Steady Induction Effects in Geomagnetism

*Part IB: Geomagnetic Estimation
of Steady Surficial Core Motions—
A Non-Linear Inverse Problem*

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1. INTRODUCTION

Geomagnetic secular change has long been attributed to an imbalance between the effects of motional induction and magnetic flux diffusion within Earth's electrically conducting liquid outer core. In Part IA (VOORHIES, 1992) attention was focused on the fluid motion near the top of the core by adopting the source-free mantle/frozen-flux core (SFM/FFC) magnetic earth model (wherein a rigid, impenetrable, electrically insulating mantle of uniform magnetic permeability surrounds a spherical, inviscid, perfectly conducting outer core in anelastic flow). Several reasons were given to further consider the geomagnetic effects of motional induction by steady flow near the top of the core (e.g., a steady flow explicates quantitatively most of the recent, observed geomagnetic secular change). The theory underlying some estimates of core surface flow was summarized. Consequences of a few kinematic and dynamic hypotheses were derived: fluid downwelling is required to change the mean square radial magnetic flux density averaged over the surface of a FFC; downwelling implies poleward flow for surficially geostrophic core motions. The solution of the forward steady motional induction problem at the top of a FFC was derived and found to be a fine example of deterministic chaos. Implications of persistent, if not steady, surficially geostrophic flow were described which apparently help explain certain features of the present broad-scale magnetic field and perhaps paleomagnetic secular change.

To investigate steady induction effects in geomagnetism, it is useful to regard the SFM/FFC model as a first approximation and to treat the supposition of steady surficial core flow as a hypothesis. To test hypotheses against observations, it seems appropriate to (a) understand both; (b) develop a satisfactory method for modeling the relevant observations in accord with the hypotheses; (c) apply the method to make quantitative predictions; and (d) subtract predicted from observed values and measure such residuals in units of the estimated uncertainty in the observations. In the context of the SFM/FFC approximation, this paper develops a method to fit the secular change indicated by geomagnetic field models in accord with the hypothesis of piecewise, statistically steady flow. Such field models represent the relevant geomagnetic observations very well and are here preferred to raw data for reasons noted in Part IA.

With enough perfectly accurate information on the normal component of the time-varying magnetic flux density at the surface of a non-diffusive core, the steady surficial core flow inducing those variations could be uniquely determined by simple linear methods—provided the interior of the cryptic set is indeed empty (VOORHIES & BACKUS, 1985). It can be assumed that geomagnetic observations are not perfectly accurate and are sparsely distributed in space and time; therefore, they are not complete in either the spatial or temporal domain. It can further be assumed that models of such data are also imperfect and incomplete (e.g., truncated spherical harmonic models are incomplete in the spectral domain despite their completeness in the spatial domain). It follows that there is not enough perfectly accurate information at Earth's surface, much less at the top of the core, to sustain the simple linear methods. One may, however, seek a steady surficial core flow which tracks that part of the total secular change indicated by real models of real data.

The inverse problem of deriving steady surficial core motions which fit imperfect models of observed geomagnetic field evolution is non-linear due to the appearance of a transcendental exponential operator in the solution of the forward steady motional induction problem (IA). Previous studies of the steady motions hypothesis (e.g., BLOXHAM, 1987a,b, 1988a,b, 1989; VOORHIES, 1986a,b, 1987a,b; WHALER & CLARKE, 1988) have noted the non-linear nature of the inverse steady motional induction problem. Methods for solving the non-linear inverse problem were developed and applied by VOORHIES (1987b) and BLOXHAM (1987b). Both methods feature iterative minimization of an objective function composed of: a square weighted residual relative to the secular change indicated by the geomagnetic field models fitted; an optional constraint requiring the flow to be as surficially geostrophic as desired; and an optional damping requiring the flow to be as smooth as desired. My method has, however, been based on a different approach which leads to differences in method, application, results, and interpretation.

2. APPROACH

For steady flow, the radial component of the induction equation at the top of a FFC of radius b is, in spherical polar coordinates (r,θ,ϕ),

$$\partial_t B_{rp}(b,t) + v_s(b) \cdot \nabla_s B_{rp}(b,t) = B_{rp}(b,t)\partial_r u(b) \ . \tag{0}$$

This special case of the ROBERTS & SCOTT (1965) equation is but (13b) of Part IA. The subscripted p on the radial component of the magnetic flux density stresses that B_{rp} is a prediction of the FFC approximation and the supposition of steady $v_s(b)$ during some time interval $t_o \le t \le t_f$; the extension to piecewise steady flow is straightforward. At the top of the free-streaming core $(r = b, |b| = 3.48$ Mm$)$ the components of the steady surficial fluid velocity $v_s(b)$ are still $[u(b)=0,v(b),w(b)]$ and $\nabla_s \cdot$ is still the surface divergence operator. Upward continuation from the core-mantle boundary (CMB) to Earth's surface (of $B_{rp}(b,t)$ to $B_{rp}(a,t)$ where $|a| \equiv 6.3712$ Mm$)$ is also straightforward in the SFM approximation; however, an initial geomagnetic condition during $[t_o,t_f]$ is needed to use evolution equation (0).

If the SFM/FFC earth model were exact, and if complete and perfect knowledge of the radial geomagnetic flux density component at Earth's surface were available during some interval, then supposition of steady surficial core flow would overdetermine the inverse motional induction problem under otherwise fairly general circumstances (VOORHIES & BACKUS, 1985). A least-squares approach to solving this problem would then be formally justified and any residual misfit would falsify the supposition. In fact the SFM/FFC model is at best an approximation; moreover, the geomagnetic field is but imperfectly known at 'points' in space and time—so neither geomagnetic data nor spherical harmonic models thereof provide either complete or perfect information on the true radial field component. In view of the approximate nature of both the underlying physical assumptions and the models of limited data, I created and applied a weighted, optionally constrained, and optionally damped iterative weighted least squares method for solving the non-linear inverse motional induction problem posed by the hypothesis of steady flow.

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The quantity to be minimized is the sum of three parts: a square weighted residual $4\pi\Delta_r^2$ measuring the misfit to the expected or input radial magnetic flux density at Earth's surface $B_r(a,t)$; an optional constraint $4\pi\lambda_g\Delta_g^2$ requiring the derived steady flow to be as nearly surficially geostrophic as desired; and an optional damping $4\pi\lambda_d\Delta_d^2$ requiring the spatial structure of the derived flow to be as smooth, or rather, as simple as desired. The total objective function is thus

$$4\pi\Delta^2 = 4\pi(\Delta_r^2 + \lambda_g\Delta_g^2 + \lambda_d\Delta_d^2) \qquad (1a)$$

where λ_g and λ_d are positive damping parameters. The magnitudes of λ_g and λ_d determine respectively the importance of surficial geostrophy and flow simplicity relative to the quality of fit. Large λ_g ensures surficial geostrophy; large λ_d ensures simple flow. Two different types of weights are investigated: radial field weights and general weights.

For radial field weights, the total square weighted residual accumulated during the interval from initial time t_o to final time t_f is $4\pi\Delta_r^2$, where semi-normalized

$$[\Delta_r(a;t_o,t_f)]^2 \equiv \int_{t_o}^{t_f} <[B_r(a,t) - B_{rp}(a,t)]^2 W(a,t)> dt , \qquad (1b)$$

$<q(\mathbf{r},t)>$ denotes the mean value of $q(\mathbf{r},t)$ averaged over the sphere of radius r (IA, equation (7)), and $W(a,t)$ is the weight function. Clearly, Δ_r^2 measures how poorly the predicted radial field B_{rp} fits the expected (or input) radial field B_r in units of the expected uncertainty $\sigma B_r(a,t) \equiv [W(a,t)]^{-1/2}$. For simplicity, the initial geomagnetic condition is taken to be $B_{rp}(a,t_o) = B_r(a,t_o)$ with the understanding that a model for $B_r(a,t_o)$ must not only be downwardly continued, but must be completed, to generate $B_{rp}(b,t_o)$. If no SV were predicted, then $B_{rp}(a,t) = B_{rp}(a,t_o) = B_r(a,t_o)$; then $4\pi\Delta_r^2$ would be the total square weighted change of the radial field accumulated during the interval (the square weighted signal). If SV were also constant, then $4\pi\Delta_r^2/|t_f-t_o|$ would increase only if the weight function grew heavier with time; therefore, Δ_r^2 is normalized for sphericity but not interval. More generally, the fit would be judged adequate if $\Delta_r^2 \leq |t_f-t_o|$ (e.g., if $[B_r(a,t) - B_{rp}(a,t)]^2 = W(a,t)^{-1}$). The integrand in (1b) is the instantaneous square weighted residual $[\delta_r(a,t;t_o)]^2 \equiv <[B_r(a,t) - B_{rp}(a,t)]^2 W(a,t)>$.

