SCALAR GAIN INTERPRETATION OF LARGE ORDER FILTERS

Paul A.C. Mason
Research Assistant
N.A.S.A. Graduate Fellow

D. Joseph Mook
Associate Professor
Member AIAA

Department of Mechanical and Aerospace Engineering
State University of New York at Buffalo, NY 14260

Abstract

A technique is developed which demonstrates how to interpret a large fully-populated filter gain matrix as a set of scalar gains. The inverse problem is also solved, namely, how to develop a large-order filter gain matrix from a specified set of scalar gains. Examples are given to illustrate the method.*

Introduction

The intent of the present work is twofold. First, the Scalar Gain Interpretation (SGI) of the gain matrix for discrete filters is developed. The scalar interpretation provides the filter designer with an easily understood description of large-order Multi-Input Multi-Output (MIMO) filters. This interpretation can be used to aid filter designers in analyzing the effects of changes in the gain matrix or other filter parameters. Second, a technique for determining a, fully-populated gain matrix which satisfies specified scalar equivalent gains is demonstrated. Thus, in the common instance that a filter designer does not know certain filter parameters, making the choice of the gain somewhat arbitrary, the gain may be selected based on the scalar equivalents directly rather than by assuming values for the unknown covariances.

The motivation for filtering is to obtain the best estimates of the true states of a dynamic system, given a (generally imperfect) model and a (generally imperfect) set of measurements[5]. To illustrate the concepts of the paper and to motivate the discussion, consider the simple linear discrete Kalman filter, which may be represented as [3]:

$$x_k^- = \Phi_k x_{k-1}^+ \tag{1a}$$

$$P_k^- = \Phi_{k-1} P_{k-1}^- \Phi_{k-1}^T + Q_{k-1} \tag{1b}$$

$$K_k = P_k^- H_k^T \left[H_k P_k^- H_k^T + R_k \right]^{-1} \tag{1c}$$

$$x_k^+ = x_k^- + K_k (z_k - H_k x_k^-) \tag{1d}$$

$$P_k^+ = [I - K_k H_k] P_k^- \tag{1e}$$

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where x is the $nx1$ state vector, Φ is the nxn state transition matrix, P_k is the nxn state error covariance matrix, Q_k is the nxn process noise, K_k is the nxm gain matrix, z_k is an $mx1$ measurement vector modeled by

$$z_k = H_k x_k + v_k \tag{2}$$

where H is the mxn measurement model, v is the measurement error vector, and R_k is the mxm measurement covariance matrix. The subscripts $k-1$, k refer to discrete times t_{k-1}, t_k. The superscript (-) refers to values based on measurements up to but not including z_k, and the superscript (+) refers to values obtained after including measurement z_k.

The operation of the filter proceeds as follows. Eqs. (1a) and (1b) are used to calculate the state estimate and its error covariance between measurements. When a new measurement set z_k is obtained, Eqs. (1d) and (1e) are used to update the values of the state estimate and its error covariance. The updated values depend on the value of the gain matrix calculated using Eq. (1c), and so the accuracy of the filter is directly tied to the determination of K_k. K_k depends on the two covariance matrices: R, representing the error covariance matrix of the measurements, and Q, representing the state error covariance introduced as a result of the approximation of the system dynamics via Eq. (1a).

Theoretically, the Kalman and related filters find the unbiased minimum variance (or maximum likelihood) estimate of the state vector. Unfortunately, this is only true if all the noise and system parameters are known exactly. In practice, neither the measurement noise covariance, R, nor the process noise covariance, Q, is perfectly known. In fact, modeling errors may be far more complex than what is theoretically modeled by the process noise. Moreover, the initial value for P_0 may not be known. Thus, the filter design problem normally requires these matrices to be assumed at least somewhat arbitrarily. For a large, fully populated, non-square gain matrix, it is very difficult to interpret the correlation between the assumed covariances and the filter performance. Often, the gain matrix itself is simply assumed directly. If the assumptions are poor, then the filter will be suboptimal, and in certain cases, the filter may itself become unstable [11].

The main motivation behind the scalar gain interpretation is to give the filter designer some insight into the process of designing a filter and an understanding of how the gain matrix affects the MIMO estimation. Although the theoretical development of classical filtering techniques is sound, the practical implementation of the theory is difficult due to the unknown numerical values of the process noise covariance matrix, and measurement noise covariance matrix. In the case of a scalar filter, the effect of the gain is readily apparent. In order to find an easily understood interpretation of the MIMO gain matrix, a parallel between the scalar filter and the MIMO filter gain is found. This parallel then allows the filter designer to visualize the MIMO filter as several scalar filters.

