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Assessment of Closure Coefficients for Compressible-Flow Turbulence Models

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Assessment of Closure Coefficients for Compressible-Flow Turbulence Models

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NOMENCLATURE

A	q_w/τ_w
a_w	sound speed based on the wall temperature
B_q	heat transfer parameter
C	wall-law parameter
C_1	sublayer parameter
c_p	specific heat
$c_{\epsilon 1}$	turbulence model constant for ϵ
$c_{\epsilon 2}$	turbulence model constant for ϵ
c_μ	turbulence model constant
c_1	turbulence model constant
c_2	turbulence model constant
D	$(A^2 + R)^{1/2}$
d_1	constant defined in equation (18)
d_2	constant defined in equation (19)
d_3	constant defined in equation (20)
k	turbulence kinetic energy
l	index defined in equation (11)
M	free-stream Mach number
M_τ	U_τ/a_w
m	index defined in equation (11)
n	index defined in equation (11)
P_k	production of kinetic energy
Pr_t	turbulence Prandtl number
q	heat flux
q_w	heat flux at the wall
R	$2c_pT_w/Pr_t$
Re_θ	Reynolds number based on the momentum thickness
T	temperature
T_w	temperature at the wall
U	velocity
U_c	transformed velocity
U_c^+	U_c/U_τ, dimensionless transformed velocity
U_τ	frictional velocity
$\overline{uv}$	shear stress
y	vertical distance from the wall
y^+	$U_\tau y/\nu_w$, dimensionless normal coordinate
ϵ	turbulence energy dissipation
κ	Van Karman constant
κ_T	Van Karman constant for temperature
μ_t	turbulence viscosity
ν_t	kinematic turbulence viscosity
ν_w	kinematic turbulence viscosity at the wall
ρ	density
ρ_w	density at the wall
σ_k	turbulence Prandtl number for k
σ_ϵ	turbulence Prandtl number for ϵ
σ_ϕ	turbulence Prandtl number for ϕ
τ	k/ϵ, turbulence time scale
τ_w	wall shear stress
ϕ	generalized turbulence length-scale variable
ω	ϵ/k

SUMMARY

A critical assessment is made of the closure coefficients used for turbulence length scale in existing models of the transport equation, with reference to the extension of these models to compressible flow. It is shown that to satisfy the compressible "law of the wall," the model coefficients must actually be functions of density gradients. The magnitude of the errors that result from neglecting this dependence on density varies with the variable used to specify the length scale. Among the models investigated, the $k-\omega$ model yields the best performance, although it is not completely free from errors associated with density terms. Models designed to reduce the density-gradient effect to an insignificant level are proposed.

INTRODUCTION

Calculations of high-Mach-number turbulent flows have become a major challenge in computational fluid dynamics in recent years. Traditionally, models developed for incompressible flows have been extended to compressible flows with little or no modification when mass-weighted dependent variables are used. Our investigation centers on the "dissipation-transport" equation and its role in predicting the compressible law of the wall. This equation, or some other equation implying a length scale for the turbulence, is used in "two-equation" eddy-viscosity transport models and in full Reynolds-stress transport models.

The Van Driest compressible law of the wall is derived from inner-layer similarity arguments that lead to the "mixing-length" formulas for velocity and temperature (Bradshaw, 1977).

$$\partial U/\partial y = (\tau_w/\rho)^{1/2}/\kappa y \quad (1)$$

$$\partial T/\partial y = -q/[\rho c_p(\tau_w/\rho)^{1/2}\kappa_T y] \quad (2)$$

Assuming $\tau = \tau_w$, integration of (1) and (2) yields the temperature and velocity profiles

$$T = C_1 T_w - Pr_t U q_w/(c_p \tau_w) - Pr_t U^2/(2c_p) \quad (3)$$

$$U_c^+ = U_c/U_\tau = \frac{1}{\kappa}\ln y^+ + C \quad (4)$$

where Pr_t is the turbulent Prandtl number and is equal to κ/κ_T. The coefficient C_1 is necessary here because, strictly, equations (1) and (2) do not apply in the viscous sublayer region. The quantities U_τ and y^+ are defined with respect to the physical properties at the wall as

$$U_\tau = (\tau_w/\rho_w)^{1/2} \quad (5)$$

and

$$y^+ = yU_\tau/\nu_w \quad (6)$$

The coefficients C_1 and C are assumed to have the following forms, recommended from a data fit by Bradshaw (1977).

$$C_1 = 1 \qquad C = 5.2 + 95M_\tau^2 + 30.7B_q + 226B_q^2 \quad (7)$$

where $M_\tau = U_\tau/a_w$ and $B_q = q_w/(\rho_w c_p U_\tau T_w)$. The Van Driest transformed velocity, U_c, is defined by

$$U_c = \int_0^U (\frac{\rho}{\rho_w})^{1/2} dU \quad (8)$$

Substituting $\rho/\rho_w = T_w/T$ and equations (3) and (7) into equation (8) yields

$$U_c = R^{1/2}[\sin^{-1}(\frac{A+U}{D}) - \sin^{-1}(\frac{A}{D})] \quad (9)$$

where $R = 2c_p T_w/Pr_t$, $A = q_w/\tau_w$ and $D = (A^2+R)^{1/2}$.

It should be noted that the Van Driest compressible law of the wall has been shown to accurately represent experimental boundary-layer velocity profiles covering a wide range of Mach numbers and Reynolds numbers (Fernholz and Finley, 1980).

