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Cost Averaging Techniques for Robust Control of Flexible Structural Systems

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Outline

- Introduction
- Modeling of Parameterized Systems
- Average Cost Analysis
- Reduction of Parameterized Systems
- Static and Dynamic Controller Synthesis
- Examples

Problem Statement

- The problem is to design a controller that provides stability robustness over a set of plants described by real parameter uncertainty.
- This type of uncertainty is common in structural plants.
  - Uncertain stiffness
  - Uncertain damping
  - Uncertain modal parameters
- Stability examined in the context of the performance-robustness trade.

Background in Robust Control Synthesis

Bounding Methods

- Develop robustness analysis test
  - Lyapunov, Kharitonov, Small Gain, μ
- Incorporate analysis test into performance metric
  - modify Lyapunov equation or develop stability index
- Find controller which minimizes modified performance metric subject to constraints.

Sensitivity Methods

- Sensitize the cost to the uncertainties, then minimize.
- Multiplicative White Noise (MEOP): Hyland, Bernstein
- Cost Sensitivity Minimization: Skelton
- Multi-Model Techniques: Bryson, Li

Approach of This Work

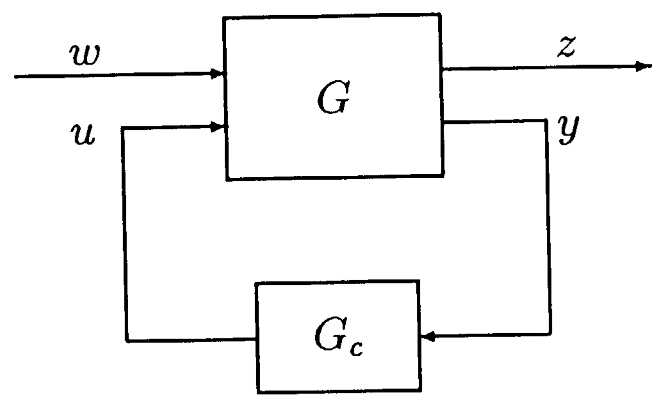
- Examine cost averaged over a continuously parameterized set of plants.
- Derive analysis tools to determine approximations and bounds to the average cost.
- Apply these analysis tools to the problem of determining critical components and uncertainties.
- Apply these analysis tools to the problem of nonconservative robust controller design.

Modeling Notation

- A system $G(s)$ can be represented in state space notation as:

$$\left[\begin{array}{c} \dot{x} \\ \hline z \\ y \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline w \\ u \end{array}\right]$$

- The controlled system can be represented schematically:



Sets of Systems

- The set, Ω, of parameters is defined on the compact interval

$$\Omega = \left\{\alpha \ : \alpha \in \mathbb{R}^r, -\delta_i^L \leq \alpha_i \leq \delta_i^U \quad i = 1, \cdots, r\right\}$$

- The set $\mathcal{G}$ of systems is parameterized as follows

$$\mathcal{G} = \{G(\alpha) \, \forall \, \alpha \ \in \ \Omega\}$$

  – for structured parameter dependence

$$G(\alpha) = \left[\begin{array}{c|cc} A_0 + \sum_{i=1}^{r} \alpha_i A_i & B_1 & B_{2_0} + \sum_{i=1}^{r} \alpha_i B_{2_i} \\ \hline C_1 & 0 & D_{12} \\ C_{2_0} + \sum_{i=1}^{r} \alpha_i C_{2_i} & D_{21} & 0 \end{array}\right]$$

The Average Cost

- The exact average cost is defined as the closed-loop $\mathcal{H}_2$-norm (quadratic cost) averaged over the model set.

$$J = \int_\Omega \|G_{zw}(\alpha)\|_2^2 \, \mathrm{d}\mu(\alpha)$$

- Finite average cost implies:
  - The closed-loop systems are stable $\forall \ \alpha \ \in \ \Omega$ except possibly at isolated points.
  - No closed-loop system can have eigenvalues with positive real parts.
- Controllers based on minimization of the average cost will guarantee stability without necessarily guaranteeing performance.