There are two major reasons why the filter designer's intuition has been removed from the design process. The first reason is due to breakthroughs in estimation theory. For example, many algorithms and theories have been developed to find the noise and filter parameters for the Kalman filter and its gain. Mehra [1] used the innovation sequence property to identify the process noise covariance matrix and the measurement noise covariance matrix. In other methods adaptive Kalman filters have been devised to determine the unknown noise and system parameters [8]. Algorithms such as least squares [8] and the dead beat process noise estimator [7] have also been used to determine the noise and system parameters. However, these algorithms do not describe the effect that the newly found Kalman gain has on the estimation. Other adaptive filters attempt to find the Kalman gain directly [2,9,10]. The accuracy of these techniques varies greatly from case to case, but in any event the interpretation of the gain matrix is difficult at best.

The second reason is that high order systems diminish a designer's general understanding of the effect the Kalman gain has on the estimation. In the scalar case, the affect of the Kalman gain on the estimation is obvious. But as the order of the system increases, the interpretation of the Kalman gain matrix, which determines the optimal estimates, becomes vague.

Incorporating the robust methods mentioned above and the scalar gains interpretation, the MIMO filter designer can determine the best gain and still retain insight as to how the gain affects the estimation.

Scalar Interpretation

The Scalar Filter

To define the problem in this paper, consider a scalar system represented by Eqs.(1). Eq.(1a) provides the state estimate, from the filter model, at time t_k, based on the estimates obtained through time t_{k-1}. At t_k, measurement z_k becomes available. Eq.(1d) is used to update the state estimate based on the residual between z_k and the predicted value $H_k x_k^-$.

In the scalar case, if $H=1$ (i.e., the state is measured directly), the gain K_k has a value between 0 and 1. At $K_k=1$, the filter relies only on the measurement

$$x_k^+ = x_k^- - 1\,(z_k - x^-{}_k) = z_k$$

At $K_k = 0$ the filter relies only on the model estimate:

$$x_k^+ = x_k^- - 0\,(z_k - x^-{}_k) = x^-{}_k$$

When the value of K_k is between one and zero, the filter takes a weighted average of the model and the measurements. In the scalar case, Eq.(1d) can be solved for K_k to obtain

$$K_k = \frac{x_k^+ - x_k^-}{z_k - H_k x^-{}_k} \tag{3}$$

To "normalize" the gain to lie within 0 and 1 we define the "physical scalar gain" (PSG) as

$$PSG = \frac{H_k x^+ - H_k x^-}{z_k - H_k x^-} \tag{4}$$

This concept is illustrated in Figure 1.

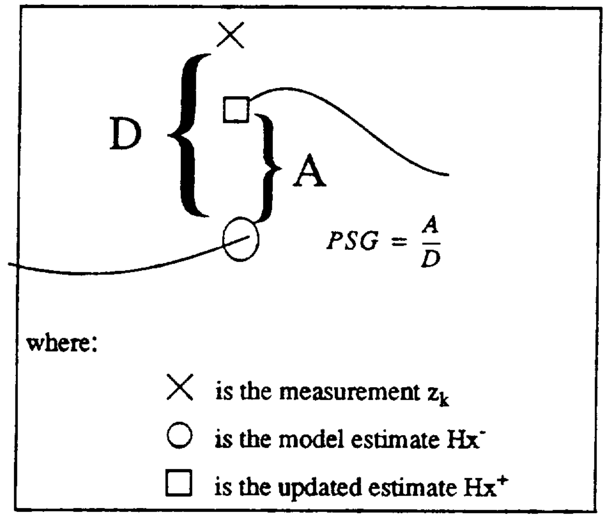

Figure 1 **Graphical interpretation of the scalar filter gain**

In Figure 1, D is the residual between the model output estimate and the measurement. It can be seen that the PSG represents the amount, A, of the residual used in the update, divided by the total residual, D, i.e.,

$$PSG = \frac{H_k x^+ - H_k x^-}{z_k - H_k x^-} = \frac{\text{total correction}}{\text{total residual}} = \frac{\text{update estimate - model estimate}}{\text{measurement - model estimate}} \tag{5}$$

Thus in the scalar case, the physical scalar gain and the Kalman gain are equivalent if the measurement is the state of the filter.

The effect of the filter parameters Q and R on the gain and state estimates can be seen by examining the scalar Kalman gain given by Eq.(1c)

$$K_k = \frac{P_k^- H_k^T}{\left[H_k P_k^- H_k^T + R_k\right]} \tag{6}$$

The Kalman gain can be written as a function of Q and R by substituting the error covariance P^-:

$$K_k = \frac{(\Phi_{k-1} P_{k-1}^+ \Phi_{k-1}^T + Q_{k-1}) H_k^T}{\left[H_k (\Phi_{k-1} P_{k-1}^+ \Phi_{k-1}^T + Q_{k-1}) H_k^T + R_k\right]} \tag{7}$$

For the scalar case where H=1 this simplifies to

$$K_k = \frac{\Phi_{k-1}P^+_{k-1}\Phi^T_{k-1} + Q_{k-1}}{\Phi_{k-1}P^+_{k-1}\Phi^T_{k-1} + Q_{k-1} + R_k} \tag{8}$$

From Eq.(8) it can be seen that if Q is large and R is small, then the denominator and the numerator will be approximately equal. This scenario describes the filtering of a good measurement with a poor model. In this situation the optimal filter relies primarily on the measurements. The other extreme occurs when Q is small and R is large (poor measurements and a good model). The gain for this case is close to zero.