Figure 1 shows computed velocity profiles at $Re_\theta = 10,000$, transformed using equation (9). The computations were obtained from the Navier-Stokes calculations of a Mach 5 flow over an insulated flat plate. Two versions of low-Reynolds-number $k-\epsilon$ models were used: Chien (1982) and Launder and Sharma (1974). Also shown is a line representing the compressible law of the wall (eq. (4)), for $\kappa = 0.41$. It can be seen that the calculated velocity profiles exhibit a steeper gradient than the expected value of $1/\kappa$. This behavior was also observed in a Reynolds-stress-model calculation based on a low-Reynolds-number version of the Launder-Reece-Rodi model (Launder et al., 1975; Launder and Shima, 1989). To eliminate the uncertainties associated with the low-Reynolds-number parts of the models, further calculations were performed that were based on the standard high-Reynolds-number version of the $k-\epsilon$ model coupled with compressible-wall-function treatments (Viegas and Rubesin, 1985; Huang, 1990).[1] Figure 2 shows that at $M = 0.1$, the predicted profile follows the Coles law of the wall, but at $M = 5$, the calculation displays an earlier rise of the profile in the log-law region.[2] It should be noted that the displacement of the profiles between the two calculations is the result of differences in the treatment of the viscous wall region and is not relevant to the present discussion. The excessive slope of the velocity profile has also been observed by other investigators, both with the $k-\epsilon$ model (Aupoix, 1990) and with a Reynolds-stress model (Viegas, J. R., private communication).

The above results clearly indicate that standard turbulence models do not accurately predict boundary layers at high Mach numbers, and since the defect occurs with both the $k-\epsilon$ and the Reynolds-stress models it appears that the trouble lies in the ϵ-equation.

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ASSESSMENT OF CLOSURE COEFFICIENTS

In this section, attention is focused on the inner layer of a constant-pressure boundary layer, where convective transport is negligible and the eddy-viscosity hypothesis is formally applied. The velocity profile is governed by

$$\frac{\partial}{\partial y}(\mu_t \frac{\partial U}{\partial y}) = 0 \tag{10}$$

where μ_t is the turbulent viscosity and, for a $k-\epsilon$ two-equation model, is defined as $\mu_t = c_\mu \rho k^2/\epsilon$. The constant c_μ is fixed at 0.09 to satisfy $-\overline{uv}/k = 0.3$ in local equilibrium.

To discuss the problem in a general manner, a new variable, ϕ, is defined in relation to the dissipation of the turbulence energy, ϵ.

$$\phi = \rho^n k^m \epsilon^l \tag{11}$$

Equation (11) allows the construction of a variety of length-scale equations by the proper choice of n, m, and l.

In the standard format of a high-Reynolds-number two-equation model, the transport equations for k and ϕ, respectively, can be written as

$$-\frac{\partial}{\partial y}(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial y}) = \rho P_k - \rho\epsilon \tag{12}$$

and

$$-\frac{\partial}{\partial y}(\frac{\mu_t}{\sigma_\phi}\frac{\partial \phi}{\partial y}) = c_1 \rho P_k \frac{\phi}{k} - c_2 \rho\epsilon \frac{\phi}{k} \tag{13}$$

where σ_k and σ_ϕ are turbulent Prandtl numbers for k and ϕ, respectively. P_k is the generation term

$$P_k = -\overline{uv}\frac{\partial U}{\partial y} = \nu_t(\frac{\partial U}{\partial y})^2 \tag{14}$$

where $-\overline{uv} = \tau/\rho = \tau_w/\rho$.

In equation (13), c_1 and c_2 are dimensionless coefficients and are related to the corresponding coefficients of the original ϵ-equation, $c_{\epsilon 1}$ and $c_{\epsilon 2}$, according to

$$c_1 = l c_{\epsilon 1} + m \tag{15}$$

$$c_2 = l c_{\epsilon 2} + m \tag{16}$$

In the standard $k-\epsilon$ model, $c_{\epsilon 2} = 1.92$; in the newer two-equation models, such as the $k-\omega$ and $k-\tau$ models, $c_{\epsilon 2}$ is adjusted to 1.8 in order to provide a better fit of the decay law of homogeneous isotropic turbulence. A constant value of $c_{\epsilon 1}$ is generally chosen by computer optimization.

To ensure that the plot of U_c^+ against ln y^+ has a slope of $1/\kappa$ in the logarithmic region, a unique relation must exist between κ and the other coefficients. The derivation of this relation is similar to that for the incompressible case, but with the following density effects taken into account. First, in this region the usual

[1] The model coefficients are $c_\mu = 0.09$, $c_{\epsilon 1} = 1.44$, $c_{\epsilon 2} = 1.92$, $\sigma_k = 1$ and $\sigma_\epsilon = 1.17$. The value of σ_ϵ is altered from 1.3 to 1.17 to give a better fit to the *incompressible* law of the wall.

[2] Experimental evidence has suggested that the correction function for the wake region is nearly the same function of the empirically chosen Reynolds number as at low speeds.

incompressible assumption of $\partial k/\partial y = 0$ is replaced by $\partial \rho k/\partial y = 0$, because $-\overline{uv}/k$ = constant and $\tau = \tau_w$ = $-\rho\overline{uv}$. Second, the dissipation relation $\epsilon = U_\tau^3/\kappa y$ is replaced by $\epsilon = (\tau_w/\rho)^{3/2}/\kappa y$, because the assumption in this region is that $\epsilon = P_k = -\overline{uv}(\partial U/\partial y)$. Finally, the density can no longer be factored out of the diffusion terms: $\partial/\partial y(\mu_t \ldots) \neq (1/\rho)\partial/\partial y(\nu_t \ldots)$.