Average Cost Calculation

- The average cost is calculated

$$J = \mathrm{tr}\left\{\left\langle \tilde{Q}(\alpha) \right\rangle \tilde{C}^T \tilde{C}\right\}$$

  - For each $\alpha \in \Omega$, $\tilde{Q}(\alpha)$ is given by

$$0 = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\tilde{A}^T(\alpha) + \tilde{B}\tilde{B}^T$$

- The problem is how to compute the average solution to a parameterized Lyapunov equation.
  - Monte-Carlo
  - Direct Integration
  - Stochastic Operator Methods

Operator Decomposition

- Can utilize techniques from the field of wave propagation in random media and turbulence modeling.
- The parameterized Lyapunov equation can be decomposed into a nominal part and a parameter dependent part.

$$\tilde{A}(\alpha)\tilde{Q} + \tilde{Q}\tilde{A}^T(\alpha) + \tilde{B}\tilde{B}^T = 0$$

is equivalent to

$$\mathbf{L}_0\left[\tilde{Q}\right] + \mathbf{L}_1(\alpha)\left[\tilde{Q}\right] + \tilde{B}\tilde{B}^T = 0$$

$$\begin{aligned} \mathbf{L}_0\left[\tilde{Q}\right] &= \tilde{A}_0\tilde{Q} + \tilde{Q}\tilde{A}_0^T \\ \mathbf{L}_1(\alpha)\left[\tilde{Q}\right] &= \tilde{A}_1(\alpha)\tilde{Q} + \tilde{Q}\tilde{A}_1^T(\alpha) \end{aligned}$$

- Two methods of computing the average: Perturbation Expansion and Dyson Equation.

Perturbation Expansion Approximation

- Perturbation expansion series can be approximated by retaining only the first two terms.

$$\tilde{Q}^P = \tilde{Q}^0 + \mathbf{A}(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha))^2\tilde{Q}^0$$

- The perturbation expansion approximate cost is given by

$$J^P = \operatorname{tr}\left\{\tilde{Q}^P\tilde{C}^T\tilde{C}\right\}$$

where $\tilde{Q}^B$ is the solution of

$$\begin{aligned}
0 &= \tilde{A}_0\tilde{Q}^P + \tilde{Q}^P\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \\
0 &= \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^0 + \tilde{Q}^0\tilde{A}_i^T\right) \quad i = 1,\ldots,r
\end{aligned}$$

- The equations for the perturbation expansion are related to those used in Skelton's cost sensitivity controller design method.

- The two sets of Lyapunov equations are coupled hierarchically and easily solved using standard techniques.

Bourret Approximation

- The Dyson Equation can be approximated by retaining only the first term of $\mathbf{M}$.

$$\tilde{Q}^B = \tilde{Q}^0 + \mathbf{A}(\mathbf{L}_0^{-1}\mathbf{L}_1(\alpha))^2\tilde{Q}^B$$