The effect of Q and R on the Kalman gain matrix is very difficult to interpret in the MIMO filter case. With the scalar gain interpretation one can clearly see the effect of Q and R on the Kalman gain matrix, by examining the scalar equivalents.

The Scalar Equivalents for the MIMO filter

In the previous section, the theory of the Kalman filter and the concept of the physical scalar gain (PSG) have been given. In this section, the scalar gain interpretation for MIMO filters is derived. The concepts of the scalar gain interpretation are applicable to any order MIMO system, and to any filter gain. Consider the matrices of Eq.(1d):

$$K_k = \begin{bmatrix} k_{11} & \cdots & k_{m1} \\ k_{21} & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ k_{n1} & \cdots & k_{nm} \end{bmatrix}_k \qquad H_k = \begin{bmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \cdots & \vdots \\ h_{m1} & \cdots & h_{mn} \end{bmatrix}_k$$

$$x^-_k = \begin{bmatrix} x^-_1 \\ \vdots \\ x^-_n \end{bmatrix}_k \qquad z_k = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}_k$$

Expanding Eq.(1d)at time t_k, we may write

$$\begin{aligned} x^+_1 &= x^-_1 + k_{11}(z_1 - h_{1,1}x^-_1 - \dots - h_{1,n}x^-_n) + \\ &\dots + k_{1,m}(z_m - h_{m,1}x^-_1 - \dots - h_{m,n}x^-_4) \\ &\vdots \qquad \vdots \qquad \vdots \\ &\vdots \qquad \vdots \qquad \vdots \\ x^+_n &= x^-_n + k_{n,1}(z_1 - h_{1,1}x^-_1 - \dots - h_{1,n}x^-_n) + \\ &\dots + k_{n,m}(z_m - h_{m,1}x^-_1 - \dots - h_{m,n}x^-_n) \end{aligned} \tag{9}$$

where the subscripts now refer to position in the vector or matrix at time t_k. Eq.(9) can be written in a simplified form by defining D_i as

$$D_i = (z_i - h_{i,1}x_1^- \ldots\ldots - h_{i,n}x_n^-) \qquad i = 1,\ldots m \tag{10}$$

D_i represents the residual between the i^{th} measurement and the i^{th} element of the predicted model output Hx^-. Eq.(1d) can now be written as

$$x_j^+ = x_j^- + K_{j,i}D_i \qquad j = 1\ldots n \tag{11}$$

Thus, the elements of the j^{th} row vector of K_k describes how much of each residual is used in the estimation of the j^{th} state.

In order to find the PSG associated with each residual D_i, the state estimates must be converted to the output estimates by multiplying by output matrix H:

$$y_j = H_j x_i^- + \sum_{i=1}^{m} (HK)_{j,i} D_i \tag{12}$$

In Eq.(12), y_j is the estimate of the j^{th} output at time t_k using all measurements including z_k, and H_j is the j^{th} row of the output matrix. Comparing Eq.(12) with Eq.(4), the physical scalar gain for a MIMO filter is described mathematically as:

$$PSG_i = \frac{(HK)_{i1}(z - Hx^-)_1 + \ldots + (HK)_{im}(z - Hx^-)_m}{(z - Hx^-)_i} \tag{13}$$

In the typical case where the i^{th} estimated output is strongly dependent on the i^{th} measurement but not on the other measurements, Eq.(13) simplifies into:

$$PSG_i \approx \frac{(HK)_{ii}(z - Hx^-)_i}{(z - Hx^-)_i} = (HK)_{ii} \tag{14}$$

If this assumption is not valid then the concept of the physical scalar gain is still valid, but the PSG is not a constant since the random elements of the numerator in Eq.(13) are not canceled by the denominator as in Eq.(14). However, for stationary measurement noise statistics, the expected value of the PSG's are constant. The HK matrix still contains the scalar percentages of the residuals used in the estimation of each output estimate. Comparing Eq.(13) and Eq.(14), the PSG's approach constants if HK becomes diagonal dominant. The concept of the scalar gain interpretation and the diagonal assumption for the HK matrix is reinforced in the expectation analysis section.

