# ON INDUCING FINITE DIMENSIONAL PHYSICAL FIELD REPRESENTATIONS FOR MASSLESS PARTICLES IN EVEN DIMENSIONS

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**Abstract**

Assuming trivial action of Euclidean translations, the method of induced representations is used to derive a correspondence between massless field representations transforming under the full generalized even dimensional Lorentz group, and highest weight states of the relevant little group. This gives a connection between 'helicity' and 'chirality' in all dimensions. We also state restrictions on 'gauge independent' representations for physical particles that this induction imposes.

## 1. Introduction

For $d = 2n + 2$, the generalized Lorentz group $SO(2n+1, 1)$ commutation relations are

$$[J_{AB}, J_{CD}] = i(\delta_{AC}J_{BD} + \delta_{BD}J_{AC} - \delta_{AD}J_{BC} - \delta_{BC}J_{AD}) \tag{1.1}$$

with $A, B = 0, \ldots (d-1)$ where $\delta$ is the $d$ dimensional Kronecker symbol ($\delta_{AB} = 1$ if $A = B$; 0 otherwise), the boosts in the $i$th direction are defined as

$$K_i = iJ_{i0} = -iJ_{0i} \tag{1.2}$$

and for $i, j \neq 0$, the rotation generator in the $i, j$ hyperplane is $J_{ij} = -J_{ji}$. Alternatively, the commutation relations may be written

$$[J_{AB}, J_{CD}] = i(g_{AC}J_{BD} + g_{BD}J_{AC} - g_{AD}J_{BC} - g_{BC}J_{AD}) \tag{1.3}$$

where $g = \mathrm{diag}(-1, 1, \ldots, 1, 1)$ with the boosts generated by

$$K_i = J_{i0} = -J_{0i} \tag{1.4}$$

and the rotations generated by $J_{ij} = -J_{ji}$, for $i \neq j$. The $-1$ in the metric in (1.3) arises from the $i$ on the boost generators in (1.1) ; to 'Wick' rotate the noncompact algebra to the compact algebra of the special orthogonal algebra $SO(2n+2)$, or vice versa,

$$\begin{aligned} SO(2n+1,1) &\leftrightarrow SO(2n+2) \\ iJ_{i0} &\leftrightarrow J_{i,d} \\ g &\leftrightarrow \delta. \end{aligned} \tag{1.5}$$

$SO(2n+2) \equiv D_{n+1}$ has rank $= n+1$, dimension $= (n+1)(2n+1)$, # of roots $= 2n(n+1)$, # of positive roots $= n(n+1)$, # of simple positive roots $= (n+1)$, where the roots are all the raising and lowering operators in the algebra, the simple roots are the linearly independent raising or lowering operators, and the positive simple roots are just the linearly independent raising operators in the algebra.

The relevant *little group* of Wigner is defined to be the maximal subgroup of the Lorentz group that leaves invariant the 'standard' momentum of a particle in $d = 2n+2$ dimensions. We choose our *massless* particle to be moving in the $d-1$ direction with $d$ momentum $k_\mu = (k, 0, \ldots, k)$; $\mu = 0, \ldots, d-1$. For $d = 2n+2$ the little group is generated by the $n$ commuting rotations $J_{12}, J_{34}, \ldots, J_{(2n-1,2n)}$ which will be the Cartan generators (and hereafter the 'weight notation', notated by a subscript $W$, will refer to the specification of a state in terms of the eigenvalues of this ordered set of Cartan generators), *and* the $2n$ 'translations'

$$L_i^+ \equiv J_{i,2n+1} + iJ_{i,0}, \; i = 1, \ldots 2n \tag{1.6}$$

which, using (1.1) can be seen to form an Abelian subalgebra

$$[L_i^+, L_j^+] = 0. \tag{1.7}$$

Each translation indexed by $i$ is a sum of a boost in the $i$th direction and a rotation in the $i, d-1$ plane. The commutation relations of the little group are

$$\begin{aligned}
&[L_i^+, L_j^+] = 0 \\
&[J_{ij}, L_k^+] = i(\delta_{ik} L_j^+ - \delta_{jk} L_i^+) \\
&[J_{ij}, J_{kl}] = i(\delta_{ik} J_{jl} - \text{permutations}).
\end{aligned} \tag{1.8}$$