The mean square ageostrophy of the flow

$$[\Delta_g(b;t_o,t_f)]^2 \equiv <[\partial_r u(b)\cos\theta + v(b)\sin\theta/b]^2> \qquad (1c)$$

measures departures from the geostrophic radial vorticity balance (IA, equation (12) whereby downwelling ($\partial_r u > 0$) implies poleward flow). The geostrophic radial vorticity constraint is not needed to derive formally unique, piecewise steady core surface motions, but it is plausible dynamically. In the limit as λ_g approaches infinity, this constraint is consistent with tangential geostrophy—which eliminates the toroidal ambiguity in $B_r\mathbf{v}_s$ (BACKUS, 1982) in some areas (BACKUS & LEMOUEL, 1986; HILLS, 1979) and reduces it everywhere on the CMB. More generally, this constraint reduces the geomagnetic information required to uniquely

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determine a steady flow. Surficially geostrophic flows have also been used to estimate the purely mechanical or topographic torque exerted by the core on the mantle (SPEITH et al., 1986) and, with the added supposition of tangential geostrophy, the perturbation pressure field at the CMB (VOORHIES, 1991). Occasional application of this interesting constraint needs no further justification.

With radial vorticity $\omega_r \equiv \hat{r} \cdot \nabla \times v$ and anelastic downwelling $\partial_r u(b) \equiv -\nabla_s \cdot v_s(b)$, the measure of spatial complexity adopted is the sum of the mean square radial vorticity and the mean square downwelling of the flow

$$[\Delta_d(b; t_o, t_f)]^2 \equiv <[\omega_r(b)]^2 + [\partial_r u(b)]^2> . \tag{1d}$$

By (1b-d), both λ_g and λ_d in (1a) must have dimensions of time cubed. Finitude of (1d) ensures piecewise continuous fluid velocity $v_s(b)$; truncated spherical harmonic models of $v_s(b)$ are continuous and smooth.

A plausible (sub-relativistic and sub-acoustic) core surface flow need not be spatially simple, nor are very smooth flows necessarily more reliable than other flows. The bias towards simple flow was introduced chiefly to speed convergence of the iteration scheme. It turns out that varying λ_d enables exploration of how well various steady flows fit secular change. For example, if the SFM/FFC earth model and steady flow admit $\Delta_r^2 \leq |t_f - t_o|$, then λ_d may be chosen so as to achieve an adequate fit and eliminate unnecessary spatial structure in the flow; if not, varying λ_d allows a reckoning of how much SV might reasonably be attributed to a steady flow at the top of a FFC surrounded by a SFM. Indeed, λ_d might be chosen to give the simplest flow yielding the 7% residuals expected in the SFM/FFC approximation (IA). Such a choice might reduce diffusive effects on the estimated downwelling anticipated by MUTH (1984, pers. comm.) and BLOXHAM (1989). Moreover, non-zero λ_d requires both the radial vorticity and the downwelling to be finite when averaged over any non-zero area—in partial accord with the condition of hydrodynamic motion (BONDI & GOLD, 1950). Non-zero λ_d further ensures a surface kinetic energy density spectrum (VOORHIES, 1986a) which falls off faster than n^{-3} at very high spherical harmonic degree n. Then the total surface kinetic energy density will converge with n—even in the absence of geomagnetic information resolving small-scale flow structure.

Other constraints could be substituted for or added to Δ_d^2. One tempting constraint imposes both finite downwelling and finite radial vorticity at all points b. This would prohibit vortex sheets, vortex lines, and singular downwellings just within the fluid core—in full accord with the condition of hydrodynamic motion. Any core analogs of fronts, tornadoes, plumes, or boundary currents would then have finite thickness—even lacking the geomagnetic information needed to resolve them. Yet tearing of the fluid is not implied by the use of truncated spherical harmonic representations of a finite fluid velocity field, nor does estimation of a truncated parameter set imply that unestimated, higher degree coefficients are zero. One might place upper bounds on fine structure thickness and reduce ringing by extending the maximum degree of such an estimate far into the damping regime (wherein damping rather than geomagnetic information and non-zero molecular diffusivities determines the scale of fine structures). This may be too burdensome computationally. Moreover, as a norm (1d) is consistent with effectively inviscid flow; norms barring sheet vorticies seem

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inconsistent with (0). So (1d) offers a good compromise between smearing out any fairly sharp-edged jets, gyres, and plumes needed to fit secular change and ringing caused by truncation. Furthermore, if localized diffusive flux eruption or decay masquerade as strong frozen-flux upwelling or downwelling plumes, then the milder constraint (1d) allows, and indeed encourages, spatial confinement of such artifacts.

Arbitrary selection of $\lambda_d \neq 0$ injects prior bias rather than genuine prior information into what some might otherwise view as a Gauss-Markov estimation problem. Following BACKUS (1988a), the damped weighted least-squares approach is not stochastic inversion nor is it properly Bayesian inference when λ_d is varied to investigate various flows rather than impose, a priori, a particular personal probability distribution upon the flow parameters. Unfortunately, with arbitrary $\lambda_d \neq 0$ the derivation of reliable uncertainty estimates for the velocity field parameters is difficult or impossible. If the contribution from $\lambda_d \Delta_d^2$ to the total information matrix were replaced by genuine prior information before inversion, then the resulting covariance could be physically meaningful. Prior information on core motions includes: (i) the time-averaged viscous dissipation within the core must not exceed the time-averaged geothermal flux; and (ii) the core flow speeds of interest must be less than Mach one everywhere and are likely much less than the rotational speed (253 m/s at b = 3.48 Mm and $\theta = \pi/2$). The former places no constraint on effectively inviscid motions at the top of the core; the latter is too weak to speed convergence of the iteration scheme. Yet the formal uniqueness problem is solved, the existence problem is of immediate interest, and the question of practical uniqueness within reliable uncertainty estimates is moot if existence cannot be established. When seeking plausible solutions to the existence problem, baseless bias towards smooth flow ought not hinder hypothesis testing. Such bias can be reduced (or eliminated) by reducing λ_d towards (or to) zero or by modifying the algorithm as described in section 3.3.

An alternative form for the square weighted residual (1b) suitable for use with 'discrete' weighted geomagnetic data $D(r_j, t_j)$ (be it D, I, H, X, Y, Z, or F) gathered between radii r_i and r_o is

$$c \int_{t_o}^{t_f} \int_{r_i}^{r_o} \int_0^{2\pi} \int_0^{\pi} [D(r,t) - D_p(r,t)]_j M_{jk} [D(r,t) - D_p(r,t)]_k r^2 \sin\theta \, d\theta \, d\phi \, dr \, dt$$

where M_{jk} is the appropriate weight matrix function reflecting any expected correlation between the jth and k^{th} data and C is the appropriate semi-normalization constant $\{3/4\pi[r_o^3 - r_i^3]\}$ (Voorhies, 1988 unpublished). LANGEL (1990, personal communication) stresses that this form leads to simultaneous estimation of both an initial geomagnetic field model and a steady surficial core flow. Although the formalism and preliminary solutions lie outside the range of the present series, it can be seen that the combination of secular change data with the steady motions hypothesis places powerful, albeit perhaps contrived, constraints on an initial geomagnetic field model at the CMB.

