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Planar reorientation of a free-free beam in space using embedded electromechanical actuators

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ABSTRACT

It is demonstrated that the planar reorientation of a free-free beam in zero gravity space can be accomplished by periodically changing the shape of the beam using embedded electromechanical actuators. The dynamics which determine the shape of the free-free beam is assumed to be characterized by the Euler-Bernoulli equation, including material damping, with apppropriate boundary conditions. The coupling between the rigid body motion and the flexible motion is explained using the angular momentum expression which includes rotatory inertia and kinematically exact effects. A control scheme is proposed where the embedded actuators excite the flexible motion of the beam so that it rotates in the desired sense with respect to a fixed inertial reference. Relations are derived which relate the average rotation rate to the amplitudes and the frequencies of the periodic actuation signal and the properties of the beam. These reorientation maneuvers can be implemented by using feedback control.

1. INTRODUCTION

Classical models of uniform free-free flexible beams in zero-gravity space result in complete decoupling of rigid body motion and flexible motion. However, conservation of the angular momentum of the beam indicates that the classical models are incomplete in the sense that there is in fact higher order nonlinear coupling between the rigid body motion and the flexible motion, if rotatory inertia and kinematically exact modeling effects are included. Assuming that the angular momentum of the beam is always zero, oscillations in the shape of the flexible beam can actually cause a rotation of the beam with respect to a fixed inertial reference. The rotation of the beam over one period depends only on the shape of the beam over the period and does not depend on the length of the period; hence this phenomenon is referred to as a geometric phase change.

These observations lead to a scheme for carrying out asymptotic reorientation of a free-free flexible beam in space using only electromechanical actuators embedded in the beam. These embedded electromechanical actuators, e.g. piezoelectric actuators, do not change the angular momentum of the free-free beam but they can be used to change the shape of the beam in a periodic way thereby resulting in a rotation of the beam in space. This reorientation scheme, based on the use of embedded actuators, does not require use of momentum wheels or gas jets and thus requires a minimal use of fuel to achieve a given beam reorientation.

In this paper, the basic modeling issues are addressed. The dynamics which characterize the shape of the free-free beam is assumed to be characterized by the Euler-Bernoulli equation, including material damping, with apppropriate boundary conditions. The coupling between the rigid body motion and the flexible motion is explained using the angular momentum expression. A control scheme is proposed where the embedded actuators excite the flexible motion of the beam so that it rotates in the desired sense. Relations are derived which relate the average rotation rate to the amplitudes and the frequencies of the periodic actuation signal and the properties of the beam. These reorientation maneuvers can be implemented by using feedback control. Important features of the approach are indicated.

2. A PLANAR FREE-FREE BEAM MODEL

Consider a uniform free–free beam of undeformed length $2L$ in space with zero angular momentum and zero linear momentum. Referring to Fig. 1 the motion of the beam is constrained to a plane defined by vectors $(\bar{e}_1, \bar{e}_3)$ where $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ is an orthonormal basis for an inertial frame whose origin is at the center of mass of the beam. Let $(\bar{i}, \bar{j}, \bar{k})$ be a rotating frame with its origin fixed at the origin of the inertial frame such that the vectors $(\bar{i}, \bar{k})$ lie in the plane $(\bar{e}_1, \bar{e}_3)$ and $\bar{j} = \bar{e}_2$. The straight line passing through the origin in the direction of vector $\bar{k}$ is called the reference line. Let the beam initially be at rest in a straight line configuration aligned with the reference line. Then, the location of each point on the line of mass centroids of the beam can be described in terms of the parameter $s \in [-L, L]$. This parameter s can be viewed as a label for each of the crossections. We assume that as the beam deforms the shape and the area of the crossections remain invariant. Following other researchers[1,2,3] we introduce three functions $u(s,t), y(s,t) : [-L, L] \times \Re \to \Re$ and $\psi(s,t) : [-L, L] \times \Re \to T^1$ such that $(u(s,t) + s, y(s,t))$ define the coordinates of the line of centroids in the deformed configuration with respect to the moving frame $(\bar{i}, \bar{k})$ at time t. The angle $\psi(s,t)$ between the normal to the crossection at s and $\bar{e}_3$ specifies the orientation of the crossection. The normal to the crossection at s is denoted by $\bar{t}_3$. We define the material basis $(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ to be orthonormal so that $\bar{t}_1$ lies in the plane $(\bar{e}_1, \bar{e}_3)$. The crossection itself can be associated with the set of points (ξ_1, ξ_2) in a compact set $A \subset \Re^2$ such that $\xi_1 \bar{t}_1 + \xi_2 \bar{t}_2 + (u(s,t) + s)\bar{k} + (y(s,t))\bar{i}$ gives the location of any point on the beam as ξ_1 and ξ_2 vary through A and s varies from $-L$ to L.