- The Bourret approximate cost is given by

$$J^B = \mathrm{tr}\left\{\tilde{Q}^B\tilde{C}^T\tilde{C}\right\}$$

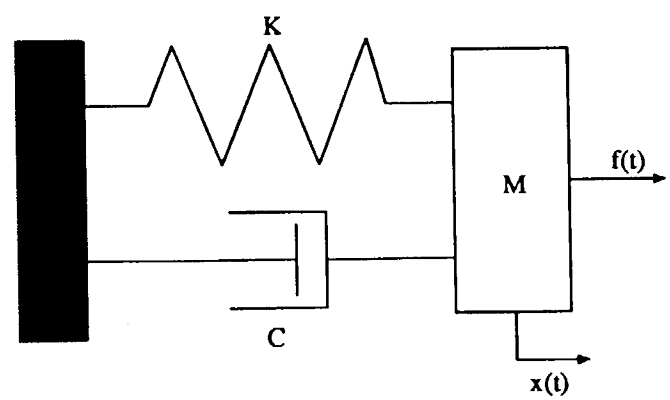
where $\tilde{Q}^B$ is the solution of

$$\begin{aligned} 0 &= \tilde{A}_0\tilde{Q}^B + \tilde{Q}^B\tilde{A}_0^T + \tilde{B}\tilde{B}^T + \sum_{i=1}^{r}\sigma_i\left(\tilde{A}_i\tilde{Q}^i + \tilde{Q}^i\tilde{A}_i^T\right) \\ 0 &= \tilde{A}_0\tilde{Q}^i + \tilde{Q}^i\tilde{A}_0^T + \sigma_i\left(\tilde{A}_i\tilde{Q}^B + \tilde{Q}^B\tilde{A}_i^T\right) \quad i = 1, \ldots, r \end{aligned}$$

- The Bourret equation represents an infinite series expansion for the approximate average solution.
- The cross-coupling complicates the solution procedure.
- Positive definite solution the Bourret equation guarantees stability over a set smaller than the design set.

Second Order System Example

- Consider the simple spring-mass-damper represented.



- The system has dynamics

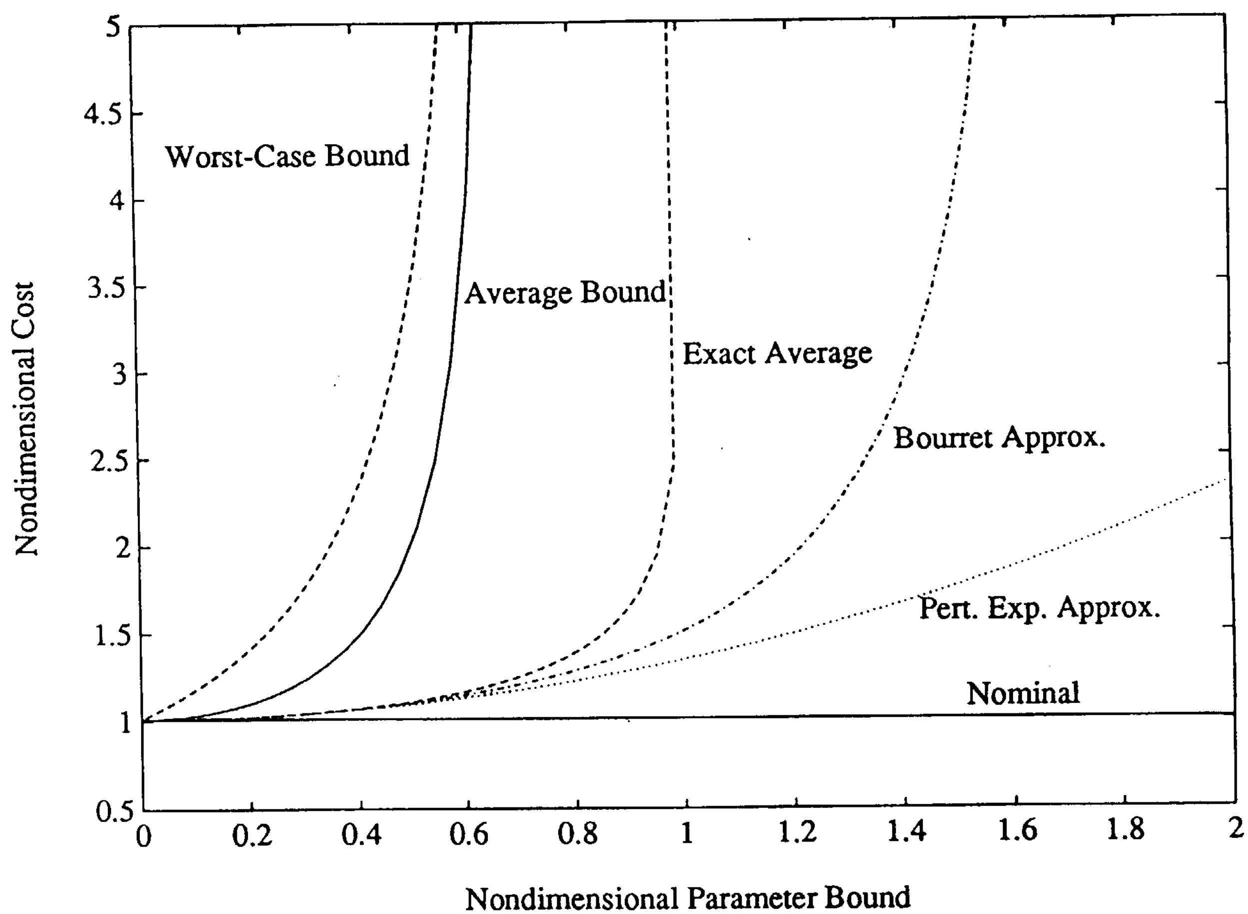
$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2 x(t) = bf(t)$$

