived having the same spatial-time dependency as a Keplerian trajectory. Thus it is entirely feasible, with caveats, that an inefficient trajectory, using thrust, could be replaced by a trajectory without thrust.

## B. The Problem of a Spacecraft Orbiting a Spherical Body

The equations of motion for a spacecraft orbiting a spherical body are:

$$\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta = - \frac{\partial V}{\partial r} \quad (19a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin \theta \cos \theta = - \frac{1}{r} \frac{\partial V}{\partial \theta} \quad (19b)$$

$$r \sin \theta \ddot{\phi} + 2\dot{r} \sin \theta \dot{\phi} + 2r \cos \theta \dot{\theta} \dot{\phi} = - \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \quad (19c)$$

where:  $\phi$  is the out-of-plane angle required for a spherical coordinate system. The gravity gradient can have the simple form:

$$V(r, \theta, \phi) = - \frac{\mu}{r} \quad (20)$$

These expanded equations include terms in both the radial and azimuthal momentum equations as well as a third equation describing momentum in a second angular plane. These three-dimensional spherical coordinate equations more accurately predict trajectories for non-thrust situations due to gravity potential variations acting outside of the original plane of motion.

If the out-of-plane angle  $\phi$  is constant regardless of orbit inclination, or if the time rate of change of this angle is zero, terms in the first two equations are zero and the third equation vanishes. Here, the problem reduces to two dimensions. Similarly, if the angular rate of change of  $\phi$  is constant, these additional terms may still appear although the third equation is greatly simplified. If it is assumed that the gravity potential consists only of terms involving radial and azimuthal variations, the constant term creates a rate of change in either radial or azimuthal variables or both.

Here, the last equation reduces to:

$$\frac{d}{dt} (r \sin \theta) = 0 \quad (21)$$

This is consistent with the two-dimensional case and may provide another 'integral' to reduce the equations of motion. Again, this is still without looking at thrust effects.

The emphasis will require examining out-of-plane thrust and subsequent effects on the spacecraft's trajectory. One can assume thrust components can be defined as a gradient acting in similar directions as the gravity potential gradient for example:

$$\nabla V' = \nabla V + \nabla F \quad (22)$$

The following insights can be gained from these equations with thrust. Out-of-plane thrust impacts both radial and azimuthal momentum adding to the non-linear mathematical coupling of these expressions. Clearly, the spacecraft's radius and its rate of angular rotation are dependent upon this thrust component as it alters the time rate of change of  $\phi$ . Thrust in either radial and azimuthal directions have either little influence on the out-of-plane momentum or no influence if there is no time variation in  $\phi$ .

Obviously, these equations are difficult to solve in closed-form. There are two alternatives. Can these equations be reduced to those in two dimensions or can the thrust term be selected such that either the coupling or non-linearities are reduced or removed?

### **B-1. Reduction of the Spherical Orbit Problem to Two-Dimensions**

The solution is straight forward. In both of the radial and azimuthal momentum equations, select the thrust term to exactly cancel the additional terms induced by the second angular coordinate variable:

$$\frac{\partial F}{\partial r} = r \dot{\phi}^2 \sin \theta \quad (23a)$$

$$\frac{1}{r} \frac{\partial F}{\partial \theta} = r \dot{\phi}^2 \sin \theta \cos \theta \quad (23b)$$

This reduces the first two equations to identical expressions of a spacecraft moving about a body with no thrust. By standard definitions, the orbits are Keplerian within the plane of motion. However, due to the third equation of motion and the rate of change of all variables, the rate of change of phi may not vanish. If this is so, then azimuthal thrust should be selected such that angular acceleration disappears and the remaining terms are compensated by the third thrust vector component.

$$\frac{\partial F}{\partial \phi} = - \dot{\phi} \frac{d}{dt} (r \sin \theta)^2 \quad (24)$$

Note that all of these thrust components depend upon  $\dot{\phi}$ ; they also contain the expression identified in equation (21).

## B-2. Removal of Coupling Terms

In a similar fashion using superposition, thrust components are selected to cancel the coupling terms. Angular momentum effects from out-of-plane motion are prevented from influencing the momentum in the remaining coordinate variables. Here, the equations of motion, based upon the two momentum integrals, are rewritten to define the force components:

$$r - \frac{k^2}{r^3} - \frac{f^2}{r} \dot{\phi}^2 = - \frac{\partial V}{\partial r} \quad (25a)$$

$$r \frac{dk}{dt} - \dot{\phi}^2 \cos \theta f = - \frac{1}{r} \frac{\partial V}{\partial \theta} \quad (25b)$$

$$r \sin \phi \ddot{\phi} + 2 \dot{\phi} \frac{df}{dt} = - \frac{1}{f} \frac{\partial V}{\partial \phi} \quad (25c)$$

where  $k = r^2 \dot{\theta}$  and  $f = r \sin \theta$ .