The little group (1.8) is not semi-simple (it has an Abelian subalgebra of translations) and is isomorphic to $E(d-2)$, the Euclidean group in $d-2$ dimensions.

$$L_i^- \equiv J_{i,2n+1} - iJ_{i,0} = L_i^+ - 2iJ_{i,0} \tag{1.9}$$

also form an Abelian subalgebra (but they do not belong to the little group for above choice of standard momentum), since

$$[L_i^-, L_j^-] = 0. \tag{1.10}$$

Note also that

$$\begin{aligned}
[L_i^+, L_j^-] &= 2iJ_{ij}, i \neq j \\
&= 2J_{2n+1,0}, i = j
\end{aligned} \tag{1.11}$$

and under complex-conjugation

$$\begin{aligned}
(L_i^+)^* &= -L_i^- \\
(L_i^-)^* &= -L_i^+.
\end{aligned} \tag{1.12}$$

Much will be said about the significance of $L^\pm$ shortly.

We now state a key assumption: **the translation generators annihilate physical states.** There are a number of reasons for this: (1) *Finite dimensional representations only:* The group transformation corresponding to the translation $L_i^+$, is written as $D_i^+(\chi_i) = e^{-i\chi_i L_i^+}$; the dimensionality of the representation is characterized by the length of the translation vector $\sum_i \chi_i^2$. For finite dimensional representations the translation parameter $\chi_i = 0, \forall i$. (2) *Gauge-independence:* In general, an eigenstate of the Cartan generators of the little group $J_{12}, \dots J_{(2n-1,2n)}$ is not an eigenstate of the translation generators $L_i^+$ since they do not commute. The eigenstates of the Cartan generators can be written as the $d$ dimensional polarization vectors (e.g. in four dimensions the generator $J_{12}$ which generates $z$-rotations has the eigenvectors $\epsilon^\mu = (0, 1, \pm i, 0)$). It can be checked, for instance, that the transversality condition required for the photon vector potential to be a Cartan eigenstate is not invariant under finite translations. In fact, the translations generate effects identical to Abelian 'gauge' transformations [1] [2] . Thus the requirement of trivial translations is the requirement that only 'gauge independent' objects be considered. We believe that these two consequences of trivial translations are desirable from the point of view of making a scattering theory for a finite number of physical degrees of freedom without auxiliary conditions, as in [3] [4]. (3) *'Factorization' of invariant operators:* It can also be shown as a rather nontrivial consequence [5] that the eigenvalue of an invariant operator in the enveloping algebra of the higher dimensional Lorentz group factorizes into the eigenvalue of a generalized Pauli-Lubanski pseudovector and and a simple factor related to the boost generator.

## 2. The Main Theorem

Only the main results are highlighted here and the interested reader is referred for further details to [6]. Define a *physical field* $\Lambda$ as a representation transforming under the full higher dimensional Lorentz group and obeying the condition of trivial translations $L_i^+\Lambda = 0$, $i = 1, 2 \dots 2n$.

**Main Theorem:** *A physical field $\Lambda$ is a highest weight of the Lorentz group.*

**Proof:** Take the full group to be $SO(2n+1, 1)$. The little group is then $E(2n)$. By definition, a physical field is annihilated by all the translations $L_1^+, \dots L_{2n}^+$. Now, a highest weight is by definition the state annihilated by *all* the linearly independent raising operators. For $SO(2n+1, 1)$, which is rank $n+1$, we need thus to find the $n+1$ positive simple roots and show that they all annihilate the physical field. To this end, we first want to prove the following fact:

**Lemma:** *All the raising operators can be made using linear combinations of the translation generators.*

**Proof of Lemma:** Since there are $2n$ available translations, and $n+1$ required linearly independent raising operators, for $n \geq 1$, i.e. for four dimensions or more, there are certainly enough translations available to make all the linearly independent raising operators.