Since the origin of the inertial frame is fixed at the center of mass of the beam we obtain

$$\int_{-L}^{L} y(s,t)ds = 0, \tag{1}$$

$$\int_{-L}^{L} u(s,t)ds = 0. \tag{2}$$

Let ρ denote the constant mass density per unit volume of the beam. We assume that the beam has a symmetric crossection so that the first moment of inertia of the crossection about the line of centroids is

$$\int_A \rho \xi_1 d\xi_1 d\xi_2 = 0. \tag{3}$$

The second moment of inertia of the crossection about the line of centroids is

$$I_2 = \int_A \rho {\xi_1}^2 d\xi_1 d\xi_2. \tag{4}$$

The mass per unit length of the crossection is given by

$$m_0 = \int_A \rho d\xi_1 d\xi_2. \tag{5}$$

We define the angle $\theta(t)$ between $\bar{e}_3$ and $\bar{k}$ so that $y(s,t)$ measured from the reference line satisfies the following orthogonality condition

$$\int_{-L}^{L} sy(s,t)ds = 0. \tag{6}$$

The existence of the angle $\theta(t)$ follows from the geometry indicated in Fig. 1. This definition provides a separation between the motion which determines the shape of the beam, given by $y(s,t)$, $-L \leq s \leq L$, and the rotation of the beam as a whole, given by $\theta(t)$.

3. EQUATIONS OF MOTION

We first develop a kinematically exact expression for the angular momentum of the free-free beam. Let $\bar{\varphi}(s,\xi_1,\xi_2,\theta,t)$ be the vector from the origin of the inertial frame to a point (s,ξ_1,ξ_2) on the beam at time t; then

$$\bar{\varphi} = (s\sin\theta + y\cos\theta + \xi_1\cos\psi + u\sin\theta)\bar{e}_1 + (\xi_2)\bar{e}_2 + (s\cos\theta - \xi_1\sin\psi - y\sin\theta + u\cos\theta)\bar{e}_3 \tag{7}$$

where $\theta = \theta(t), y = y(s,t)$ and $\psi = \psi(s,t)$. The angular momentum about the origin of the inertial frame at time t is zero so that

$$\int_{-L}^{L}\int_{A}\rho\bar{\varphi}\times\frac{d\bar{\varphi}}{dt}d\xi_1 d\xi_2 ds = 0. \tag{8}$$

Substituting equation (7) into equation (8) and using equations (4) and (5) we can express $\dot{\theta}$ in terms of y, u and α as

$$\dot{\theta} = \frac{\int_{-L}^{L}\{m_0 s\frac{\partial y}{\partial t} + I_2\dot{\alpha} + m_0(\frac{\partial y}{\partial t}u - \frac{\partial u}{\partial t}y)\}ds}{\int_{-L}^{L}\{-m_0 s^2 - m_0 y^2 - I_2\}ds} \tag{9}$$

where $\alpha = \psi - \theta$ is the angle between the normal $\bar{t}_3$ to the crossection at s and the reference line.