By substituting these assumptions into equation (13) and performing some mathematical manipulations, one can obtain

$$\frac{c_\mu^{1/2}\sigma_\phi}{l^2\kappa^2}(c_2 - c_1) = 1 + \frac{1}{l^2}\left[d_1\frac{y}{\rho}\frac{\partial \rho}{\partial y} + d_2\frac{y^2}{\rho}\frac{\partial^2 \rho}{\partial y^2} + d_3\left(\frac{y}{\rho}\frac{\partial \rho}{\partial y}\right)^2\right] \quad (17)$$

where

$$d_1 = n - m - 2l + 3l^2 + 2ml - 2nl + c_1\frac{\sigma_\phi}{\sigma_k} \quad (18)$$

$$d_2 = n - m - \frac{3}{2}l + c_1\frac{\sigma_\phi}{\sigma_k} \quad (19)$$

$$d_3 = (n - m - \frac{3}{2}l)(n - m - \frac{3}{2}l - \frac{1}{2}) - \frac{3}{2}c_1\frac{\sigma_\phi}{\sigma_k} \quad (20)$$

It can be seen that in order to extend the models for incompressible flows to compressible flows without having to adjust the closure coefficients, the second term on the right-hand side of equation (17) must be negligible compared to unity. This can be accomplished either with small values of d_1, d_2, and d_3 or with small density-gradient terms, $(y/\rho)\partial\rho/\partial y$, $-(y^2/\rho)\partial^2\rho/\partial y^2$, and $(y^2/\rho^2)(\partial\rho/\partial y)^2$.

Table 1 lists the coefficients d_1 to d_3 for different turbulence models. The magnitude of the density-gradient terms associated with d_1 to d_3 can be obtained by assuming that equations (3) and (4) are valid in the range $y^+ = 30$ to 1000 at the Reynolds number of the calculation in figure 1. The density-gradient terms are shown in figure 3 for $M = 5$ and an insulated wall. It can be seen clearly that the density effects at the Reynolds number of the computation are not negligible and have to be taken into account in order to correctly predict the law-of-the-wall behavior.

Table 1. Coefficients associated with density-gradient terms

Model	n	m	l	d_1/l^2	d_2/l^2	d_3/l^2
$k-\epsilon$	0	0	1	2.68	0.18	0.48
$k-\omega^2$	0	−2	2	0.63	−0.13	0.19
$k-\omega$	0	−1	1	0.56	0.06	−0.33
$k-\tau^a$	0	1	−1	2.47	0.97	0.70

[a]The τ-equation ($\tau = k/\epsilon$) (Speziale et al., 1990) written in the form of equation (13) will lead to $\sigma_\tau = -1.44$.

Equations (3), (10), (12), and (13) have been solved between $y^+ \approx 50$ and $y^+ \approx 1000$ for a Mach 5 flow over an insulated wall. The temperature profile is given by equation (3), with the free-stream temperature given at 15°C. The velocities at two boundaries are given, one fixed at the free-stream condition and the other calculated from the law of the wall (eq. (4)). The boundary conditions for k and ϕ are estimated from $k = (\tau_w/\rho)/c_\mu^{1/2}$ and $\phi = (\rho^{n-3l/2-m}\tau_w^{3l/2+m}/c_\mu^{m/2})/(\kappa y)^l$.

Figure 4 shows the calculated density-gradient terms obtained using the $k-\epsilon$ model. These results should be compared with those of figure 3, in which the profiles were obtained under the assumption that the compressible law of the wall has been satisfied. The square root of the right-hand side of equation (17) can be viewed as the ratio of the theoretical κ (= 0.41) to the calculated κ. This ratio is shown in figure 5. The value can be as high as 1.6 in the $k-\epsilon$ model; in contrast, the $k-\omega$ ($\omega = \epsilon/k$) model of Wilcox (1988; also Wilcox and Traci, 1976; and Wilcox, D. C., private communication) displays only a mild increase of $1/\kappa$. This is because the coefficients d_1 to d_3 associated with density-gradient terms are all relatively small in the $k-\omega$ model, as shown in table 1. Figures 6(a) and 6(b) show the calculated velocity profiles for the $k-\epsilon$ and $k-\omega$ models, respectively. These profiles agree with the behavior of κ shown in figure 5.

POSSIBLE REMEDIES

One obvious way to predict the correct Van Driest law-of-the-wall profile is to incorporate the density-gradient terms directly into the coefficient c_1.[3] This is done by an iterative procedure and has been found to return the $k-\epsilon$ model to the expected law-of-the-wall profile. As shown in figure 7, the value of c_1 required in the ϵ-equation drops to 50% of its original value at $y^+ = 1000$. This is too drastic a change to be acceptable.

[3]It appears that c_1 is a better candidate to adjust than σ_ϕ because equation (17) is derived under the assumption that σ_ϕ is a constant.

Another feasible remedy is to find a new model that will make d_1, d_2, and d_3 all zero. Unfortunately, substituting equation (15) into equations (18) through (20) gives the solution $n = m = l = 0$. Since this is not a meaningful solution, it seems best to choose a model that will allow two leading coefficients to equal zero, such as d_1 and $d_2 = 0$. Assuming $n = 0$ and $m = -1$, the solution $l = 5/6$ and $c_1(\sigma_\phi/\sigma_k) = 0.25$ makes both d_1 and d_2 equal to zero. On the other hand, if one assumes $n = 0$ and $m = -l$, the requirement for d_1 and d_2 to equal zero will lead to a $k - \sqrt{\omega}$ model. These models give rise to an additional constraint on the relation between σ_k and σ_ϕ because $c_1(\sigma_\phi/\sigma_k)$ is a known value obtainable from the solution satisfying d_1 and d_2 equal to zero. Calculations have demonstrated that these models do indeed reduce the density effects to insignificant levels: 0.43% and 0.15% for $k - \sqrt{\omega}$ and $k - \epsilon^{5/6}/k$ models, respectively, at $y^+ = 1000$.