- Let the natural frequency and damping ratios of the the system be uniformly distributed uncertain parameters of the form

$$\begin{aligned} \omega^2 &= {\omega_0}^2 + \tilde{\omega}^2 \qquad -\delta_{\omega^2} \le \tilde{\omega}^2 \le \delta_{\omega^2} \\ \zeta &= \zeta_0 + \tilde{\zeta} \qquad -\delta_\zeta \le \tilde{\zeta} \le \delta_\zeta \end{aligned}$$

Cost vs. Model Uncertainty

- The cost as a function of the uncertain damping bound.



Robust Control Synthesis

- Fixed-form controllers

$$G_c = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_c \end{array}\right] \text{ or } \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$$

- Controller Parameter Optimization Design Proceedure

Step 1: Define cost based on exact average, bounding or approximating cost.

Step 2: Append the appropriate equation to the cost using a matrix of Lagrange multipliers.

Step 3: Determine the necessary conditions for optimality using matrix calculus.

Step 4: Minimize the cost numerically using necessary condition for gradient information.

Step 5: Evaluate the resulting controllers for stability and performance robustness.

Necessary Conditions

- For exact average cost minimization with static output feedback.

$$D_c = -R^{-1}B_2^T \left\langle \tilde{P}(\alpha)\,\tilde{Q}(\alpha)\right\rangle \left\langle \tilde{Q}(\alpha)\right\rangle^{-1}$$

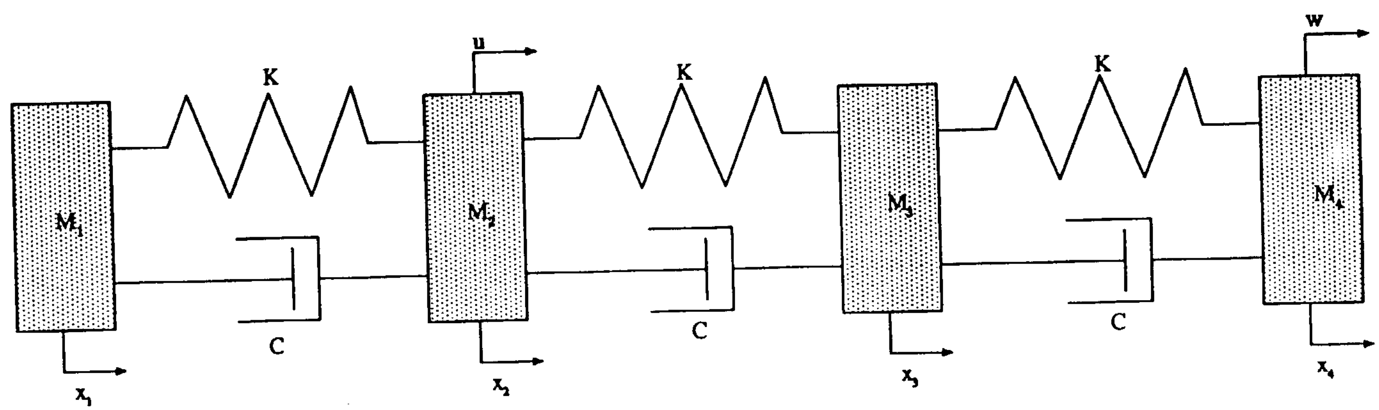
  where

$$0 = \tilde{A}(\alpha)\tilde{Q}(\alpha) + \tilde{Q}(\alpha)\,\tilde{A}^T(\alpha) + \tilde{B}\tilde{B}^T$$
$$0 = \tilde{A}^T(\alpha)\tilde{P}(\alpha) + \tilde{P}(\alpha)\,\tilde{A}(\alpha) + \tilde{C}^T\tilde{C}$$