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3. METHOD

3.1 The Square Weighted Residual

With initial condition $B_r(a,t_o) = B_{rp}(a,t_o)$, (1b) is rewritten as

$$\Delta_r^2 = \int_{t_o}^{t_f} <(\int_{t_o}^{t} [\partial_\tau B_r(a,\tau) - \partial_\tau B_{rp}(a,\tau)]d\tau)^2 W(a,t)>dt \qquad (2)$$

where the dependence of Δ_r^2 on $(a;t_o,t_f)$ is understood. For $r \geq b$, the input radial field $B_r(r,t)$, the predicted radial field $B_{rp}(r,t)$, and their time derivatives are expressed in terms of their compact spherical harmonic expansions

$$B_r(r,t) = g_i(r,t)S_i(\theta,\phi) \qquad \partial_t B_r(r,t) = \partial_t g_i(r,t)S_i(\theta,\phi) \qquad (3a,b)$$

$$B_{rp}(r,t) = \gamma_i(r,t)S_i(\theta,\phi) \qquad \partial_t B_{rp}(r,t) = \partial_t \gamma_i(r,t)S_i(\theta,\phi) . \qquad (4a,b)$$

Repeated subscripts are summed over (Einstein convention); the spherical harmonics $S_i(\theta,\phi)$ and radial field coefficients are defined as follows. Let P_n^m represent the Schmidt normalized associated Legendre function of degree n and order m (CHAPMAN & BARTELS, 1940; JACOBS, 1987). For index $i = n^2$, $S_i(\theta,\phi) = P_n^0(\cos\theta)$; for $i = n^2+2m-1$ and $m \neq 0$, $S_i(\theta,\phi) = [\cos m\phi]P_n^m(\cos\theta)$; for $i = n^2+2m$ and $m \neq 0$, $S_i(\theta,\phi) = [\sin m\phi]P_n^m(\cos\theta)$. Clearly, $n(i)$ and $m(i)$ are specified by i. In the SFM approximation, the expected radial field coefficients $g_i(r,t)$ are the corresponding input Gauss coefficients (g_n^m,h_n^m) multiplied by $[n+1][a/r]^{n+2}$. The predicted radial field coefficients $\gamma_i(r,t)$ are similarly defined and are derived via spherical harmonic analysis of the radial field predicted by steady motional induction at the CMB for $t \neq t_o$. Radii a and b are of primary interest, so let

$$g_i(a,t) = g_i(t) = g_i \qquad\qquad g_i(b,t) = G_i(t) = G_i \qquad (5a,b)$$

$$\gamma_i(a,t) = \gamma_i(t) = \gamma_i \qquad\qquad \gamma_i(b,t) = \Gamma_i(t) = \Gamma_i \qquad (6a,b)$$

$$g_i = \Upsilon_{ij}G_j \qquad\qquad\qquad \gamma_i = \Upsilon_{ij}\Gamma_j \qquad (7a,b)$$

where the time dependence is understood and, with δ_{ij} denoting the Kroenecker delta, the upward continuation operator is a diagonal matrix with elements $\Upsilon_{ij} = [b/a]^{n(i)+2}\delta_{ij}$. For a SFM, the Υ_{ij} map the radial field coefficients, and thus the scaloidal core field, from the CMB to Earth's surface. Curiously, a diagonal upward continuation filter with elements that depend upon n and the temporal frequency ω_k of the discrete Fourier transformed $g_i(r,t) \rightarrow g_{ik}(r,\omega_k)$ can account for the effects of non-zero, laterally homogeneous, mantle electrical conductivity; however, mere inclusion of m-dependent and off-diagonal elements will not account for the toroidal-poloidal coupling expected for laterally heterogeneous mantle conductivity (Voorhies, 1988 unpublished manuscript).

The streamfunction $-T(b)$ and the velocity potential $-U(b)$ describing the steady surficial fluid velocity field $v(b;t_o,t_f)$ (hence $B_{rp}(b,t\neq t_o)$) are also expanded in terms of spherical harmonics:

$$\mathbf{v}(\mathbf{b}) = \nabla_s T(b,\theta,\phi) \times \hat{\mathbf{r}} + \nabla_s U(b,\theta,\phi) \tag{8a}$$

$$T(\mathbf{b}) = \alpha_i S_i(\theta,\phi) \qquad\qquad U(\mathbf{b}) = \beta_i S_i(\theta,\phi) . \tag{8b,c}$$

This ensures $\nabla \cdot \mathbf{v}_s = 0$ at $\mathbf{b}$-as required by the kinematic boundary and anelastic flow conditions (IA).

Spherical harmonics S_i are orthogonal on spheres, so (3a-7b) can be used to rewrite (2) as

$$\Delta_r^2 = \int_{t_o}^{t_f} <[g_i S_i - \gamma_i S_i]^2 W(a,t)> dt \tag{9a}$$

$$= \int_{t_o}^{t_f} <(g_i - \gamma_i)[S_i W(a,t) S_j](g_j - \gamma_j)> dt \tag{9b}$$

or

$$4\pi\Delta_r^2 = \int_{t_o}^{t_f} [g_i - \gamma_i] W_{ij} [g_j - \gamma_j] dt \tag{9c}$$

where the dependence of the S_i on (θ,ϕ) is understood and $W_{ij}(t) \equiv 4\pi <S_i W(a,t) S_j>$ defines the elements of the time-dependent radial field weight matrix. In the case of equal weighting (VOORHIES, 1986b), W_{ij} reduces to the diagonal normalization matrix for Schmidt normalized spherical harmonics, $N_{ij} \equiv 4\pi[2n(i)+1]^{-1}\delta_{ij}$. If the g_i describing $B_r(a,t)$ are not all equally well determined, not independent, or both, then W_{ij} does not reduce to N_{ij}.

For general weights, Ω_{ij}/f replaces $W_{ij}/4\pi$ in (9c): with matrix inversion preceding assignation of element indices, $\Omega_{ij} \equiv E^{-1}_{ij}$; $E_{kl} \equiv E_o\{e_k e_l\}$ defines the time-varying covariance associated with the input radial field model; the e_k are the unknown true errors in the input g_k; and the E_o operator yields the expected value (see Appendix).

Radial field weights rely upon a scalar weight function $W(a,t)$ which is the inverse of the expected squared uncertainty in $B_r(a,t)$. If uncorrelated observations of B_r were used, then W would be the inverse squared uncertainty of the observations when and where observations exist; W would be zero elsewhere and elsewhen as non-existent data enjoy zero weight. When spherical harmonic models of the radial field are used the weight function is

$$W(a,t) \equiv [\sigma B_r(a,\theta,\phi,t)]^{-2} = [S_k(\theta,\phi) E_{kl}(a,t) S_l(\theta,\phi)]^{-1} . \tag{10}$$

If the covariance for the Gauss coefficients at (a,t) is V_{kl}, then $E_{kl}(a,t) = [n(k)+1]V_{kl}[n(1)+1] \equiv R_{ik}V_{ij}R_{jl}$, where $R_{ik} \equiv [n(i)+1]\delta_{ik}$.

Substitution of (10) into (9b) yields

$$\Delta_r^2 = \int_{t_o}^{t_f} <(g_i - \gamma_i)S_i[S_k E_{kl} S_l]^{-1} S_j(g_j - \gamma_j)> dt \tag{11a}$$

or

$$4\pi\Delta_r^2 = \int_{t_o}^{t_f} [(g_i - \gamma_i)W_{ij}(g_j - \gamma_j)]dt \qquad (11b)$$

$$= \int_{t_o}^{t_f} \{[\int_{t_o}^{t} (\partial_\tau g_i - \partial_\tau\gamma_i)d\tau]W_{ij}[\int_{t_o}^{t} (\partial_\tau g_j - \partial_\tau\gamma_j)d\tau]\}dt \qquad (11c)$$

where the radial field weight matrix elements are

$$W_{ij}(a,t) = 4\pi\langle S_i[S_kE_{kl}S_l]^{-1}S_j\rangle \qquad (12a)$$

$$= 4\pi\langle S_i[w_k(a,t)S_k]S_j\rangle \qquad (12b)$$

$$= 4\pi Z_{ijk}w_k , \qquad (12c)$$

the time-dependent spherical harmonic representation of the weight function (including its non-zero mean value) is

$$W(a,t) \equiv w_k(a,t)S_k(\theta,\phi) = [S_kE_{kl}(a,t)S_l]^{-1} \qquad (13a)$$

and the symmetric third-rank tensor has elements

$$Z_{ijk} \equiv \langle S_iS_jS_k\rangle = Z_{jik} = Z_{ikj} . \qquad (13b)$$

When the weight function is independent of position, then W_{ij} is proportional to N_{ij}. If $W(a,t)$ is everywhere and always equal to unity, then W_{ij} reduces to N_{ij}. The latter conditions were in effect presumed by VOORHIES (1986b); such presumptions are avoided here. Although $W(a,t)$ should be nearly laterally homogeneous for broad-scale models of satellite data, such data are not always available; when they are, $W(a,t)$ can be $(16\ nT)^{-2}$ (LANGEL, ESTES, & SABAKA, 1988a; 1989). A derivation of weight matrices for the Definitive Geomagnetic Reference Field (DGRF) models (IAGA, 1988) will be described in Part IC.