CONCLUSIONS

The present study has shown that the extension of incompressible turbulence models to compressible flow requires density corrections to the closure coefficients in order to satisfy the law of the wall (logarithmic law in Van Driest transformed coordinates). Equation (17) provides a way to estimate the error in boundary layer calculations. The $k - \omega$ model appears to be more attractive than the $k-\epsilon$ model at high Mach numbers, because the coefficients of the unwanted density-gradient terms are smaller. A length-scale transport equation can be devised to minimize the density effects and has proved successful, at least in boundary-layer flow. Further investigation of the proposed models in other flows is under way.

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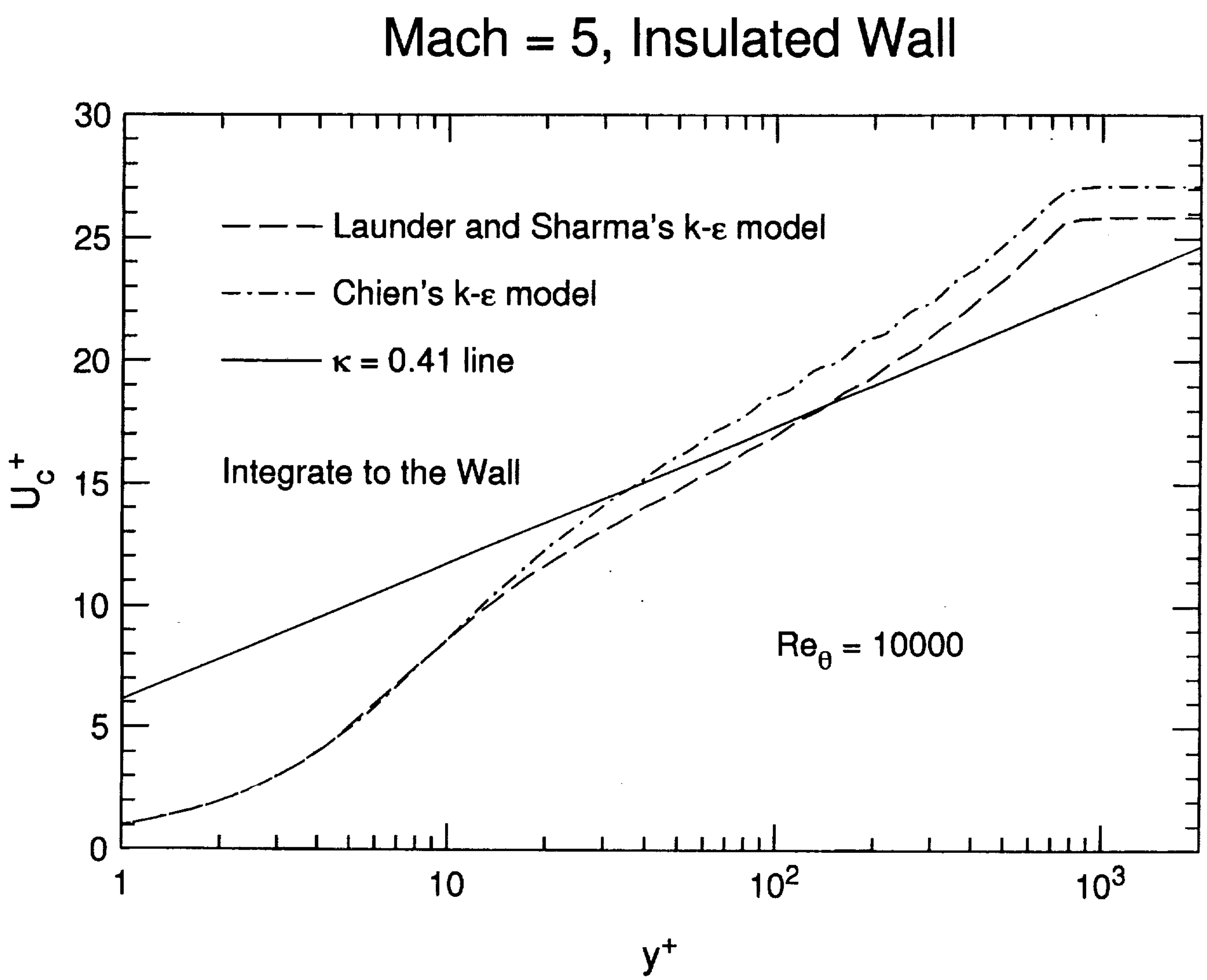

Figure 1. Transformed velocity profiles obtained by low-Reynolds number $k - \epsilon$ models; Navier-Stokes calculations.