## C. The Restricted Three-Body Problem

In an earlier effort<sup>17</sup>, the thesis was presented that a potential of motion could be defined which reduced the coupling and complexity of the two-dimensional equations of motion governing a spacecraft in motion about two larger bodies. The potential was not a Hamiltonian in the purest sense and required several mathematical restrictions in its definition.

First, the potential has to be analytical in a complex variable context. Second, the potential would satisfy rules of partial differentiation, and third, the potential possesses an integration property that did not violate energy considerations. If this potential is admissible, pseudo-analytical terms can be defined that allow for the principle of superposition. This accounts for effects from gravity potential perturbations or the influence of additional larger bodies at considerably far distances. The problem is extended to consider thrust.

By pseudo-analytical, the functions solve a similar relation-

ship as the Cauchy-Reiman conditions for analytical functions. They do, however, represent solutions to the inhomogeneous Laplace equation. Briefly, psuedo-analytical functions consist of analytical functions which are solutions to Laplace's equation and may be multiplied by a complex function based upon the inhomogeneous source term, cross-product term(s), or first-order derivatives; they represent solutions to elliptical partial differential equations.

The equations of motion in three-dimensional rotating cartesian coordinates for a spacecraft having thrust moving about two larger bodies are:

$$\ddot{x} - 2\dot{y} - x = -V_x + F_x \quad (26a)$$

$$\ddot{y} + 2\dot{x} - y = -V_y + F_y \quad (26b)$$

$$\ddot{z} = -V_z + F_z \quad (26c)$$

where acceleration components are:  $F_x$ ,  $F_y$  and  $F_z$ . The gravity potential for the two large primaries, located on the x axis, is defined as:

$$V(x, y, z) = -\frac{(1-\mu)}{r_1} - \frac{\mu}{r_2}; \quad r_1^2 = (x-x_1)^2 + y^2 + z^2 \\ r_2^2 = (x-x_2)^2 + y^2 + z^2 \quad (27)$$

and the energy integral for no thrust accelerations is defined as:

$$E = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} (x^2 + y^2) + V(x, y, z) \quad (28)$$

### C-1. The Two-Dimensional Case Without Thrust

Accordingly, a potential may be defined such that:

$$\dot{x} = \frac{dx}{dt} = \Psi_x, \text{ and } \dot{y} = \frac{dy}{dt} = -\Psi_y \quad (29)$$

where the potential is a perfect differential which means the cross-derivatives are equal. The derivative is defined as:

$$d\Psi = \Psi_t dt + \Psi_x dx + \Psi_y dy \quad (30)$$

then the cross-derivatives imply:

$$\Psi_{xy} = \Psi_{yx} \text{ or } \frac{\ddot{x}}{\dot{y}} = -\frac{\dot{y}}{\dot{x}} \quad (31a)$$

and

$$\dot{x} \frac{d\dot{x}}{dt} + \dot{y} \frac{d\dot{y}}{dt} = 0 \quad (31b)$$

When this is integrated, the results reveal the kinetic energy portion of the energy integral and a constant of integration that is a function of both potential energy and the gravity potential. Thus, this definition possess both mathematical properties and also satisfies energy considerations. Results satisfy the energy integral requirement and compatibility suggesting that the expression is admissible.

The potential is a function of both spatial variables and time. The second derivative or acceleration in the x direction can be defined as:

$$\ddot{x} = \Psi_{xt} + \dot{x} \Psi_{xx} + \dot{y} \Psi_{xy} = \Psi_{xt} + \Psi_x \Psi_{xx} - \Psi_y \Psi_{yx} \quad (32)$$

with a similar expression for acceleration in the y component.