We choose the Cartan generators in $SO(2n+1,1)$ to be $J_{12},\ldots,J_{2n-3,2n-2},J_{2n+1,0}$. We claim that the complete set of linearly independent raising operators for $n>1$ is

$$
\begin{aligned}
&L_1^+ \pm iL_2^+ \\
&L_3^+ + iL_4^+ \\
&\vdots \\
&L_{2n-1}^+ + iL_{2n}^+ .
\end{aligned}
\tag{2.1}
$$

For instance, for the four dimensional Lorentz group ($n=1$) the two linearly independent raising operators are $L_1^+ \pm iL_2^+$. To check, in any dimension that (2.1) is indeed the complete set of linearly independent raising operators, commute each member of the set with the Cartan generators using (1.1) and (1.8) to obtain the coordinates of the positive simple roots in the Cartan basis. Translating this to the Dynkin basis [6] [7] obtain the rows of the Cartan matrix. But since the rows of the Cartan matrix are defined to be the coordinates of the linearly independent raising operators under commutation with the Cartan generators, (2.1) is in fact all of them. With the physical field condition and (2.1) the theorem is proved. **QED**

Note that using (1.11) we obtain

$$[L_i^+ + iL_{i+1}^+, L_i^- - iL_{i+1}^-] = 4(J_{2n+1,0} + J_{i,i+1}),\ i = 1,3,5,2n-1 \tag{2.2}$$

and

$$[L_1^+ - iL_2^+, L_1^- + iL_2^-] = 4(J_{12} - J_{30}). \tag{2.3}$$

Since the right hand side lies in the Cartan subalgebra, the lowering operator corresponding to each of the raising operators is obtained by replacing $L_i^+ \to L_i^-$, $i \to -i$ in (2.1). With the definition

$$
\begin{aligned}
E^+_{1,1,0,\ldots,0} &= A_+^+ \equiv L_1^+ + iL_2^+ \\
E^+_{-1,1,0,\ldots,0} &= A_-^+ \equiv L_1^+ - iL_2^+ \\
E^-_{1,-1,0,\ldots,0} &= A_-^- \equiv L_1^- - iL_2^- \\
E^-_{-1,-1,0,\ldots,0} &= A_+^- \equiv L_1^- + iL_2^-
\end{aligned}
\tag{2.4}
$$

where the subscripts on the roots $E$ denote the eigenvalue under commutation with the Cartan generators,

$$[A_+^+, A_-^+] = [A_+^+, A_+^-] = [A_+^-, A_-^-] = [A_-^+, A_-^-] = 0 \tag{2.5}$$

we note that $A_+^+$ and $A_-^+$ are raising operators in two orthogonal directions (with $A_-^-$ and $A_+^-$ the orthogonal lowering operators). In four dimensions, $A_+^+, A_-^+$ are the two linearly independent raising operators, and $A_-^-, A_+^-$ are the corresponding lowering operators. From the physical field condition

$$L_1^+\Lambda = L_2^+\Lambda = 0 \Leftrightarrow A_+^+\Lambda = A_-^+\Lambda = 0 \tag{2.6}$$

which means that the physical field is a highest weight, and complex conjugating the last two equations with the use of (1.12)

$$A_{-}^{-}\Lambda^{*} = A_{+}^{-}\Lambda^{*} = 0 \tag{2.7}$$

which shows that the complex conjugate is the lowest weight. Looking, for instance, at the weight diagram for the left and right handed spinors in four dimensions (in Dynkin notation)

$$\begin{array}{cc} \text{Left} & \text{Right} \\ (+1\ \ 0) & (0\ \ +1) \\ (-1\ \ 0) & (0\ \ -1) \end{array} \tag{2.8}$$

we confirm the result of (2.7) that indeed the spinors are inequivalent and self-conjugate (i.e. the weight diagrams reflect to minus themselves).

## 3. Helicity-Chirality Correlation in Higher Dimensions

In one of his classic papers on 'Feynman Rules for Any Spin', Weinberg [8] proves that the annihilation operator for a massless particle of helicity $\lambda$ and the creation operator for the antiparticle with helicity $-\lambda$ can only be used to form a field transforming as $U[\Lambda]\psi_n(x)U[\Lambda]^{-1} = \sum_m D_{nm}[\Lambda^{-1}]\psi_m(\Lambda x)$ under those representations $[A, B]$ of $SO(3,1)$ such that $\lambda = B - A$[1]. This restriction arises solely due to the non-semi-simplicity of the little group for massless particles; in particular due to the requirement that the Euclidean 'translations' of the little group act trivially on the physical Hilbert space. As a direct consequence of Weinberg's result it is observed that in four-dimensions a physical left-handed, helicity $-j$ particle can only correspond to a representation $[j+n, n]$, ($n$ is an integral multiple of $\frac{1}{2}$) whereas a physical right-handed, helicity $j$ particle can only correspond to the representation $[n, j+n]$. The generalization of the statement to higher even dimensions will be stated as the following corollary to the main theorem:

**Helicity-Chirality Correlation:** *A physical field of the full group corresponds to a highest weight state of the little group (given trivial action of translations),* and *the eigenvalue of each generator common to the full group and the little group remains unchanged under the projection of a representation of the full group to a representation of the little group.*

**Proof:** Since the little group with trivial translations is just the orthogonal group in two lower dimensions, a subset of the linearly independent raising operators of the full group is exactly the complete set of linearly independent raising operators of the little group, with the caveat that they have to be appropriately Wick rotated to obtain the compact form. For $SO(2n+1,1)$ ($n > 1$) as the full group, the subset of (2.1) without the last raising operator, $L_{2n-1} + iL_{2n}$, upon Wick rotation, is the complete set of linearly independent raising operators for $SO(2n)$. For example, for $SO(5,1)$ the raising operators are $L_1^+ \pm iL_2^+, L_3^+ + iL_4^+$, whereas for the little group of trivial translations $\sim SO(4)$ they are $L_1^+ \pm L_2^+$. Since the full set of raising operators annihilates the full group highest weight,

[1] $A$ and $B$ correspond to independent $SU(2)'s$.

*and* the projection from the full group to the little group is an orthogonal projection (since the boost generator is the only non-common generator and it commutes with the Cartan generators of the little group), it follows that the little group state is a highest weight state of the little group, and most importantly, the eigenvalue of the Cartan generators of the little group is invariant under the projection (hence the consequences of the corollary are most explicit in the weight notation). To recapitulate, the helicity-chirality correlation in higher dimensions is nothing but the fact that under projection of the field representation from the full Lorentz group to a little group state, the eigenvalue of the Cartan generators remains unchanged.

In conclusion, we demonstrate that our main theorem and corollaries reproduce some familiar results of four dimensions:

•In weight notation, the last result shows why a chiral left-handed field transforming as $(-\frac{1}{2},\frac{1}{2})_W$ ($[\frac{1}{2},0]$ in conventional $SU(2)\times SU(2)$ notation) corresponds to a helicity $-\frac{1}{2}$ particle and a chiral right-handed field transforming as $(\frac{1}{2},\frac{1}{2})_W$ ($[0,\frac{1}{2}]$ in conventional notation) corresponds to a helicity $\frac{1}{2}$ particle.

•Using $\frac{1}{2}(\mathbf{J}\pm i\mathbf{K})=\mathbf{A},\mathbf{B}$ and the definition $J_3\Lambda\equiv J_{12}\Lambda=\lambda\Lambda$, with $\lambda$ the helicity, we get, on using the physical field condition and (2.1)

$$\begin{aligned}(L_1^+-iL_2^+)\Lambda=0\rightarrow(A_1-iA_2)\Lambda=0\rightarrow A_3=-A\\(L_1^++iL_2^+)\Lambda=0\rightarrow(B_1+iB_2)\Lambda=0\rightarrow B_3=B\end{aligned}\tag{3.1}$$

since $\mathbf{A},\mathbf{B}$ generate independent $SU(2)$ algebras and $A_1-iA_2$ is the lowering operator for one and $B_1+iB_2$ is the raising operator for the other. Then, by definition $J_3=A_3+B_3=B-A=\lambda$ which is Weinberg's condition [8] .

•As mentioned earlier, the translations are also generators of Abelian gauge transformations. Requiring them to be trivial restricts us to gauge independent, finite dimensional repesentations of the full group. Our theorem and the corollary then tell us what is the little group representation corresponding to this gauge independent full group representation. For example, the representation corresponding to the field strength tensor in the conventional $(SU(2)\times SU(2))$, Dynkin and weight basis is:

$$F^{\mu\nu}=[1,0]+[0,1]=(2,0)_D+(0,2)_D=(-1,1)_W+(1,1)_W\tag{3.2}$$

which has $J_{12}$ eigenvalues $\pm1$ and so is admissible as the $J_{12}$ eigenvalue remains unchanged and corresponds to the correct helicity of the photon as we project to the little group state. However, the vector potential corresponds to

$$A^\mu=[\frac{1}{2},\frac{1}{2}]=(1,1)_D=(0,1)_W\tag{3.3}$$

which has a $J_{12}$ eigenvalue of 0 which does not correspond to a transversely polarized helicity $\pm1$ photon.

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