If SV coefficients were used as input and if an expected error covariance for the time rate of change of the radial field coefficients $(E_{ij}^{SV} \equiv E_o\{(\partial_t e_i)(\partial_t e_j)\})$ were available, then (11c) would become

$$4\pi\Delta_r^2 = \int_{t_o}^{t_f}\{[\int_{t_o}^{t} \omega_{ki}^{SV}(\partial_\tau g_i - \partial_\tau\gamma_i)d\tau][\int_{t_o}^{t} \omega_{kj}^{SV}(\partial_\tau g_j - \partial_\tau\gamma_j)d\tau]\}dt$$

where the time (τ)-dependent matrix $\underline{\underline{\omega}}^{SV}$ is the upper triangular matrix square root of the SV weight matrix defined as in (12a) but with E_{kl}^{SV} replacing E_{kl}. This approach was not pursued. Though the DGRF models employed were used to derive dummy SV models for the τ integration, main field weights seem more appropriate to the fitting of a sequence of main field models and were thus used to weight the residuals.

In order to minimize the objective function with respect to the flow parameters α_i and β_i, Δ_r^2 in (11c) must be expressed in terms of these parameters. By analogy with VOORHIES (1986b), write (0) as

$$\partial_t B_{rp}(b,t) = -\nabla_s\bullet[B_{rp}(b,t)v_s(b)] \qquad (14a)$$

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or, using equations (2) through (8),

$$\partial_t \Gamma_k S_k = -\nabla_s \cdot (\Gamma_i S_i [\nabla_s (\alpha_j S_j) \times \hat{r} + \nabla_s (\beta_j S_j)]) \tag{14b}$$

$$= b^{-2} \Gamma_i \{ \frac{\partial_\phi S_i \partial_\theta S_j}{\sin\theta} - \frac{\partial_\theta S_i \partial_\phi S_j}{\sin\theta} \} \alpha_j +$$

$$b^{-2} \Gamma_i \{ S_i [n(j)][n(j)+1] S_j - \partial_\theta S_i \partial_\theta S_j - \frac{\partial_\phi S_i \partial_\phi S_j}{\sin^2\theta} \} \beta_j . \tag{14c}$$

Left multiply the scalar equation (14c) by $S_1 \sin\theta / 4\pi$ and integrate over θ and ϕ. Then use the orthogonality of the spherical harmonics to evaluate the left-hand side, relabel $1 \to k$, and reorder the integrand on the right-hand side noting terms like $S_k S_j \beta_j = S_j S_k \beta_j$ because there is no sum over k. The result is written

$$\partial_t \Gamma_k = (\Gamma_i X_{ijk}) \alpha_j + (\Gamma_i Y_{ijk}) \beta_j = P_{kj} \alpha_j + Q_{kj} \beta_j \tag{15}$$

where to develop the third-order coupling, the third-rank tensors are

$$X_{ijk} \equiv b^{-2} [2n(k)+1] < (\partial_\phi S_i \partial_\theta S_j - \partial_\theta S_i \partial_\phi S_j) S_k / \sin\theta> = -X_{jik} \tag{16a}$$

$$Y_{ijk} \equiv b^{-2} [2n(k)+1] < (S_i [n(j)][n(j)+1] S_j - \partial_\theta S_i \partial_\theta S_j - \frac{\partial_\phi S_i \partial_\phi S_j}{\sin^2\theta}) S_k> \tag{16b}$$

and, for numerical integration over the CMB, the second-rank tensors are

$$P_{kj} \equiv b^{-2} [2n(k)+1] < S_k \{ \partial_\phi B_{rp} \partial_\theta S_j - \partial_\theta B_{rp} \partial_\phi S_j \} / \sin\theta> \tag{17a}$$

$$Q_{kj} \equiv b^{-2} [2n(k)+1] < S_k \{ B_{rp} [n(j)][n(j)+1] S_j - \partial_\theta B_{rp} \partial_\theta S_j - \frac{\partial_\phi B_{rp} \partial_\phi S_j}{\sin^2\theta} \}> . \tag{17b}$$

Upward continuation of (15) via (7b) yields

$$\partial_t \gamma_i = \Upsilon_{ik} \partial_t \Gamma_k = \Upsilon_{ik} [(\Gamma_i X_{ijk}) \alpha_j + (\Gamma_i Y_{ijk}) \beta_j] \tag{18a}$$

$$= \Upsilon_{ik} [P_{kj} \alpha_j + Q_{kj} \beta_j] \tag{18b}$$

$$\equiv p_{ij} \alpha_j + q_{ij} \beta_j \equiv A_{il} \xi_l . \tag{18c}$$

In (18c) $p_{ij} \equiv \Upsilon_{ik} P_{kj}$; $q_{ij} \equiv \Upsilon_{ik} Q_{kj}$; $\xi_l \equiv \{\alpha_j; \beta_j\}$ defines the extended vector obtained by concatenating the coefficients of T(b) with those of U(b); and time-varying $A_{il} \equiv \{p_{ij}; q_{ij}\}$ defines the extended matrix obtained by concatenating the corresponding sub-matrices. Substitution of (18c) into (11c) and relabeling yields

$$4\pi \Delta_r^2 = \int_{t_o}^{t_f} \{ [\int_{t_o}^{t} (\partial_\tau g_i - A_{ik} \xi_k) d\tau] W_{ij} [\int_{t_o}^{t} (\partial_\tau g_j - A_{jk} \xi_k) d\tau] \} dt . \tag{19}$$

The square weighted residual (19) will be minimal only if it is extreme, in which case its total derivatives with respect to the parameters ξ_l vanish. The first approximation to this condition was to set the partial derivatives of (19) with respect to ξ_l zero. Then

$$0_l = -2\int_{t_o}^{t_f} \{[\int_{t_o}^{t} (A_{ik}\delta_{kl})d\tau]W_{ij}[\int_{t_o}^{t} (\partial_\tau g_j - A_{jk}\xi_k)d\tau]\}dt \qquad (20a)$$

where the ultimate dependence of the A_{ik} upon $B_{rp}(b,t)$, hence on $v(b)$ and the initial condition (i.e., on ξ_l and $G_i(t_o)$), has been temporarily omitted to achieve linearization. (Recent tests show this is a good approximation, particularly for slow flows over short intervals). In matrix notation equation (20a) is

$$\underline{0} = -2\int_{t_o}^{t_f} \{[\int_{t_o}^{t} \underline{\underline{A}}\, d\tau]^T \underline{\underline{W}} [\int_{t_o}^{t} (\partial_\tau \underline{g} - \underline{\underline{A}}\,\underline{\xi})\, d\tau]\}dt$$

where a single underline denotes a column vector, a double underline denotes a matrix, and the T superscript indicates the transpose. The linear least-squares (LLS) estimate of the parameters is

$$\underline{\xi}^{LLS} = (\int_{t_o}^{t_f} \{[\int_{t_o}^{t} \underline{\underline{A}}\, d\tau]^T \underline{\underline{W}} [\int_{t_o}^{t} \underline{\underline{A}}\, d\tau]\}dt)^{-1}$$

$$(\int_{t_o}^{t_f} \{[\int_{t_o}^{t} \underline{\underline{A}}\, d\tau] \underline{\underline{W}} [\int_{t_o}^{t} \partial_\tau \underline{g}\, d\tau]\}dt) \qquad (20b)$$

$$= (\overline{\overline{\underline{A}^T\, \underline{\underline{W}}\, \underline{\underline{A}}}})^{-1}(\overline{\overline{\underline{A}^T\, \underline{\underline{W}}\, \partial_\tau \underline{g}}}) \qquad (20c)$$

where the lower (single) overbar indicates dummy time integration over τ from t_o to t, and the upper (double) overbar indicates total time integration over t from t_o to t_f. Equation (20c) provides the linear least-squares estimate of the steady streamfunction and velocity potential coefficients which best fit the expected evolution of the radial magnetic flux density at Earth's surface during the interval.

There are three crucial differences between (14a-20c) above and equations (10-17) of VOORHIES (1986b). Firstly, the squared residual is non-uniformly weighted. Secondly, the square weighted residual includes the double time integral needed to fit the evolving main field rather than simple SV. Thirdly, the predicted radial field (B_{rp} or its coefficients Γ_i) appears on the right-hand sides instead of the input field (B_r or its coefficients G_i). The A_{ik} in (19) thus depend upon the predicted main field at the CMB instead of the input field model. The elements of $\underline{\underline{A}}$ in (20) also depend upon the predicted field at the CMB instead of the input field, but this dependence was suppressed to obtain a system of linear equations for the model parameters $\underline{\xi}$.