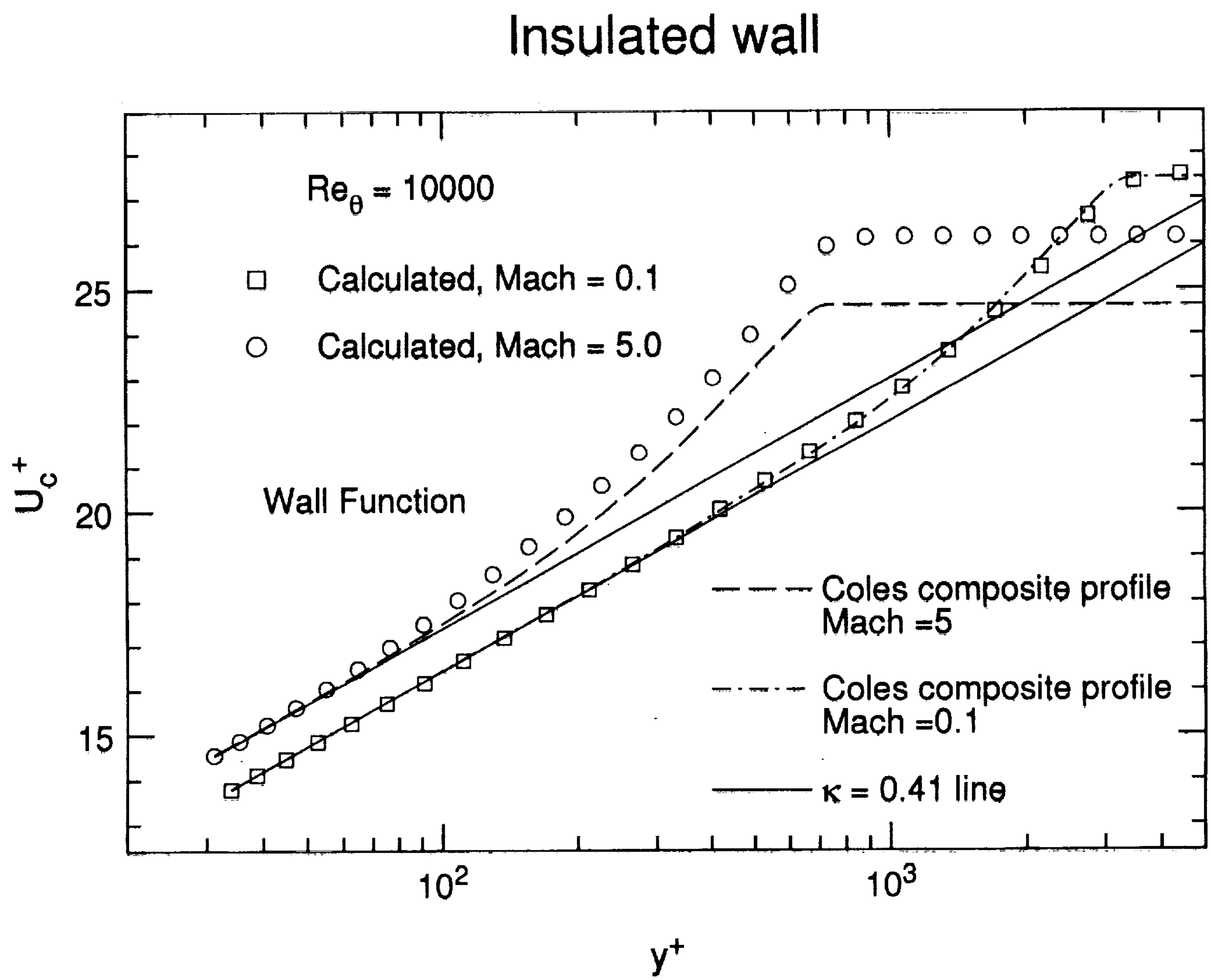

Figure 2. Transformed velocity profiles obtained by the high-Reynolds number $k - \epsilon$ model with compressible wall functions; Navier-Stokes calculations.