- The necessary conditions have a form very similar to the necessary conditions derived for the static output feedback problem.
- Uncertainty couples the two parameterized Lyapunov equations.
- The same form is present with the approximations and bounds but with parameter independent equations.
- For dynamic output feedback the necessary condition comprise
  - three gain equations obtained by taking derivatives with respect to A_c, B_c, and C_c
  - two Lyapunov-based equations

Example 2: The Cannon-Rosenthal Problem

- Consider the four mass/spring/damper system with uncertain body one mass.



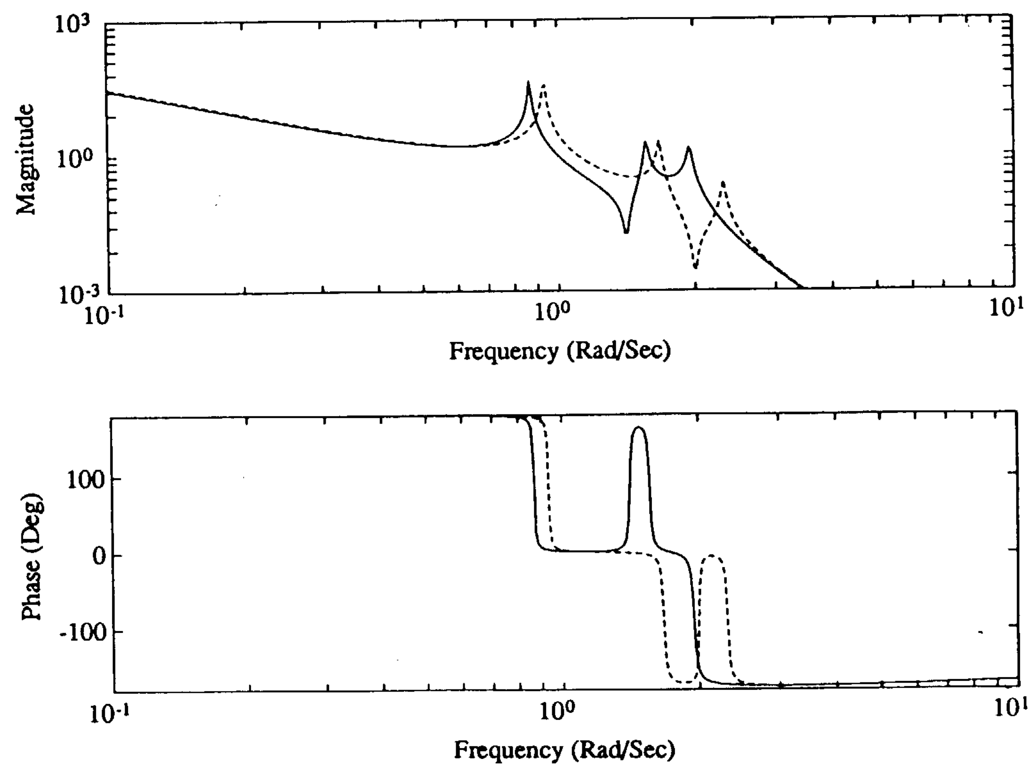
- The uncertain mass is represented

$$1/m_1 = 1/m_{1_0} + \tilde{m} \quad m_0 = 0.5 \quad |\tilde{m}| \leq \delta_m$$

- Spring and mass uncertainty treated by numerous researchers.