To solve the non-linear problem, I developed an iterative method wherein the linearized problem is first posed by supposing $\Gamma_i = G_i$ for purposes of estimating the elements of $\underline{\underline{A}}$. These can be computed using equations (15), either (16) or (17), and (18). The linearized least-squares problem (20c) is then solved for $\underline{\xi}$ and thus $\mathbf{v}(\mathbf{b})$ by back substitution into (8). $B_{rp}(\mathbf{b},t)$ is computed by solving the forward motional induction problem (14a) from the initial condition $B_{rp}(\mathbf{b},t_o) = B_r(\mathbf{b},t_o)$. These values for $B_{rp}(\mathbf{b},t)$ are then used to compute $\bar{\Gamma}$ and γ; residuals $(\underline{g} - \gamma)$; both weighted and unweighted residuals; and new elements for $\underline{\underline{A}}$ matrix elements via (15), either (16) or (17), and (18). The new $\underline{\underline{A}}$ matrices and the residuals comprise the input for the next iteration. Let j indicate the jth such deep matrix iteration, let $\underline{\xi}(j+1) = \underline{\xi}(j) + \delta\underline{\xi}(j+1)$, and let $\delta\underline{g}(j+1) = \underline{g} - \underline{\gamma}(j)$ so that $\delta\partial_\tau\underline{g}(j+1) = (\partial_\tau\underline{g} - \partial_\tau\underline{\gamma}(j))$. Then

$$\delta\underline{\xi}^{NLLS}(j+1) = [\underline{\underline{A}}^T(j)\ \underline{\underline{W}}\ \underline{\underline{A}}(j)]^{-1}[\underline{\underline{A}}^T(j)\ \underline{\underline{W}}\ \delta\partial_\tau\underline{g}(j)] \tag{21}$$

describes this non-linear least-squares iteration procedure.

For $t \neq t_o$ the spherical harmonic content of B_{rp} typically extends to far higher degree than that of B_r; such narrow-scale structure must be preserved in the computation of the new $\underline{\underline{A}}$, though it will typically not contribute to the weighted residuals due to truncation of g_i, hence W_{ij}.

3.2 The Geostrophic Radial Vorticity Constraint

Parameterization of the geostrophic radial vorticity constraint proceeds by writing the zero-mean geostrophic radial vorticity balance at the top of the core in terms (ε_k) of its spherical harmonic expansion

$$\varepsilon_k S_k = (\partial_r u\cos\theta + \frac{v}{b}\sin\theta) \tag{22a}$$

$$= b^{-2}[\alpha_j\partial_\phi S_j + \beta_j(\partial_\theta S_j\sin\theta + n(j)[n(j)+1]S_j\cos\theta)] \ . \tag{22b}$$

Left multiply (22b) by $S_i\sin\theta$, integrate over a spherical CMB, and use the orthogonality of the spherical harmonics to obtain

$$\varepsilon_i = C_{ij}\alpha_j + D_{ij}\beta_j = B_{ik}\xi_k \tag{23a}$$

where, corresponding to ξ_k, B_{ik} is the appropriate element of either

$$C_{ij} = b^{-2}[2n(i)+1]<S_i\partial_\phi S_j> \tag{23b}$$

or

$$D_{ij} = b^{-2}[2n(i)+1]<S_i(\partial_\theta S_j\sin\theta + n(j)[n(j)+1]S_j\cos\theta)> \ . \tag{23c}$$

Only spherical harmonics of like degree and order contribute to the C_{ij}. Indeed, only certain elements adjacent to the diagonal of matrix C (corresponding to $\cos m\phi[\partial_\phi\sin m\phi]$ and $\sin m\phi[\partial_\phi\cos m\phi]$) are non-trivial; these are readily evaluated analytically. Only spherical harmonics of like order contribute to the D_{ij}. However, if the spherical harmonic

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expansions of streamfunction $-T(\mathbf{b})$ and velocity potential $-U(\mathbf{b})$ are truncated at degree and order $N_T = N_U = N_V$, then D_{ij} (hence B_{ik}) can have non-trivial elements for $i \leq [N_V+1][N_V+3]$ – which can include $i > N_V[N_V+2]$. This is due immediately to the factors of $\sin\theta$ and $\cos\theta$ appearing in (22) and ultimately to the latitudinal variation of the radial planetary vorticity $\Omega_r = 2\Omega_b\cos\theta$. Neglect of this fact (e.g., assuming $D_{ij} = 0$ for $i \geq N_V(N_V+2)$) may lead to a truncated velocity field which fails to be even approximately surficially geostrophic.

Equations (22) and (23) allow the geostrophic radial vorticity constraint (1c) to be rewritten

$$4\pi\lambda_g\Delta_g(b;t_o,t_f)^2 = 4\pi\lambda_g<[\varepsilon_iS_i][\varepsilon_jS_j]> \qquad (24a)$$

$$= \lambda_g[\varepsilon_iN_{ij}\varepsilon_j] \qquad (24b)$$

$$= \xi_kB_{ik}(\lambda_gN_{ij})B_{jl}\xi_l \qquad (24c)$$

$$= (B\,\xi)^T\Lambda_g(B\,\xi) \qquad (24d)$$

where Λ_g is the normalization matrix multiplied by λ_g. Adding (24d) to (20c) gives the constrained, weighted objective function

$$4\pi[\Delta_r^2 + \lambda_g\Delta_g^2] = (\partial_\tau g - A\,\xi)^T W(\partial_\tau g - A\,\xi) + (B\,\xi)^T\Lambda_g(B\,\xi) . \qquad (25)$$

This function is minimal only if it is extreme, in which case its partial derivative with respect to any element of ξ vanishes. Again omitting the weak dependence of A on ξ, the constrained, weighted linear least-squares estimate of the parameters is

$$\xi^{CLLS} = (A^T W A + B^T \Lambda_g B)^{-1}(A^T W \partial_\tau g) . \qquad (26)$$

In the context of the iterative solution to the non-linear, constrained, weighted inverse problem

$$\delta\xi^{CNLLS}(j+1) = [A^T(j)\ W\ A(j) + B^T\ \Lambda_g\ B]^{-1}(A^T(j)\ W\ \delta\partial_\tau g(j)$$

$$- [B^T \Lambda_g B][\xi(j) - \xi_{go}]) \qquad (27)$$

replaces equation (21). The prior estimate of the parameters is ξ_{go}. If $\xi_{go} = 0$ the bias is fixed on zero mean with intensity λ_g. If $\xi_{go} = \xi(j)$ the bias floats with the iteration scheme–yielding a learning algorithm. In the latter case, small λ_g may yield large corrections $\delta\xi(j+1)$ which result in a rather ageostrophic velocity field far from that used to calculate $A(j)$ on iteration j; one might prefer an iteration-dependent constraint parameter with such a learning algorithm.

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Here attention is focused on $\xi_{2go} = 0$ and values of λ_g which are either zero for the unconstrained piecewise steady inverse motional induction problem or so large that the normalized mean square ageostrophy (VOORHIES, 1986c)

$$\Lambda_g \equiv \frac{<[\partial_r u \cos\theta + v\sin\theta/b]^2>}{<[\partial_r u \cos\theta]^2 + [v\sin\theta/b]^2>}$$

is less than 10^{-4}. In the latter case, the constrained weighted least-squares solution will equal that obtained using stochastic inversion with prior information matrix $\underline{B}^T\underline{\Lambda_g}\underline{B}$. Because the geostrophic constraint is viewed as a plausible approximation rather than an absolute necessity required by genuine prior information, solutions (26) or (27) are not considered to be stochastic estimates as described by MCLEOD (1986). These estimates involve 'soft bias' rather than soft or hard bounds as described by BACKUS (1988a,b). A hard version of the geostrophic radial vorticity constraint might be imposed using the geostrophic basis functions of BACKUS & LEMOUEL (1986).

Intermediate values of λ_g can be used to study the tradeoff between the constraint and the length of the interval in which the flow is assumed to be steady. This seems appropriate when the surficially geostrophic flow hypothesis is tested in the context of a SFM/FFC model with piecewise steady surficial core motions. This approach was taken using the unweighted, non-iterative linearized method of VOORHIES (1986b). Analysis of the resulting, severely truncated ($N_v = 5$) fluid velocity fields (VOORHIES, 1986c) showed that the constraint reduces the tightness of fit; yet a mild constraint can increase slightly the accuracy of geomagnetic forecasts made by extrapolating the effects of steady motional induction at the CMB outside the interval $t_o \le t \le t_f$ and subsequent upward continuation. The former point has been confirmed using superior methods [VOORHIES, 1987c, 1988, 1989; BLOXHAM, 1988b].