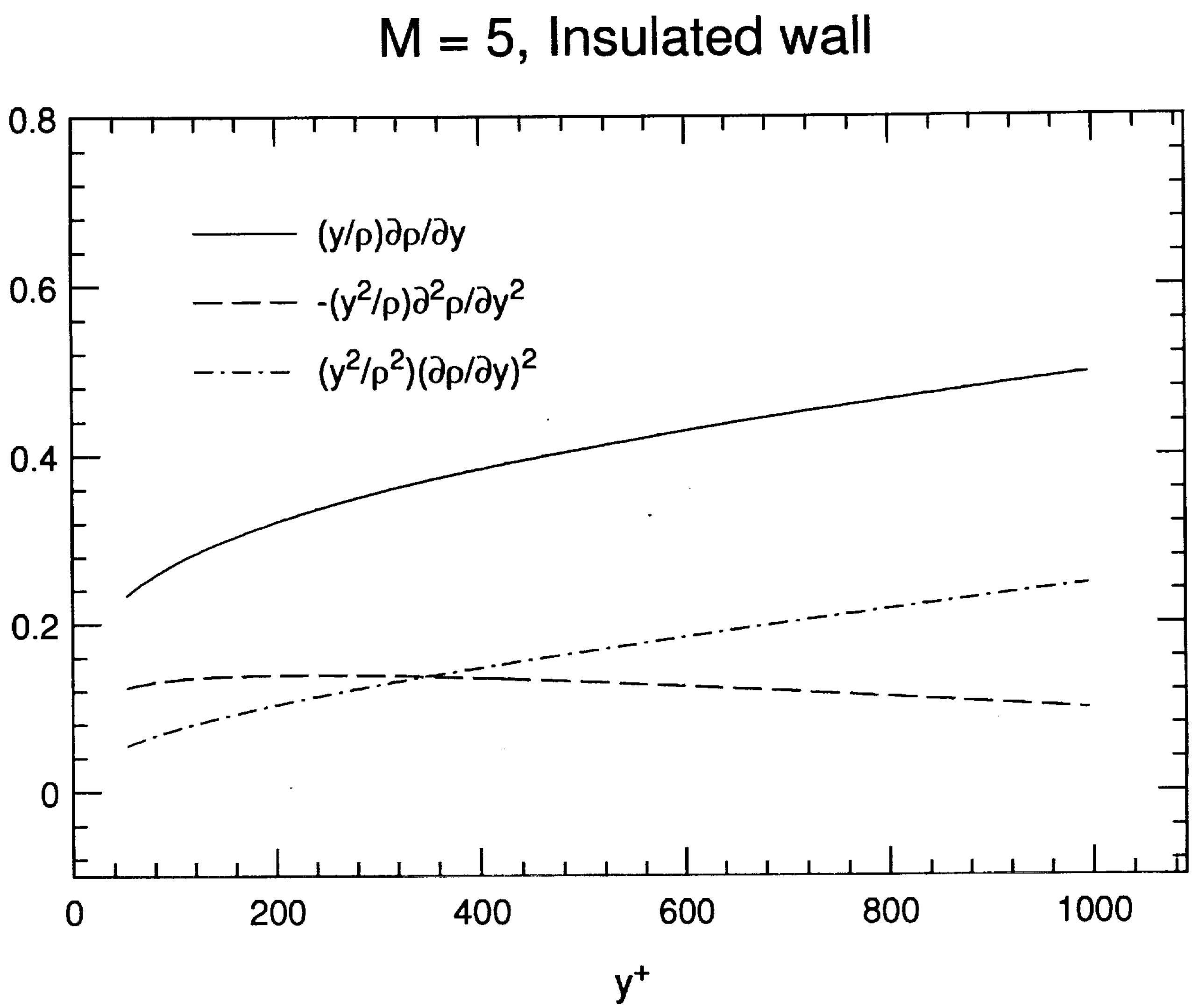

Figure 3. Density related terms calculated from the Van Driest compressible law of the wall.

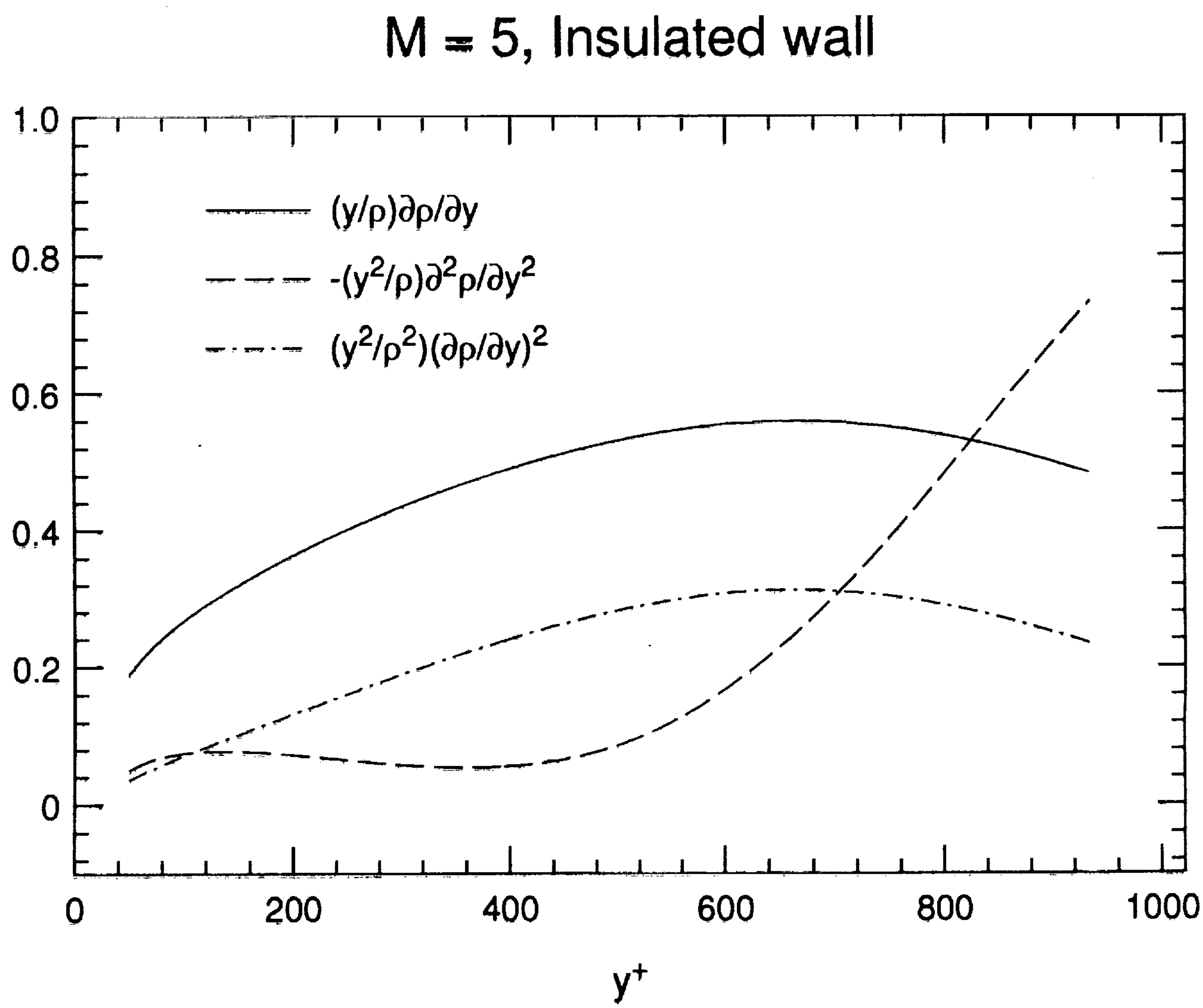

Figure 4. Density related terms obtained by the $k - \epsilon$ model.