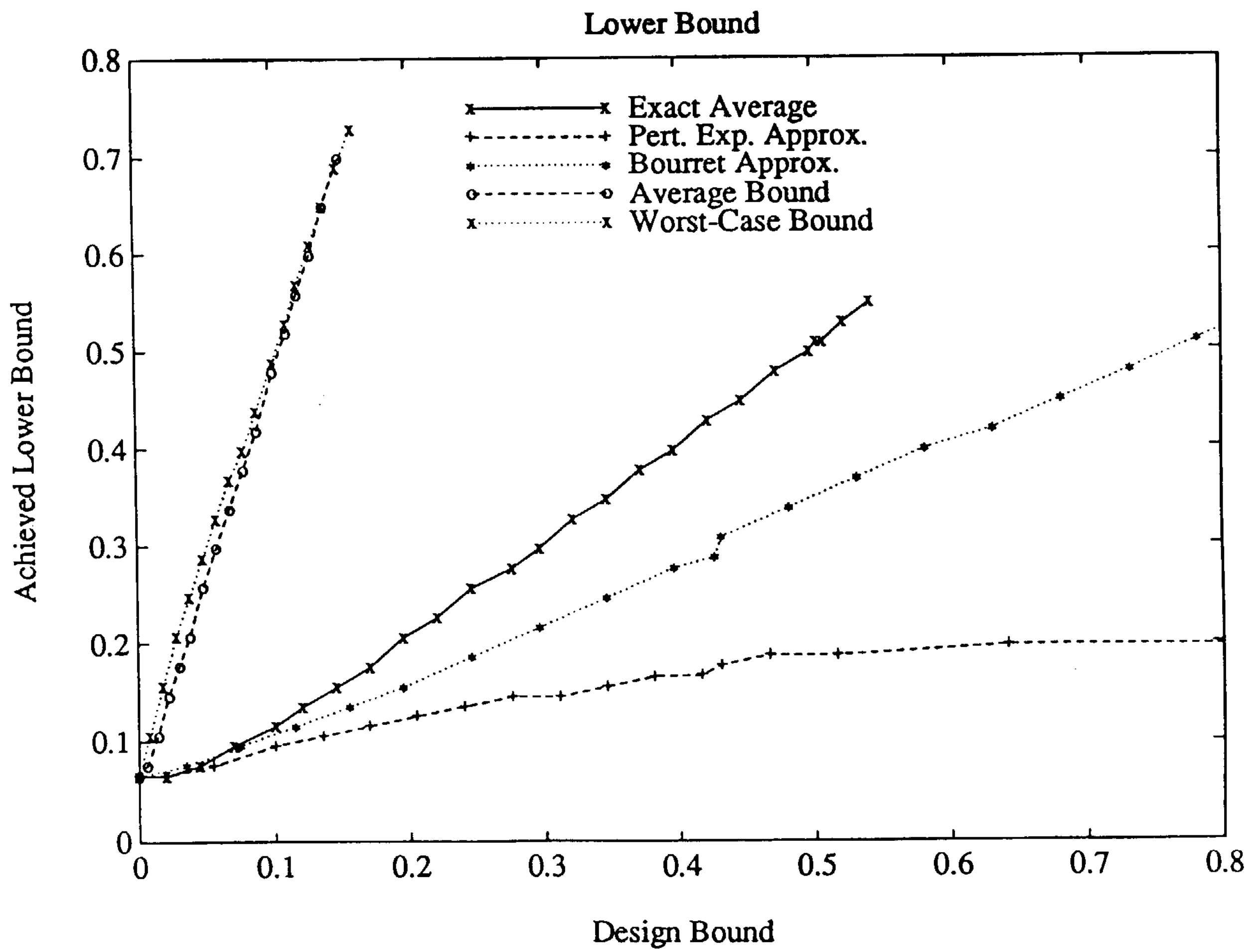
Pole-Zero Flip

- The problem was chosen because the plant zero and second mode change relative positions as the parameter is increased to $\tilde{m} = 0.6$
- The open loop u-y transfer functions for the Cannon-Rosenthal problem for $m_1 = 0.5$ (solid) and $m_1 = 0.25$ (dashed)