The scalar gains can be used to monitor the effect of Q and R. If Q is large and R is small then the corresponding scalar equivalent gains should all be close to 1. In the inverse case if R is large and Q is small the scalar gains should be close to zero. In the common situation where Q and R are not well known, checking the scalar gains can aid in interpreting the effect of assumed values of Q and R.

To illustrate the equivalence between the physical scalar gains and the full gain matrix consider the following simple example, where n=4 and m=2.

Let

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad z = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$$

$$k = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad x^- = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Substituting into Eqs.(1), we find

$$Hx^- = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \qquad D = (z - Hx^-) = \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix}$$

$$KD = \begin{bmatrix} 0.2 \\ 0.05 \\ 0.2 \\ 0.05 \end{bmatrix}$$

$$x^+ = x^- + KD = \begin{bmatrix} 0.3 \\ 0.25 \\ 0.5 \\ 0.45 \end{bmatrix}$$

$$y = Hx^+ = \begin{bmatrix} 0.3 \\ 0.25 \end{bmatrix}$$

$$HK = \begin{bmatrix} 0.5 & 0. \\ 0. & 0.5 \end{bmatrix}$$

The physical scalar gains (PSG) are

$$\frac{Hx^+ - Hx^-}{z - Hx^-} = \frac{\begin{bmatrix} 0.3 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}}{\begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} - \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

The physical scalar gains are the same as the diagonal elements of the diagonally dominant HK matrix.

The Expectation Analysis

The PSG described in Eq.(14) can be written in the following simplified form:

$$(PSG)_j = \frac{\sum_{i=1}^{m} (HK)_{ji} [z_i - (Hx^-)_i]}{[z_j - (Hx^-)_j]} \tag{15}$$

Performing the expectation analysis of the PSG results in the predicted numerical value of the PSG at steady state. This value describes how much weight is placed on the measurement vs. the model of a particular measurement estimate. The expectation analysis of the PSG is similar to the analysis of the error covariance [3]. As in the case of the error covariances the PSG is squared and then the expectation is taken:

$$E[(PSG)_j^2] = E\left[\left(\frac{\sum_{i=1}^{m} (HK)_{ji} [z_i - (Hx^-)_i]}{[z_j - (Hx^-)_j]}\right)^2\right] \tag{16}$$

Since H and K matrices are time invariant (constants), they can be taken outside the expectation

$$E[(PSG)_j^2] = \frac{\sum_{i=1}^{m}\sum_{l=1}^{m} (HK)_{ji} (HK)_{jl} E\{[z_i - (Hx^-)_i][z_l - (Hx^-)_l]\}}{E\{[z_j - (Hx^-)_j][z_j - (Hx^-)_j]\}} \tag{17}$$

The expected value in the numerator and denominator is expanded by substituting for $z=Hx+v$.

$$E\{[Hx+v-Hx^-][Hx+v-Hx^-]^T\} = E\{[H(x-x^-)+v][H(x-x^-)+v]^T\} = RHS$$

The right-hand side can be expanded by multiplying out the internal terms to obtain

$$RHS = E\{H(x-x^-)(x-x^-)^T H^T + vv + H(x-x^-)v^T + v(x-x^-)^T H^T\} \tag{18}$$

Note that $(x-x^-) = e^-$, $E[vv^T] = R$ and $E[e^-(e^-)^T] = P_{xx}^-$. Eq.(18) now simplifies into

$$RHS = HP_{xx}^- H^T + R + HE\{v(e^-)^T\} + E\{(e^-)v^T\}H^T \tag{19}$$

Assuming that the noise vector, v, and the estimation error, e, are uncorrelated, the last two terms in Eq.(19) are zero. Eq.(19) can now be written as:

$$RHS = HP_{xx}^- H^T + R \tag{20}$$

Substituting this back into Eq.(17) yields

$$E[(PSG)_j^2] = \frac{\sum_{i=1}^{m}\sum_{l=1}^{m} (HK)_{ji} (HK)_{jl} (HP_{xx}^- H^T + R)}{[H_j (P_{xx}^-)_{jj} H_j^T + R_{jj}]} = \frac{(HK)_j [HP_{xx}^- H^T + R] (HK)_j^T}{[H_j (P_{xx}^-)_{jj} H_j^T + R_{jj}]} \tag{21}$$

If HK, R and $HP_{xx}^- H^T$ are diagonal matrices, then Eq.(21) may be simplified to

$$E[(PSG)_j^2] = (HK)_{jj}^2 \tag{22}$$

This assumption is valid for systems where the covariance error matrix is diagonal; therefore, the errors associated with the states are not coupled. Taking the square root of both sides gives the expression for the approximation of the expected magnitude of the PSG.

$$\sqrt{E[(PSG)_j^2]} = \sqrt{\frac{(HK)_j [HP_{xx}^{-} H^T + R] (HK)_j^T}{[H_j (P_{xx}^{-})_{jj} H_j^T + R_{jj}]}} \tag{23}$$

The square root of the squared expected value PSG is a function of not only the j^{th} measurement but also of the other measurements and the residuals associated with them [Eq.(23)].