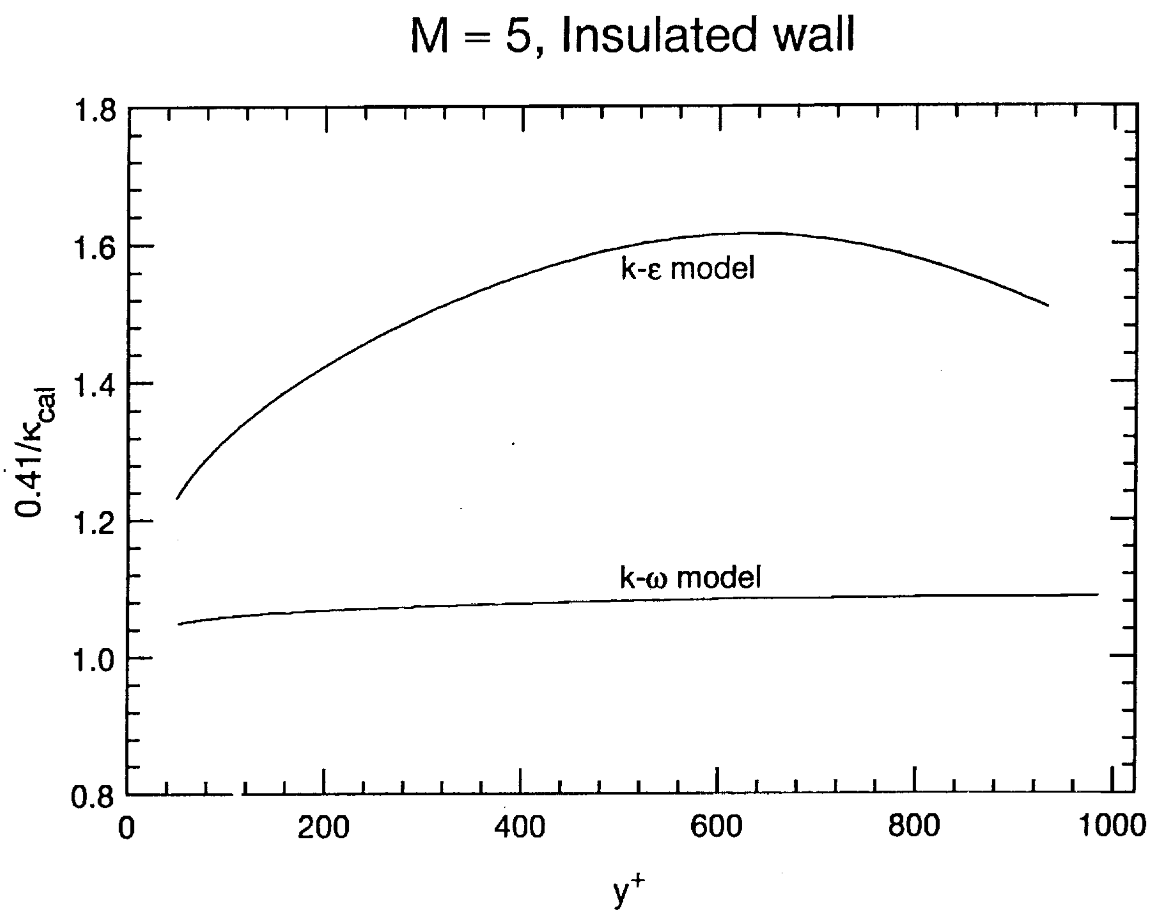

Figure 5. Ratio of the theoretical κ (= 0.41) to the calculated κ.