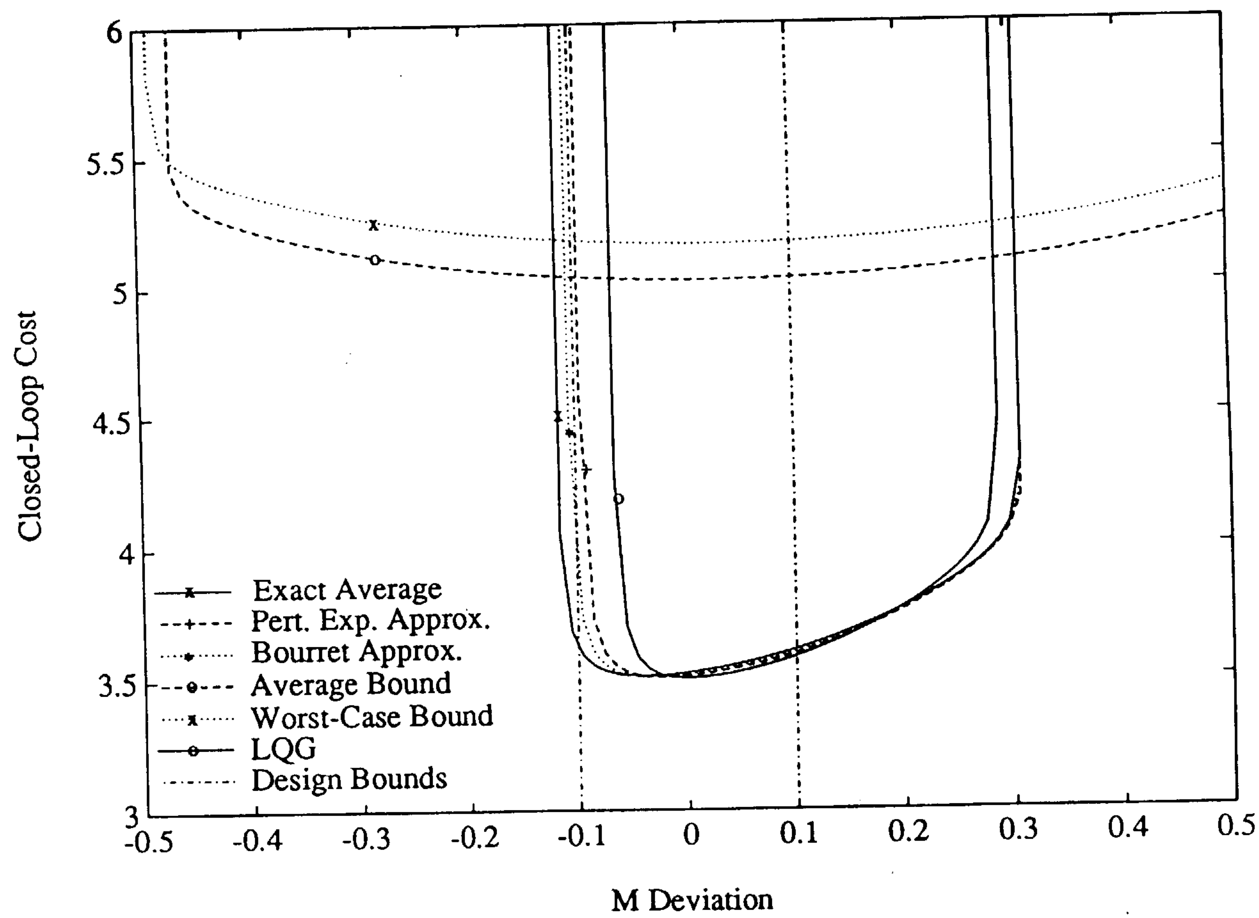
Achieved vs. Designed-for Robustness

- Achieved closed loop stability bounds as a function of the design bound, δ_m.