Finding a Fully Populated Gain Matrix

Determination of Q and R (or directly, K) is rarely simple for a large order filter. Since it is easier to determine and understand a scalar gain rather than a fully populated gain matrix, MIMO filters are often written as sets of decoupled scalar filters. The process of transforming a MIMO filter into several scalar ones is clumsy and may result in errors. This is the primary motivation for finding a method to determine an equivalent fully populated gain matrix from a set of scalar gains.

Let the specified scalar gains be placed in the diagonal G_d matrix (mxm). The fully populated K matrix is determined by equating the scalar gains to the HK matrix. If H is a square matrix, then premultiplying both sides by the inverse of H yields an equation for the gain matrix as a function of the scalar gains:

$$\begin{aligned} H^{-1} H K_d &= H^{-1} G_d \\ K_d &= H^{-1} G_d \end{aligned} \tag{24}$$

where K_d is the MIMO filter gain matrix designed from the scalar gains. Since H is generally not a square matrix one can not use this procedure to determine K_d. To circumvent this problem the Moore -Penrose Pseudoinverse [4,6] is utilized to determine the pseudoinverse of a non-square matrix H:

$$K_d = (H)^{*-1} G_d \tag{25}$$

where $(\ )^{*-1}$ represents the pseudoinverse.

This method of determining a fully populated gain matrix from a set of scalar gains, constrains the output of the filter. It does not constrain the states of the filter. Since the scalar gain interpretation is associated with the output of the filter and not the states, then the fully populated gain matrix found from these gains is forced to have the same output as dictated by the scalar gains. A design gain matrix that has the same scalar gains as another fully populated gain matrix will have the same output estimate but not necessarily the same states.

Simulation Results

A filter simulation is used to verify the theory of the scalar gain interpretation. There are two objectives of this section. The first objective is to demonstrate that the SGI does approximate the scalar gains of the filter. The second objective is to verify that a fully populated gain matrix can be determined from a set of specified scalar gains.

To accomplish the first objective an 8th-order time-varying Kalman filter is used. During the execution of this filter the average physical scalar gains are calculated. These averages are compared to the expected scalar gains to validate the scalar gain interpretation. To ensure a fair comparison, the constant gain filter is implemented with the steady-state gain matrix, K. The physical scalar gains of both filters are compared to prove the equivalence of the scalar gains interpretation. The parameters and model of the filter are described in a problem statement.

The second part of this section uses the scalar gains from part 1 to determine the fully populated design gain matrix, K_d A constant gain filter is implemented using K_d. The physical scalar gains are calculated during the execution of the filter. The physical scalar gains are compared to expected scalar gains to prove the equivalence of the design gain matrix to the set of scalar gains. Then, the output estimates of both filters are compared to prove that the K_d and the original steady-state gain matrix produces the same output estimates.

Consider the following example to illustrate the scalar gain interpretation. The measurements are created from the following state space equations.

$$\begin{gathered} \dot{x} = Ax \\ measurements = Cx + noise \end{gathered} \tag{26}$$

A is the model of the states and C is the output matrix of a 8 state system. The true model of the system is

A =

0.0	0.0	0.0	0.0	1.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	1.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0
-10.0	5.0	0.0	0.0	0.0	0.0	0.0	0.0
5.0	-10.0	5.0	0.0	0.0	0.0	0.0	0.0
0.0	5.0	-10.0	5.0	0.0	0.0	0.0	0.0
0.0	0.0	5.0	-5.0	0.0	0.0	0.0	0.0

The state transition matrix is use to find the state trajectories of the true model. A perturbed model is used as the filter model. This perturbed model is created by changing the last four elements in the true A matrix to -3.5 and changing -10.0 , -5.0 to -9.5 , -4.5 respectively. After the states have been converted to output via the output matrix, 26% noise is added to the output to create the measurements. The measurement noise covariances is $R \approx 2.2 \cdot I_{4x4}$. These perturbations and simulated measurements noise are arbitrarily; the values given here are simply for demonstration purposes.

The simulation is implemented for 500 and 1000 measurement points. The Kalman gain reaches steady state at approximately the 200th time step. The steady state Kalman gain matrix is.