Substituting these terms into equations (26a) and (26b), with no force components, these equations are further differentiated and when combined, the resulting equation has the form:

$$\nabla^2 \Psi = \Psi_{xx} + \Psi_{yy} = -V_{xy} \quad (33)$$

This resulting equation is elliptical in the canonical partial differential sense and suggests this transformation is a pseudo-analytical function. Due to superposition, the potential can consist of an analytical function and an inhomogeneous term accounting for the gravity potential. This additional term can also be a pseudo-analytical function. A general solution to this equation has the form:

$$\Psi(x, y) = - \iint_D G(\xi, \eta; x, y) V_{\xi\eta} d\xi d\eta + \dots \quad (34)$$

where additional terms satisfy boundary conditions and  $G(\xi, \eta; x, y)$  is the Greens function:

$$G(\xi, \eta; x, y) = - \frac{(1-\mu)}{2\pi} \log [(x-x_1-\xi)^2 + (y-\eta)^2] - \frac{\mu}{2\pi} \log [(x-x_2-\xi)^2 + (y-\eta)^2] \quad (35)$$

These two terms represent point source distributions. The Greens function retains the mathematical behavior near the origins of the primaries. Integration should be performed over the domain bound by the zero-velocity curves. No contributions are added to this expression from the region beyond the zero-velocity curve because the spacecraft can not cross into this forbidden zone on the basis of energy considerations. Thus there is consistency between the mathematics and physics of the problem.

### C-2. No Thrust in Three-Dimensions

The potential for this problem is defined such that:  $\dot{x} = \Psi_x$ ,  $\dot{y} = -\Psi_y$  and  $\dot{z} = \Psi_z$ . Using similar substitution into eqs (34a)-(34c) and cross-differentiation results in several partial differential equations:

$$\Psi_{xx} + \Psi_{yy} = -V_{xy} \quad (36a)$$

$$\Psi_{xz} = -V_{yz} \quad (36b)$$

$$\Psi_{yz} = 0. \quad (36c)$$

Note that (36a) is the same as previously derived. The latter two equations are additional expressions that show the gravity potential drives the motion.

### C-3. Thrust in Two-Dimensions

With such simplifications, the problem is reduced to altering the partial differential equation form by specifying thrust. This eliminates coupling appearing in the momentum equation in a given direction or removes coupling in another momentum equation.

Results are shown in Table I for several forms of thrust components. Basically, the elliptical canonical nature of these expressions is preserved. For the third case, the results is equivalent to motion in a simplistic linear potential field and there is no clearcut way of accurately predicting the spacecraft's motion. In the last case, thrust is selected to nullify force from the gravity potential reflecting earlier comments regarding large sustained thrust-to-weight ratios. Consequently in this situation, the potential is truly analytical.

Table I

$\dot{x}$	$\dot{y}$	$F_x$	$F_y$	Functional Form
$\Psi_x$	$-\Psi_y$	$+2\dot{x}$	$-2\dot{y}$	$\Psi_{xx} + 2\Psi_{xy} + \Psi_{yy} = -V_{xy}$
$\Psi_x$	$-\Psi_y$	$-2\dot{x}$	$+2\dot{y}$	$\Psi_{xx} - 2\Psi_{xy} + \Psi_{yy} = -V_{xy}$
$\Psi_x$	$-\Psi_y$	$+2\dot{y}$	$-2\dot{x}$	$V_{xy} = 0.$
$\Psi_x$	$-\Psi_y$	$V_y - x$	$V_y - y$	$\Psi_{xx} + \Psi_{yy} = 0.$

Depending upon the judicious selection of thrust, the governing equations are reduced to an equation having the form:

$$\nabla^2 \Psi + \gamma \Psi_{xy} + \xi \Psi = -V_{xy} \quad (37)$$

where the constants depend upon the transformation function and thrust terms.

### C-4. Analytical/Pseudo-Analytical Functions

Another means of solving the equation (37) would be to introduce a direct relationship between the velocity potential and gravity potential. This expression can be expanded to include a potential representing the thrust components. A direct relationship can be defined between the velocity and gravity potentials in a Beltrami equation:

$$\begin{aligned} \Psi_x &= \alpha V_x + \beta V_y \\ \Psi_y &= \delta V_x + \gamma V_y. \end{aligned} \quad (38)$$

Note the similarity with the Cauchy-Reimann equations governing complex variables. The problem is to determine the value of the constants to define the desired potential.

Inversely, when certain derivatives are taken, the resulting equation reduces to the inhomogeneous equation. However, when these derivatives are taken in reverse order, the resulting expres-

sion is a hyperbolic canonical partial differential equation that is a two-dimensional wave equation.