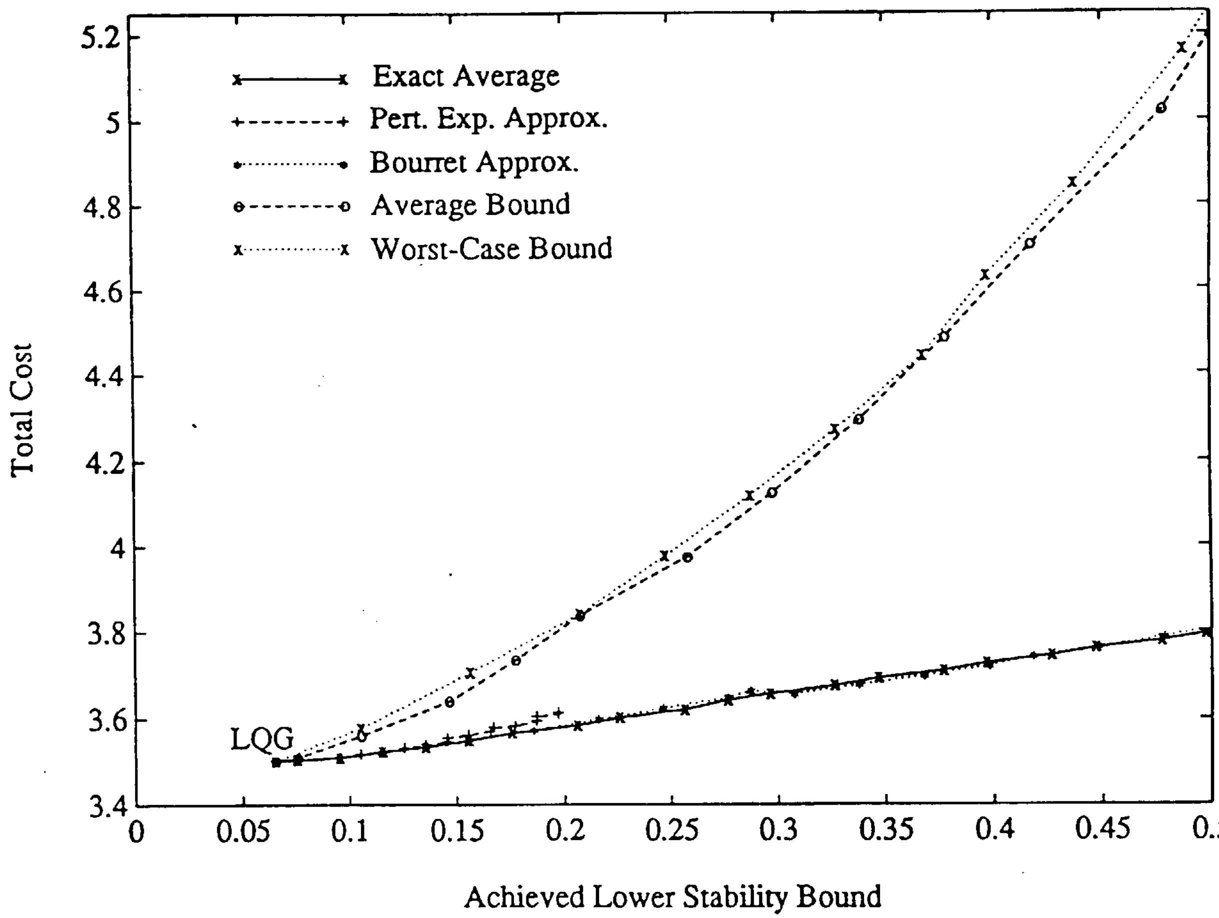
Cost vs. Parameter

- System closed-loop $\mathcal{H}_2$-norm as a function of $\tilde{m}$, the deviation about $1/m_1$, for controllers designed Using $\delta_m = 0.1$.



Efficiency Plot

- Nominal cost as a function of the achieved stability bound.



Conclusions

- Have investigated a new class of controllers based on minimizing quantities related to the cost averaged over a parameterized set of plants.
- While possessing useful properties, the average cost based controllers were difficult to compute.
- The perturbation based controllers were easy to compute but performed well only at low uncertainty levels.
- The average bound designs were essentially equivalent to the worst case bound designs.
- The worst case bound performed as predicted but gave low efficiency due to conservatism.
- The Bourret approximation based designs were overall best in computability and efficiency.