K =

0.5124	0.0068	0.0033	-0.0051
0.0115	0.6262	0.0057	-0.0002
0.0032	0.0039	0.5161	-0.0012
-0.0052	0.0001	-0.0013	0.5175
-0.0913	0.0622	0.0054	-0.0018
0.0740	-0.0829	0.0722	0.0058
0.0052	0.0615	-0.0847	0.0767
-0.0021	0.0023	0.0753	0.0407

The H*K matrix at steady state is

0.5124	0.0068	0.0033	-0.0051
0.0115	0.6262	0.0057	-0.0002
0.0032	0.0039	0.5161	-0.0012
-0.0052	0.0001	-0.0013	0.5175

The expected PSGs and the diagonal elements of the H*K matrix are almost identical, since the noise covariance, error covariance, and process noise are assumed to be diagonal. Since the H*K matrix and the expectation gain are diagonally dominant the scalar gains can be found by Eq.(23) or by taking the ii^{th} element of H*K. The expected scalar gains are

Table 1: The Scalar Gain

gain 1	gain 2	gain 3	gain 4
0.5122	0.6261	0.5161	0.5174

The average physical scalar gains (PSGs), of the time-varying filter simulation, are calculated using the matrix form of Eq.(4), which is equivalent to Eq.(13). The average physical scalar gains of the time-varying filter are

Table 2: The average PSG of the variable gain filter

ave PSG	gain 1	gain 2	gain 3	gain 4
500 pts	0.5851	0.6191	0.5743	0.5211
1000 pts	0.5591	0.6188	0.5156	0.5190

Notice the similarity between these gains and the expected gains.

Next a constant gain filter is executed with the steady-state gain matrix shown above. The average PSGs of the constant gains filter are

Table 3: The average PSG of the constant gain filter

ave PSG	gain 1	gain 2	gain 3	gain 4
500 pts	0.5842	0.6189	0.5747	0.5202
1000 pts	0.5586	0.6187	0.5158	0.5185

The PSGs of the constant gain filter and the time-varying filter approximate the expected scalar gains in Table 1. The average PSGs from the constant gain filter are closer to the predicted physical scalar gains, since the gain matrix of the time-varying filter does not reach steady state instantly. As the number of cycles increases this average approaches the predicted value. This trend can be seen in the comparison between the 500 and 1000 point average. The physical scalar gains of these filters are constantly fluctuating, but the average of these gains approach the expected PSGs as time goes to infinity. This fluctuation is due to the off-diagonal residuals of the $H*K$ matrix.

The second objective of this section is to prove that the design gain matrix can be found from the scalar gains. This gain matrix will produce the same estimated outputs. The set of scalar gains used in this part are taken from Table 1. The design gain, K_d is determined for a diagonal dominant $H*K$ matrix and the diagonal G_d. This is done to test the uniqueness of the method.

The K_d is found from the diagonally dominant $H*K$ matrix. A constant gain filter is executed with this K_d. matrix and the same initial conditions as in part 1. The average physical scalar gains of this filter are

Table 4: The average PSG of the design gain filter

ave PSG	gain 1	gain 2	gain 3	gain 4
500 pts	0.5948	0.6175	0.5109	0.5288
1000 pts	0.5510	0.5936	0.5176	0.5216

Like the constant gain and time varying gain filter, the PSGs of this filter are not constant.

Next, the K_d for a diagonal G_d is found. The gain matrix that is produced from this method only uses the i^{th} residual to determine the i^{th} estimate. The PSGs are not a function of the off-diagonal residual effects, therefore physical scalar gains of the diagonal gain filter are constant. The diagonal elements of the G_d matrix are taken from Table 1. The average PSGs are

Table 5: The average PSG of design-diagonal gain filter

ave PSG	gain 1	gain 2	gain 3	gain 4
500 pts	0.5122	0.6261	0.5161	0.5174
1000 pts	0.5122	0.6261	0.5161	0.5174

The physical scalar gains of the diagonal gain filter are constant; therefore, the average PSG is also constant.

Tables 2,3,5 illustrate how all three constant gain filters approach the predicted scalar gains. The comparison of 500 and 1000 points sets of average gains illustrate how the accuracy of approximation increases as time goes to infinity. Therefore it can be inferred that the SGI is a good approximation of the true physical gains of the filter.

Since the scalar gains are the same, the output estimates should be the same. Figures 2-4 contain the first output estimates of the three constant gains filters.