With these thoughts, define the psuedo-analytical function as:

$$\Psi_x = \phi_y - \frac{1}{2} V_y \quad (39)$$

$$\Psi_y = -\phi_x - \frac{1}{2} V_x.$$

If different cross-derivatives are taken, the results yield partial differential equations that depend upon the gravity potential:

$$\begin{aligned} \Psi_{xx} + \Psi_{yy} &= -V_{xy} \text{ and} \\ \phi_{xx} + \phi_{yy} &= -\frac{1}{2} (V_{xx} - V_{yy}) \end{aligned} \quad (40)$$

To a degree this explains why these equations tend to demonstrate an elliptical and hyperbolic nature. For example, a spacecraft's trajectory near the zero-velocity curve domain tends to resemble mixed characteristics in the sense of a Tricomi partial differential equation.

Since this activity focuses upon finding a means for changing the nature of the spacecraft's trajectory, it is not clear how changes in the canonical form of the partial differential equation produces change in the spacecraft's trajectory. The above is provided only to demonstrate that the governing equations can be altered to result in real as well as imaginary characteristics which influence the type of spacecraft orbit.

A more lucid approach is available. Here the governing equations are reduced by phase-space notation into an inhomogeneous vector-matrix equation. The gravity potential represents the inhomogeneous expression which will be referred to in a similar sense as a control vector.

Using the following definitions:

$$\begin{aligned} x_1 &= x & y_1 &= y \\ x_2 &= \dot{x}_1 = \dot{x} & y_2 &= \dot{y}_1 = \dot{y} \end{aligned} \quad (41)$$

This transforms equation(26a) and (26b) into:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 0 \\ V_x \\ 0 \\ V_y \end{bmatrix} - \begin{bmatrix} 0 \\ F_x \\ 0 \\ F_y \end{bmatrix} \quad (42)$$

or the vector-matrix equation:

$$\dot{\bar{x}} = \tilde{\bar{A}} \bar{x} + \bar{u} \quad (43)$$

The dot denotes time differentiation and the matrix has constant coefficients. A bar denotes a vector and a double bar signifies a matrix.

This vector-matrix equation is subject to boundary conditions as a function of the control vector. Due to the elliptical nature of some orbits, one should expect periodic solutions. The solution

of this equation has the form:

$$\bar{x}(t) = \bar{X}_0 e^{\tilde{A}t} + \int_0^t e^{\tilde{A}(t-\xi)} \bar{u}(\xi) d\xi \quad (44)$$

where the vector,  $\bar{X}_0$ , represents initial conditions. To evaluate the degenerate kernel in the integral, let:

$$e^{\tilde{A}(t-\xi)} = \alpha_0 \tilde{I} + \alpha_1 \tilde{A} + \alpha_2 \tilde{A}^2 + \alpha_3 \tilde{A}^3 \quad (45)$$

where the constants are determined by the eigenvalues of the constant matrix. For this particular matrix, the eigenvalues are repeated according to the following characteristic expression:

$$\lambda^4 + 2\lambda^2 + 1 = 0 \quad \text{Then:} \quad \lambda = \pm i, \pm i \quad (46)$$

Since the eigenvalues repeat, the problem is to solve for the coefficients in:

$$(t - \xi)e^{\tilde{A}(t-\xi)} = \alpha_0 \tilde{I} + \alpha_1 \tilde{A} \quad (47)$$

where  $\tilde{I}$  is the identity matrix.

After finding the coefficients and using the Cayley-Hamilton theorem, the final matrix becomes:

$$e^{\tilde{A}t} = \begin{vmatrix} \cos t & \sin t & 0 & 0 \\ \sin t & \cos t & 0 & 2 \sin t \\ 0 & 0 & \cos t & \sin t \\ 0 & -2 \sin t & \sin t & \cos t \end{vmatrix} \quad (48)$$

Subsequently, the resulting matrix has the desired features of periodicity due to the embedded circular functions within the kernel displaying an elliptical nature. However, to examine changes to the 'type' of trajectory with thrust, eigenvalue decomposition is necessary. If the vector defining thrust is provided as a function of the initial state vector (i.e.: thrust as a function of either position or velocity), the matrix is altered by including additional coefficients to those within the A matrix. Here, the thrust acceleration term can have the form:

$$\begin{vmatrix} 0 \\ F_x \\ 0 \\ F_y \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 & 0 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{vmatrix} \quad (49)$$

The resulting characteristic equation has the form:

$$\lambda^4 + \gamma_0 \lambda^3 + (2 + \gamma_1) \lambda^2 + \gamma_2 \lambda + \gamma_3 = 0. \quad (50)$$

This provides several interesting insights. For real solutions, coefficients of the odd powers of the eigenvalue should not vanish. This eliminates eigenvalue multiplicity. If these particular terms are negative, eigenvalues are no longer imaginary but



real. Solution for these real eigenvalues results in hyperbolic sine and hyperbolic cosine terms as a function of time. Similar changes could provide eigenvalues producing parabolic solutions. In this fashion, changing thrust can produce trajectories which can linearly vary as a function of time, or vary in a hyperbolic fashion. Again, as mentioned earlier in the original analysis, the form of the equation can easily be altered without a strong dependency upon an initial velocity constraint.