3.3 Damping Mean Square Radial Vorticity and Mean Square Downwelling

If the SFM/FFC earth model and the working hypothesis of steady (optionally geostrophic) flow were correct, and if complete, albeit imperfect, information on the evolving geomagnetic field were available, then the weighted (constrained) least-squares estimate (26-27) would uniquely determine the (constrained) steady flow to within uncertainties implied by the inverse of the total information matrix. This covariance of the flow parameters $[\underline{A}^T\underline{W}\underline{A} + \underline{B}^T\underline{\Lambda_g}\underline{B}]^{-1}$ is mere expectation; it depends upon neither the flow parameters nor the residuals. Because complete information is not available, if one steady flow adequately fitted the incomplete, imperfect information, there may well be others which do so. In seeking whether one such flow exists, it seems reasonable to initially restrict attention to solutions which are spatially simple.

I chose to seek flows characterized by low values of the mean square surficial curvature of both the streamfunction and the velocity potential in hopes of eliminating unecessary flow structure—particularly on small spatial scales. The radial vorticity

$$\omega_r(b) = \hat{r}\cdot\nabla\times v_s(b) = -\nabla_s^2 T(b) = \alpha_i\{n(i)[n(i)+1]S_i\} \tag{28}$$

is the surface Laplacian of the streamfunction; the downwelling

$$\partial_r u(\mathbf{b}) = -\nabla_s \cdot \mathbf{v}_s = -\nabla_s^2 U(\mathbf{b}) = \beta_i \{n(i)[n(i)+1]S_i\} \qquad (29)$$

is the surface Laplacian of the velocity potential. A velocity field with small mean square radial vorticity and small mean square surface divergence has small mean square surface curvature in both T and U. This choice will tend to fill in any regions where the velocity field is relatively poorly determined by interpolation from surrounding regions without introducing unecessary flow sources and without smearing out isolated jets, gyres, or plumes.

Parameterization of equation (1d) using (28) and (29) yields

$$4\pi\lambda_d\Delta_d(\mathbf{b};t_o,t_f)^2 = 4\pi\lambda_d <[\nabla_s^2 T]^2 + [\nabla_s^2 U]^2> \qquad (30a)$$

$$= \alpha_i F_{ij}\alpha_j + \beta_i F_{ij}\beta_j = \underline{\xi}\,\underline{\underline{\Lambda}}_d\,\underline{\xi} \qquad (30b)$$

where $F_{ij} \equiv \delta_{ij}4\pi\lambda_d[n(i)]^2[n(i)+1]^2/[2n(i)+1]$ and Λ_d is the extended diagonal matrix with elements in both the upper left and lower right quarters equal to those of F_{ij}. Either $<[\omega_r(\mathbf{b})]^2>$ or $<[\partial_r u(\mathbf{b})]^2>$ may be damped more strongly by adjusting the elements of Λ_d. This option was not pursued despite earlier findings (VOORHIES, 1984, 1986a) suggesting more energetic toroidal flow.

Adding (30c) to (25) gives the damped, constrained, weighted objective function (1)

$$4\pi[\Delta_r^2 + \lambda_g\Delta_g^2 + \lambda_d\Delta_d^2] = (\overline{\partial_\tau\underline{g}} - \underline{\underline{A}}\,\underline{\xi})^T W(\overline{\partial_\tau\underline{g}} - \underline{\underline{A}}\,\underline{\xi}) +$$

$$(\underline{\underline{B}}\,\underline{\xi})^T\Lambda_g(\underline{\underline{B}}\,\underline{\xi}) + (\underline{\xi}^T\,\underline{\underline{\Lambda}}_d\,\underline{\xi}) . \qquad (31)$$

This total objective function is minimal only if it is extreme, in which case its total derivatives with respect to the elements of $\underline{\xi}$ all vanish. The weighted, constrained, and damped linearized least-squares estimate of the parameters puts the partial derivatives to zero and is given by

$$\underline{\xi} = (\overline{\underline{\underline{A}}^T W \underline{\underline{A}}} + \underline{\underline{B}}^T \Lambda_g \underline{\underline{B}} + \Lambda_d)^{-1}(\overline{\underline{\underline{A}}^T W \partial_\tau\underline{g}}) . \qquad (32)$$

In the context of the iterative solution to the weighted, constrained, and damped non-linear inverse problem,

$$\delta\underline{\xi}(j+1) = [\overline{\underline{\underline{A}}^T(j) W \underline{\underline{A}}(j)} + \underline{\underline{B}}^T \Lambda_g \underline{\underline{B}} + \Lambda_d]^{-1}(\overline{\underline{\underline{A}}^T(j) W \delta\partial_\tau\underline{g}(j)}$$

$$- [\underline{\underline{B}}^T \Lambda_g \underline{\underline{B}}][\underline{\xi}(j) - \underline{\xi}_{go}] - \Lambda_d[\underline{\xi}(j) - \underline{\xi}_{do}]) \qquad (33a)$$

replaces equation (21) or (27).

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The value of deep matrix element iteration depends upon how much $A(j)$ changes with j. This in turn depends upon how incompatible the input field models, hence $A(j=0)$, are with the earth model, and upon the intitial condition, the estimated fluid velocity, and the length of the interval during which steady flow is presumed. Faster flows tend to generate appreciable non-linear feedback more rapidly; yet even a slow flow may do so eventually. Rough flows tend to generate small-scale structure in $B_{rp}(\mathbf{b},t)$ via a chaotic cascade to ever higher wavenumbers. This need not degrade the fit to broad-scale input field models because such models do not specify small-scale field structure (i.e., because unknown high-degree Gauss coefficients are assigned zero weight). In fact, high-degree structure in $B_{rp}(\mathbf{b},t)$ can be exploited by the flow to improve the fit to the broad-scale field models (via a reverse cascade). Unlike broad-scale spherical harmonic models, surface data may be used to test (or constrain) high-degree structure in $B_{rp}(\mathbf{a},t)$; measures of $B_{rp}(\mathbf{b},t)$ and its time rate of change showed that the steady flows derived from the DGRFs were not rough enough to grossly violate such constraints during the several-decade interval targeted.

As noted above, ξ_{go} was taken to be $\underline{0}$ in the actual calculations. Though ξ_{do} is commonly taken to be 0, many calculations have been performed with floating bias $\xi_{do} \to \xi(j)$ and an adjustable convergence factor $\lambda_d \to \lambda_d(j+1)$. The resulting learning algorithm is useful for deriving rougher flows which otherwise require values of fixed λ_d so small as to inhibit convergence of the iteration scheme or even allow convergence towards local minima where Δ_r^2 exceeds values found using larger λ_d or the learning algorithm. The learning algorithm thus relaxes restrictions imposed by otherwise-fixed bias toward zero flow roughness on tests of the steady flow hypothesis. Flows derived using the learning algorithm are, of course, not optimally simple; they can be smoothed by switching to the fixed λ_d iteration scheme with bias towards zero flow roughness. Some may prefer this strategy to imposing both a fixed bias toward no flow and a floating bias toward the previous estimate governed by a convergence factor.

The use of a non-trivial flow estimate $\xi(j)$ to predict $B_{rp}(\mathbf{b},t)$ and calculate new $A(j)$ may seem inconsistent with a fixed bias towards parameters ξ_{do} which are zero; however, the fixed bias strategy need not be inferior to the learning algorithm; indeed, the former may yield flows which represent any steady part of the true flow near the top of the core more accurately than those derived by excessive repetition of the learning algorithm due to errors in the SFM/FFC approximation.

Because deep matrix element iteration can be computationally burdensome when equations (16) are integrated numerically over the CMB, it seems worth ensuring that the best estimate of $\xi(j)$ is used to obtain $B_{rp}(\mathbf{b},t)$ for the subsequent calculation of $A(j+1)$. In principle, this can be accomplished by introducing shallow iteration whereby the correction vector of streamfunction and velocity potential coefficients determined on sub-iteration i+1 of deep matrix element iteration j is

$$\delta\xi(i+1,j) = [A^T(j)\ W\ A(j)\ +\ B^T\ \Lambda_g\ B\ +\ \Lambda_d]^{-1}\ \{A^T(j)\ W\ \delta\partial_r g(i,j)$$

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$$- [\underline{\underline{B}}^T \, \underline{\underline{\Lambda}}_g \, \underline{\underline{B}}] \, [\underline{\xi}(i,j) - \underline{\xi}_{go}] - \underline{\underline{\Lambda}}_d [\underline{\xi}(i,j) - \underline{\xi}_{do}] \} \, . \qquad (33b)$$

Shallow iteration proved to be of but slight use in practice.