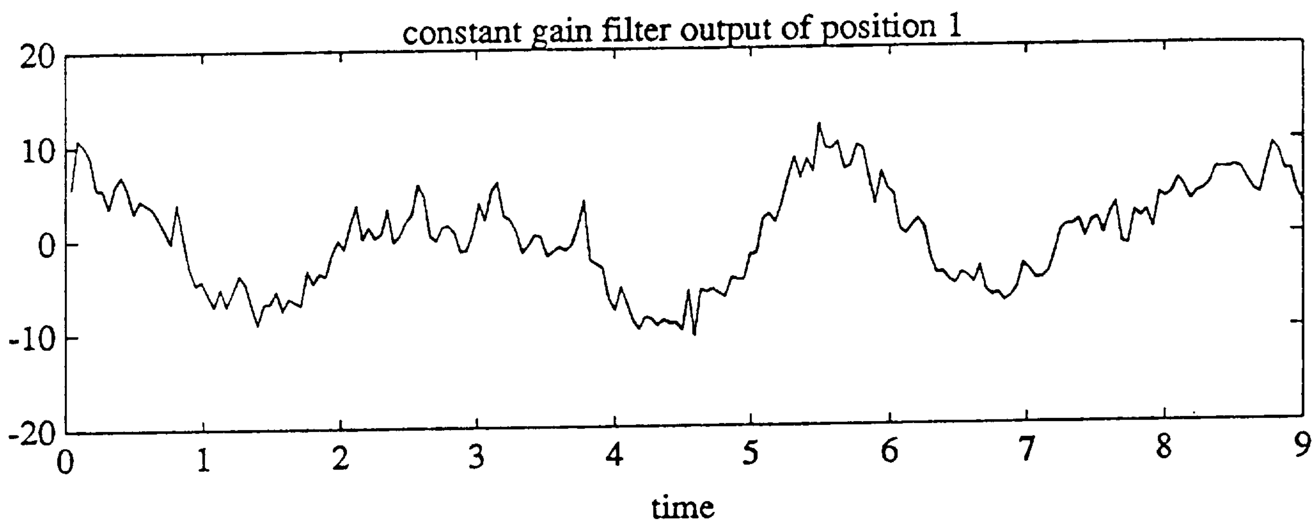

Figure 2 **The output estimates of the constant filter**

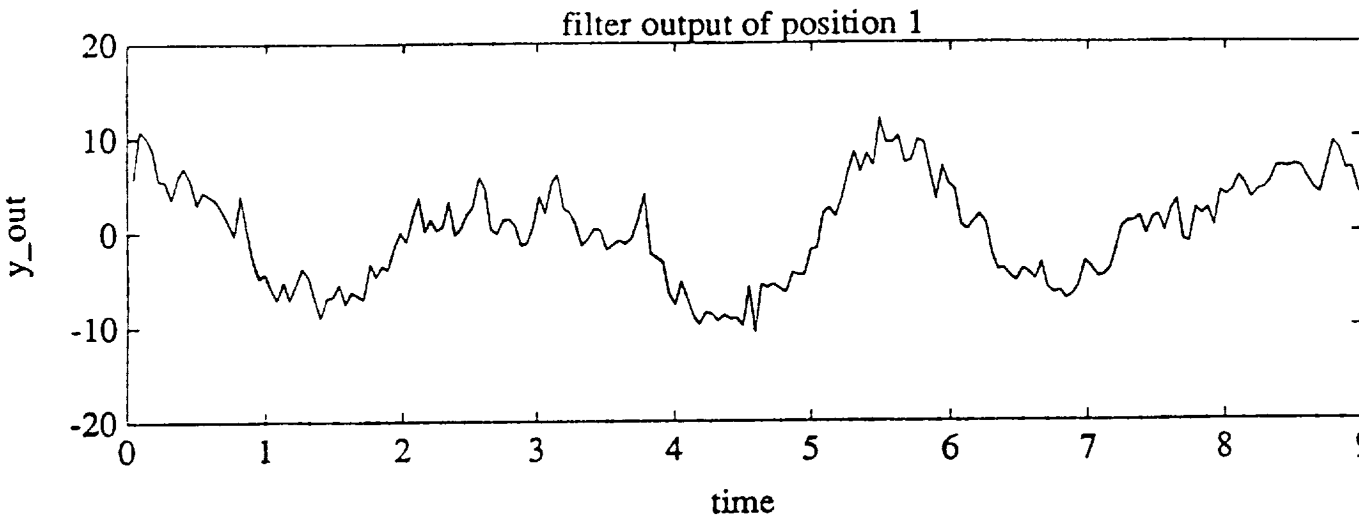

Figure 3 **The output estimates of the constant design gain filter**

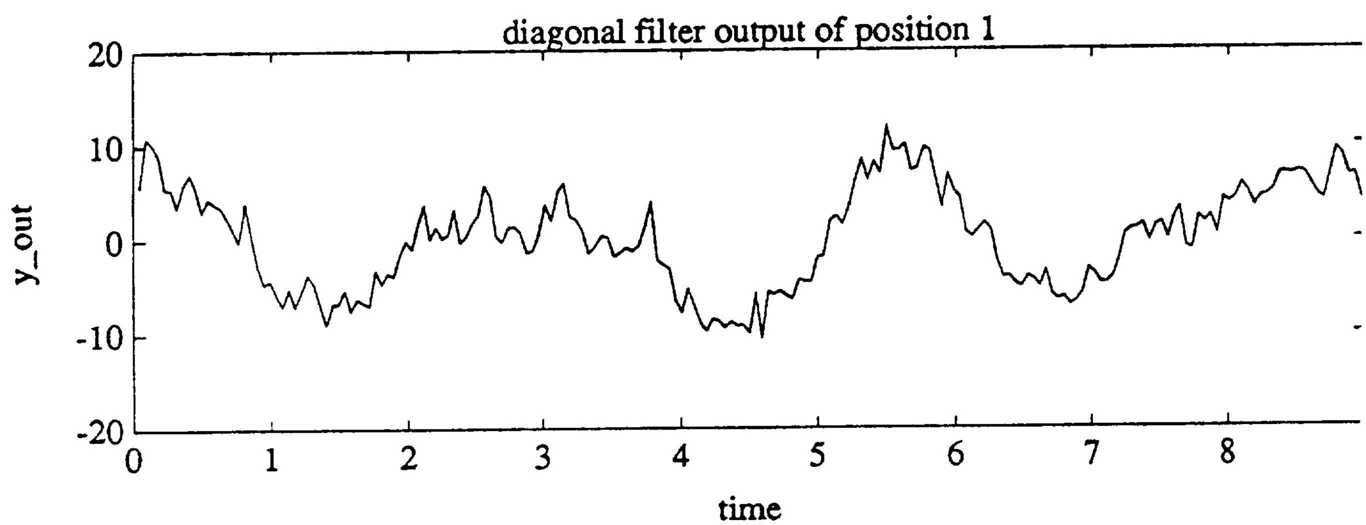

Figure 4 **The output estimates of the constant diagonal design gain filter**

Figures 2-4, which contain the first output estimate, are essentially identical. Therefore the three constant gain filters have an equivalent effect on the output estimates and it can be inferred that a gain matrix can be found from a set of scalar gains. This accomplishes the second objective.

Conclusion

This paper has demonstrated that a large fully populated gain matrix can be interpreted as a set of scalar gains; and, conversely, a fully populated gain matrix can be developed from a specified set of scalar equivalences. The simulation results verify that the scalar gain interpretation is a good approximation of the true filter scalar gains associated with the gain matrix. The accuracy of this approximation increases as the number of measurements samples approaches infinity. Also, the results showed that a fully populated gain matrix can be found from a set of scalar gains. The fully populated gain matrix found from this method is not unique. This was demonstrated by the comparison of the filtering results of the Kalman gains found by the diagonal G_d and the diagonal dominant $H*K$ matrix.

With the scalar gain interpretation, filter designers can easily interpret the effect of assumed values for the covariance matrices Q and R (or, the gain matrix itself). Alternatively, the scalar gains may be specified directly and the equivalent fully populated gain matrix may be found.