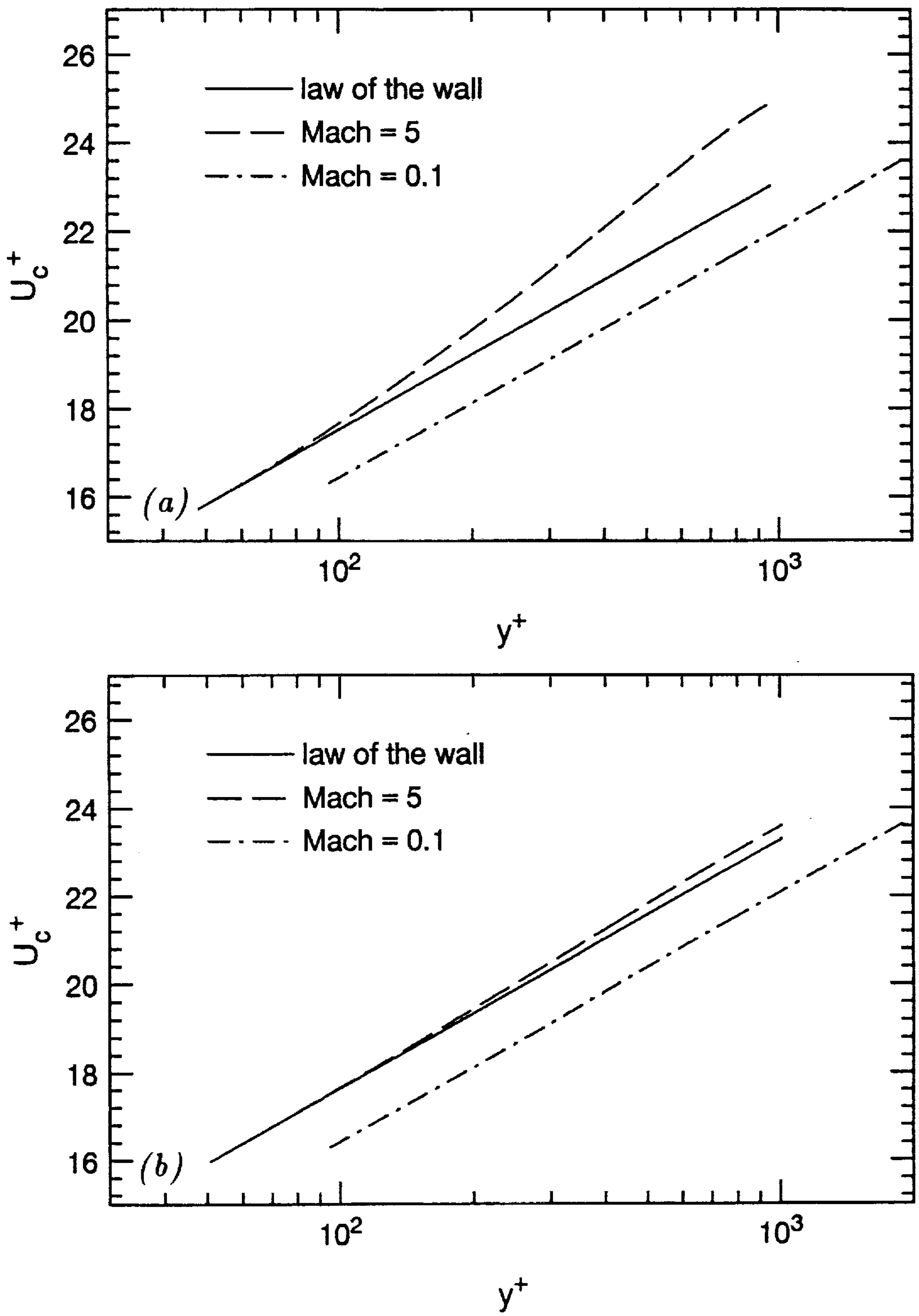

Figure 6. Transformed velocity profiles obtained from simple couette flow equations. (a) $k - \epsilon$ model, (b) $k - \omega$ model.

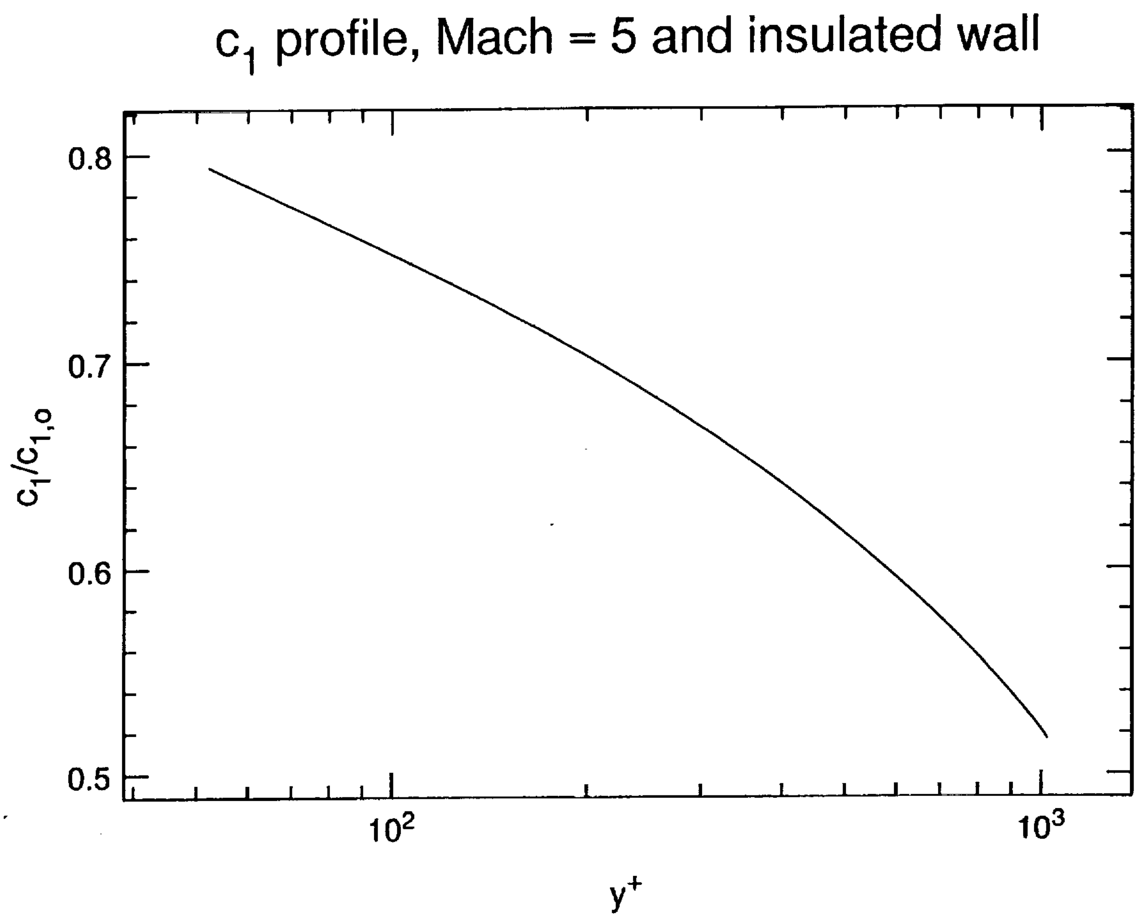

Figure 7. Modified c_1 distribution of the $k - \epsilon$ model.

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