### III. CONCLUSIONS

This generalized approach demonstrates that Tsien's method leads to a class of solutions where thrust and other acceleration effects change the trajectory classification. In addition to explaining deviate behavior when viewed from the classical sense, constraints placed upon a trajectory based upon energy considerations may no longer be valid under certain thrust applications. The zero-order solution, without consideration of thrust, for classical Keplerian dynamics should be viewed as a small subset of a much larger non-Keplerian domain.

### REFERENCES

1. P. A. Murad: **"Tsien's Method for Generating Non-Keplerian Trajectories"**, AIAA Paper 91-0678, presented at the AIAA 29th Aerospace Sciences Meeting, Reno, Nevada, 7-10 January 1991
2. R. R. Bate, et al: **Fundamentals of Astrodynamics**, 1970, Dover Press.
3. R. M. L. Baker, jr.: **Astronautics- Applications and Advanced Topics**, 1967, Academic Press, New York and London.
4. R. H. Battin: **An Introduction to the Mathematics and Methods of Astrodynamics**, AIAA Education Series, 1987.
5. This effort is a result of a private conversation with Victor R. Bond from McDonnell Douglas Space Systems Company-Houston Division, Texas. The author wishes to acknowledge and deeply appreciates the insights gained during this chance meeting.
6. Major I. Postovit: **"Will There be a 'Shuttle-C'?"**, Prospects for New U.S. Heavy Launch Vehicle Surveyed, 91SV0011R Moscow Aviatsiya I Kosmonavtika, No. 12, Dec 90 pp 42-43, JPRS-UAC-91-005, 15 Aug 1991.
7. A. A. Ivanov, V. D. Kolganov, V. A. Pavshuk, V. F. Semenov, I. V. Kurchatov Institute of Atomic Energy, Moscow: **Conception of the Nuclear Electromotive Power Plant of the Manned Mars System**, presented at the VI symposium on Cosmic/Space Nuclear Energy Systems, Albuquerque, USA, 8-12 January 1989.
8. M. R. LaPointe: **"Antiproton Powered Propulsion with Magnetically Confined Plasma Engines"**, AIAA Journal of Propulsion & Power, Vol

7, Number 5, pp 749-759, Sept-Oct 1991.

9. C. Powell: **"Propulsion by Tachyon Beams"**, AIAA Paper 89-2335, presented at the AIAA/ ASME/SAE/ASEE 25th Joint Propulsion Conference, Monterey, California, 10-12 July 1989.

10. R. Forward: **Space Warps: "A Review of one Form of Propulsionless Transport"**, AIAA Paper 89-2332, presented at the AIAA/ASME/SAE/ASEE 25th Joint Propulsion Conference, Monterey, California, 10-12 July 1989.

11. C. Tarpley, M. J. Lewis, A. P. Kothari: **"Radiation Safety Issues in Single-Stage-To-Orbit Spacecraft Powered by Antimatter Rocket Engines"**, AIAA Journal of Propulsion & Power, Vol 8, Number 1, pp 127-135, Jan-Feb 1991.

12. H. S. Tsien: **"Optimum Thrust Programming for a Sounding Rocket"**, J. Am. Rocket Soc. 21, No. 5(1951) 99.

13. H. S. Tsien: **"Take-off From a Satellite Orbit"**, Jet Propulsion 23, 233, 1953.

14. H. S. Tsien: **"Escape From a Circular Orbit Using Tangential Thrust"**, Jet Propulsion, Vol 28, March, 1958, pp. 167-169.

15. B. E. Baxter: **"Keplerian Representation of a Non-Keplerian Orbit"**, J. Guidance and Control, Vol. 3, No. 2, March-April 1980.

16. R. H. Battin and T. J. Fill: **"Extension of Gauss' Method for the Solution of Kepler's Equation"**, Journal of Guidance and Control, Vol. 2, No. 3, May-June 1979.

17. P. A. Murad: **"A Mathematical Treatise on the Restricted Three-Body Problem of Celestial Mechanics"**, AIAA Paper 75-8, presented at the AIAA 13th Aerospace Sciences Meeting, Pasadena, California, 20-22 January 1975.