References

[1] Mehra, R. K.," On The Identification Of Variance And Adaptive Kalman Filtering", **IEEE Transactions Automatic Control,** vol. AC-15, pp. 175-184. 1970.

[2] Leondes, C. T., and Siu, T. K. "Identification Of Both The Unknown Plant And Noise Parameters Of The Kalman Filter", **Inter. J. Sys. Sci.**,vol II, no. 6, pp. 711-720., 1980.

[3] Gelb, A., **Applied Optimal Estimation**, MIT Press,1974.

[4] Golub, G.H., and Van Loan, C., **MATRIX Computations**, John Hopkins University Press, 1985.

[5] Chui, C. K., and Chen, G, **Kalman Filtering With Real-Time Applications**, Springer Series in Information Science 17, Springer-Verlag, 1987.

[6] Albert, A., **Regression And The Moore-Penrose Pseudoinverse**, Academics Press 1972.

[7] Gutman, P., and Velger, M., "Tracking Targets With Unknown Process Noise Using Adaptive Kalman Filtering", **Proceedings of the 27th IEEE Conference on Decision and Control**, vol 1., pp. 869-874, 1988.

[8] Moghaddamjoo, A., and Kirlin, R.L., "Robust Adaptive Kalman Filtering With Unknown Inputs", **IEEE Transactions on Acoustics, Speech, and Signal Processing**, v27, n8, pp. 1166-1175, Aug 1989.

[9] Sinha, N.K., and Alvan, T., " Adaptive State Estimation For Systems With Unknown Noise Covariance", **Int. J. System Sci.**, v 8, n4, pp. 377-384, 1977.

[10] Carew, B., and Belanger, P. R., "Identification Of Optimal Filter Steady State Gain For Systems With Unknown Noise Covariance," **IEEE Tran. A.C.**, vol. AC-18, pp. 582-589, 1973.

[11] Fitzgerald, R. J., "Divergence Of The Kalman Filter", **IEEE Tran. A.C.**, vol. AC-16, pp. 736, 1971.