4. SUMMARY

In the SFM/FFC earth model derivation of a (piecewise, statistically) steady fluid flow near the top of the core from imperfect models of the time-varying geomagnetic field requires solving a non-linear geophysical inverse problem. In 1987, a method was developed to solve this problem. The method attempts iterative minimization of an objective function composed of the square weighted residual in the geomagnetic secular change relative to a reference epoch; the ageostrophy of the flow as measured by the mean square departure from a geostrophic radial vorticity balance; and the spatial complexity of the flow as measured by flow source amplitudes (the mean square radial vorticity and the mean square downwelling of the flow). The geostrophic constraint and the damping of spatial structure are optionally imposed with variable-strength damping parameters. In order to mitigate the artificial restrictions imposed by prior bias towards zero flow sources on the investigation of steady flows, a learning algorithm was also developed in which the bias is shifted towards the result of the previous iteration and departures therefrom governed by a convergence factor.

When combined with numerical forward solution of the surficial motional induction equation (14a), equation (33a) defines a method for solving iteratively the non-linear geophysical inverse problem posed by the simple suppositions of a source-free mantle surrounding a frozen-flux core in surficially steady motion. The method involves two levels of iteration, double time integrals of matrices whose elements are surface integrals, a weight function which varies with both position and time, and includes two optional constraints: one for imposing the geostrophic radial vorticity balance and one for damping the spatial complexity of the flow. Additionally, two kinds of weights have been explored with fixed bias and learning algorithms. The reader may appreciate the practical difficulties of applying this method and keeping track of the various solutions, the irony of having such a simple set of working hypotheses yield so complicated a method, and most importantly, the complexity of the real Earth when stripped of such simplifying suppositions.

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Consider the use of general weight matrix elements Ω_{ij}/f instead of $W_{ij}/4\pi$ in (9c). The generalized weight matrix $\underline{\underline{\Omega}}$ is the inverse of the error covariance matrix for the radial field coefficients $\underline{\underline{E}}$:

$$E_{ij} \equiv E_0\{e_i e_j\} \tag{A1a}$$

$$\Omega_{ij} E_{jk} = \delta_{ik} \tag{A1b}$$

where e_i is the (unknown) true error associated with the use of expected radial field coefficients g_i and only regular expectations (non-singular $\underline{\underline{E}}$) are considered. This transforms $\underline{\underline{\Delta}}_r^2$ into the generalized square weighted residual

$$f[\Delta_r^{*}(a;t_o,t_f)]^2 \equiv \int_{t_o}^{t_f} f(t) [\delta_r^{*}(a,t;t_o)]^2 dt$$

$$= \int_{t_o}^{t_f} (g_i - \gamma_i)\Omega_{ij}(g_j - \gamma_j) dt \tag{A2}$$

$$= \int_{t_o}^{t_f} (\underline{g} - \underline{\gamma})^T \underline{\underline{\Omega}} \, (\underline{g} - \underline{\gamma}) dt$$

where matrix notation is employed in the last step. The scalar $f(t)$ renormalizes the instantaneous generalized square weighted residual $[\delta_r^{*}(a,t;t_o)]^2$, hence Δ_r^{*2}. It is taken to be the trace of $\underline{\underline{\Omega}}(t)\underline{\underline{E}}(t)$ —typically the number of radial field coefficients fitted at time t. No correction for the number of degrees of freedom of the flow model is included because it is the significance of the residuals, rather than the efficiency of the model, which is of interest here (see Part IC).

Let ω_{kj} denote the elements of the upper triangular matrix square root of the positive definite generalized weight matrix

$$\underline{\underline{\Omega}} = \underline{\underline{\omega}}^T \underline{\underline{\omega}} \tag{A3}$$

(BIERMAN, 1977, p40). Then

$$f[\Delta_r^{*}(a;t_o,t_f)]^2 = \int_{t_o}^{t_f} (\underline{g} - \underline{\gamma})^T \underline{\underline{\omega}}^T \underline{\underline{\omega}}(\underline{g} - \underline{\gamma}) dt$$

$$= \int_{t_o}^{t_f} 4\pi<(\underline{g} - \underline{\gamma})^T \underline{\omega}^T (\underline{S} \, \underline{\underline{N}}^{-1} \underline{S}) \, \underline{\underline{\omega}} \, (\underline{g} - \underline{\gamma})>dt$$

$$= \int_{t_o}^{t_f} 4\pi<[\tilde{B}_r(a,t) - \tilde{B}_{rp}(a,t)]^2>dt$$

$$= \int_{t_o}^{t_f} (\tilde{g}_k - \tilde{\gamma}_k)(\tilde{g}_k - \tilde{\gamma}_k)\,dt \qquad\qquad (A4)$$

where

$$\tilde{B}_r \equiv \tilde{g}_k([\frac{2n(k)+1}{4\pi}]^{1/2} S_k) \qquad\qquad \tilde{B}_{rp} \equiv \tilde{\gamma}_k([\frac{2n(k)+1}{4\pi}]^{1/2} S_k)$$

and where $\tilde{g}_k \equiv \omega_{ki}g_i$ or $\tilde{\gamma}_k \equiv \omega_{ki}\gamma_i$ are respectively the input or predicted radial field coefficients measured in units of expected uncertainty.

To better understand the general weights, introduce the vector of expected or input Gauss coefficients $p_j(t)$ and recall the diagonal matrix $\underline{\underline{R}}$ with elements $R_{ij} = [n(i)+1)]\delta_{ij}$:

$$\underline{g} = \underline{\underline{R}}\,\underline{p} . \qquad\qquad , \qquad\qquad (A5)$$

Recall that $\underline{\underline{V}} = \underline{\underline{R}}^{-1}\underline{\underline{E}}\underline{\underline{R}}^{-1}$ represents the error covariance matrix for the Gauss coefficients which, to the extent that the modeler's expectations are realized, measures the (square correlated) uncertainties in the Gauss coefficients. Then, in matrix notation, the generalized square weighted signal in the radial field coefficients

$$[S(t)]^2 = \underline{g}^T\underline{\underline{\Omega}}\,\underline{g} = \underline{p}^T\underline{\underline{R}}^T\underline{\underline{\Omega}}\underline{\underline{R}}\,\underline{p} = \underline{p}^T\underline{\underline{R}}^T(\underline{\underline{R}}\underline{\underline{V}}\underline{\underline{R}}^T)^{-1}\underline{\underline{R}}\,\underline{p} = \underline{p}^T\underline{\underline{V}}^{-1}\underline{p} \qquad (A6)$$

is the weighted signal in the Gauss coefficients.

Now, let $\delta g_i \equiv g_i(t) - g_i(t_o)$, let $\delta p_i \equiv p_i(t) - p_i(t_o)$, and omit the uncertainty in the initial conditions at time t_o. Relative to t_o, the generalized weighted signal in the secular change at t is then

$$[\Delta S(t;t_o)]^2 = \delta\underline{g}^T\,\underline{\underline{\Omega}}\,\delta\underline{g} = \delta\underline{p}^T\,\underline{\underline{V}}^{-1}\,\delta\underline{p} \equiv f(t)\,[\delta_r^*(a,t;t_o)]^2 \qquad (A7)$$

With initial condition $\underline{\gamma}(t_o) = \underline{g}(t_o)$, hence predicted Gauss coefficients $\underline{\rho}(t_o) = \underline{p}(t_o)$, the instantaneous generalized square weighted residual in the secular change at time t relative to t_o is

$$f(t)\,[\delta_r^*(a,t;t_o)]^2 = (\underline{g} - \underline{\gamma})^T\underline{\underline{\Omega}}(\underline{g} - \underline{\gamma}) = (\underline{p} - \underline{\rho})^T\underline{\underline{V}}^{-1}(\underline{p} - \underline{\rho}) \qquad (A8)$$

The time integral of (A8) from t_o to t_f is (A2)—the total generalized square weighted residual in the secular change accumulated during the interval $[t_o,t_f]$.

By replacing $W_{ij}/4\pi$ with Ω_{ij}/f, one transforms the objective from an attempt to fit the evolution of the weighted radial magnetic flux density called for by a time series of geomagnetic field models into an attempt to fit the evolution of the scalar geomagnetic potential called for by a time series of weighted Gauss coefficients. The former may seem more sensible because field components are observable, unlike the scalar potential, and because horizontal components of the induction equation are not used (0); however, the latter has appreciable merit.

For example, if the expected Gauss coefficients at time t could be obtained from a geomagnetic data vector $\underline{d}$, a symmetric data weight

matrix M, and a normal equations matrix A such that

$$p = (A^TMA)^{-1}A^TMd = VA^TMd \tag{A9}$$

then by (A6)

$$S(t)^2 = g^T\Omega g = p^Tv^{-1}p = d^TM^TAVV^{-1}VA^TMd = d^TM^TAVA^TMd \tag{A10}$$

Now $AVA^T = A(A^TMA)^{-1}A^T = Q$ implies $QMA = A$; but Q is not necessarily M^{-1} because A is typically rectangular and not invertible. However, if Q were M^{-1}, then S^2 would equal d^TMd-the weighted signal in the data from which the geomagnetic field model could be derived; then (A8) would equal the instantaneous signal in the secular change indicated by the weighted data, and its time integral would equal the total signal in the secular change called for by the weighted data accumulated during the interval.

Unfortunately such data might not exist due, for example, to the use of damping, prior bias in deriving the field model, averaging of coefficients, or roundoff errors. If such data do exist, then their weighting might be suspect; moreover, the data may well contain contributions from fields other than the broad-scale core field of interest (e.g., crustal fields). The latter seem problematic due to the time-varying spatial distribution of geomagnetic survey data. Inclusion of such extraneous fields in the objective function seems inappropriate, so a suitable modification of Ω might be considered. Fortunately, great compensation for crustal and external fields can be achieved using the correlated data weight matrix technique developed by LANGEL, ESTES & SABAKA (1988a, 1989). This kind of approach was used to derive the DGRF models for epochs 1945 through 1960 (LANGEL et al., 1988b). Yet it is still not clear that general weights are preferable to radial field weights-particularly as the correspondence between the weighted signal in the data and the weighted signal in the field model is not rigorous.

As pointed out by G. BACKUS (1987, personal communication), it is useful to introduce the quantity

$$x_k \equiv \omega_{ki}e_i \tag{A11}$$

with covariance matrix elements

$$E_o\{x_kx_l\} = E_o\{\omega_{ki}e_ie_j\omega_{lj}\} = \omega_{ki}E_o\{e_ie_j\}\omega_{lj}$$

$$= \omega_{ki}E_{ij}\omega_{lj} = \delta_{kl} \ . \tag{A12}$$

For a tenth-degree DGRF model at epoch t_n the trace of (A12) is

$$Tr(E_o\{x_kx_l\}) = E_o\{Tr(x_kx_l)\} = E_o\{x_kx_k\} = 120 = f(t_n)$$

$$= E_o\{(\omega_{ki}e_i)\omega_{kj}e_j\} = E_o\{e_iE^{-1}_{ij}e_j\} = E_o\{e_i\Omega_{ij}e_j\} \tag{A13}$$

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To the extent that the fitting residuals $[g_i - \gamma_i]$ resemble the expected uncertainty, the suitably normalized generalized square weighted residual at time t_n

$$[\delta_r^*(a,t_n;t_o)]^2 = [g_i(t_n) - \gamma_i(t_n)]\Omega_{ij}(t_n)[g_j(t_n) - \gamma_j(t_n)]/120 \quad (A14)$$

is expected to be unity. This quantity is readily evaluated provided $\Omega = E^{-1}$ can be obtained (see Part IC).

Of course, other kinds of geomagnetic field models, notably the harmonic spline models of SHURE, PARKER, & BACKUS (1982), may not have truncated spherical harmonic representations. In such cases, a total change of basis functions seems less computationally burdensome than transforming the finite dimensional expected error covariance for the modeled parameters (say, the harmonic spline coefficients) into an equivalent, apparently infinite dimensional, expected error covariance for spherical harmonic coefficients of the radial field.

ERRATA: PART IA.

Page 9, line 30. The reference should be upper case and in parentheses rather than square brackets.

Page 22, lines 28-34. The passage "On Earth's surface ... the main field. Nevertheless, ..." should be replaced with "Near Earth's surface (and apparently within the Earth) the high frequency electromagnetic oscillations of solar and terrestrial origin have far greater energy density than the main geomagnetic field, so Ampere's law is broken. However, ...".
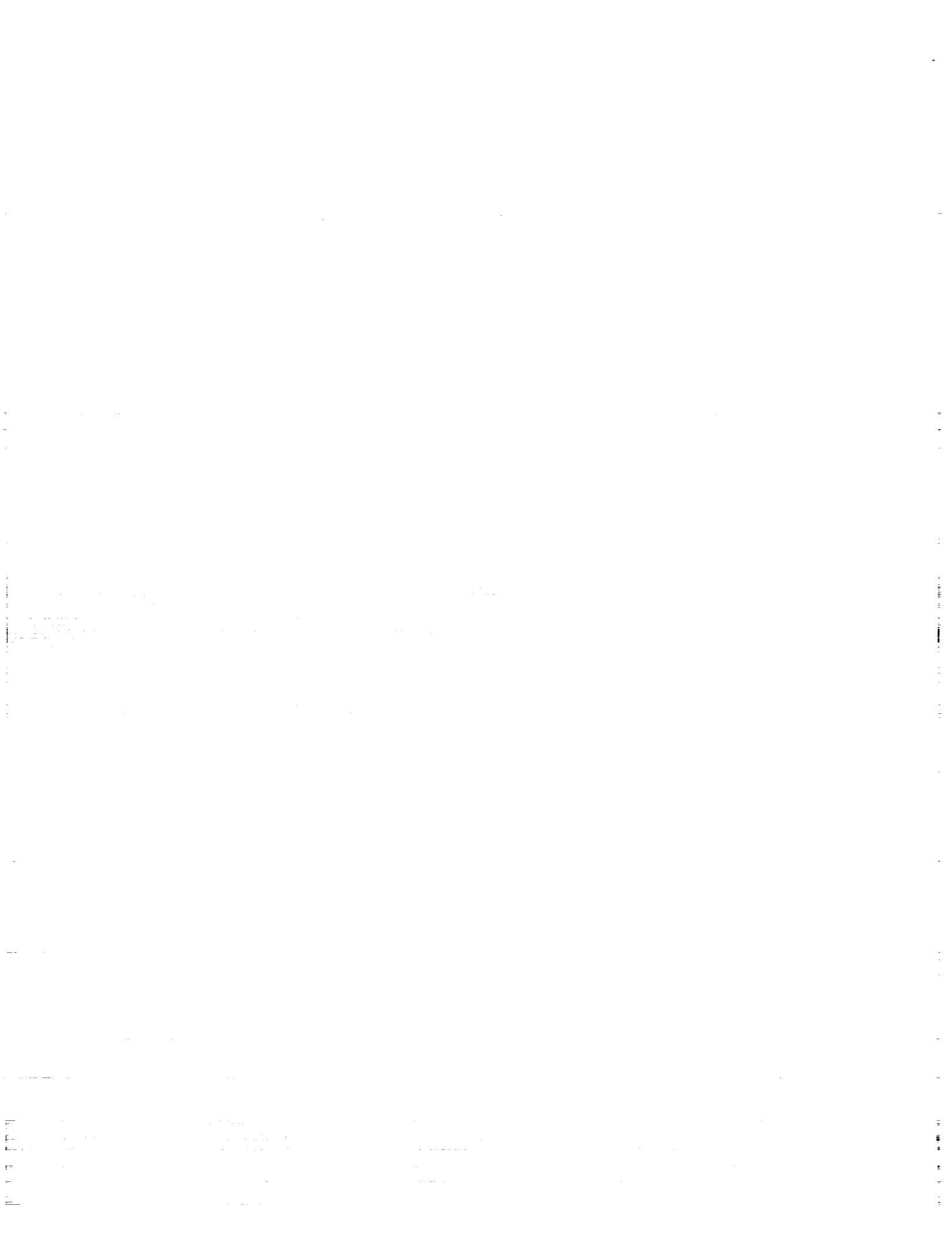
Page 25, line 11-12. Note that non-subscripted $\mu(r,t) = \Sigma_i \mu_i(r,t)$ can be a sum of matrices.

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The problem of estimating a steady fluid velocity field near the top of Earth's core which induces the secular variation (SV) indicated by models of the observed geomagnetic field is examined in the source–free mantle/frozen–flux core (SFM/FFC) approximation. This inverse problem is non–linear because solutions of the forward problem are deterministically chaotic, the SFM/FFC approximation is inexact, and neither the models nor the observations they represent are either complete or perfect. A method is developed for solving the non–linear inverse motional induction problem posed by the hypothesis of (piecewise, statistically) steady core surface flow and the supposition of a complete initial geomagnetic condition. The method features iterative solution of the weighted, linearized least–squares problem and admits optional biases favoring surficially geostrophic flow and/or spatially simple flow. Two types of weights are advanced: radial field weights for fitting the evolution of the broad–scale portion of the radial field component near Earth's surface implied by the models, and generalized weights for fitting the evolution of the broad–scale portion of the scalar potential specified by the models.

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