Assume that the beam is unshearable and inextensible and that the deformations are small. This implies, using equation (2), that

$$u(s,t) = 0. \tag{10}$$

and that

$$\alpha \approx y_s. \tag{11}$$

We use the Euler–Bernoulli beam model to characterize the shape of the beam.[4,5] Thus $y(s,t)$ satisfies the Euler-Bernoulli equation of the form

$$m_0 y_{tt} + \gamma y_{tssss} + EI y_{ssss} = -\sum_{j=1}^{m} v_j(t)\delta'(s-s_j) \tag{12}$$

with the boundary conditions

$$y_{ss}(-L) = y_{ss}(L) = 0, \tag{13}$$

$$y_{sss}(-L) = y_{sss}(L) = 0 \tag{14}$$

where $I = I_2/\rho$, E is Young's elasticity modulus, δ' is the distributional derivative of the delta function and where for simplicity we assume Kelvin-Voigt damping with a positive damping coefficient γ. In addition, $y(s,t)$ must satisfy conditions (1) and (6). Internal bending torques $v_j(t)$, $j = 1,\ldots,m$ are produced by m point actuators located at $s = s_j$ on the beam where $s_j \in [-L, L]$. These embedded electromechanical actuators change the shape of the beam but at the same time preserve the angular momentum. Although such actuators are capable of inducing relatively small displacements one can excite the beam periodically at a frequency near one of the lower resonant frequencies of the beam to obtain relatively large periodic shape change.

Using expressions (6), (10) and (11) in equation (9) we obtain

$$\dot{\theta} = \frac{-\int_{-L}^{L} I_2 y_{ts} ds}{\tau + \int_{-L}^{L} m_0 y^2 ds} \tag{15}$$

where $\tau = \frac{2}{3}m_0 L^3 + 2I_2 L$. This expression demonstrates the nonlinear coupling between the beam's shape and its rigid body motion. Expression (15) is non-integrable in the sense that if $y(s,t)$ is a periodic function of time, the integral of $\dot{\theta}$ over one period is, in general, non-zero.

Remark 3.1 If in the above derivation we had not used the kinematically exact expression for the angular momentum but had used the linearized strain assumptions we would have obtained the expression

$$\dot{\theta} = -\frac{1}{\tau}\int_{-L}^{L} I_2 y_{st} ds. \tag{16}$$

As can be seen expression (16) leads to the incorrect conclusion that a periodic change in the shape of the beam does not result in rotation of the beam. Note that inclusion of rotatory inertia effects and the use of the kinematically exact expression for the angular momentum is necessary in order to demonstrate that the beam can rotate in space due to periodic shape change.

We expand the solution $y(s,t)$ to equation (12) in the series

$$y(s,t) = \sum_{i=1}^{\infty} w_i(s) q_i(t) \tag{17}$$

where $w_i(s), i = 1, 2, \ldots$ are the orthonormal elastic mode shapes of the Euler-Bernoulli model. The elastic mode shapes are given by

$$w_i(s) = \begin{cases} \cos(\beta_i s) - \frac{\sin(\beta_i L)}{\sinh(\beta_i L)}\cosh(\beta_i s) & \text{if } i = 1,3,5\ldots \\ \sin(\beta_i s) + \frac{\cos(\beta_i L)}{\cosh(\beta_i L)}\sinh(\beta_i s) & \text{if } i = 2,4,6\ldots \end{cases}$$

where β_i are the positive roots of the equation

$$\cos(2\beta L)\cosh(2\beta L) = 1$$

ordered according to their magnitude.

Expansion (17) provides the modal decomposition

$$\ddot{q}_i + c_i\dot{q} + {\omega_i}^2 q = \sum_{j=1}^{m} b_{ij} v_j(t), i = 1, 2\ldots \tag{18}$$

where ${\omega_i}^2 = \frac{EI\beta_i^4}{m_0}$, $c_i = \frac{\gamma {\omega_i}^2}{EI}$ and $b_{ij} = \frac{\partial w_i}{\partial s}|_{s=s_j}$. Equation (12), or equivalently equation (18), determines the shape of the beam and is called the shape space equation. Substituting equation (17) into equation (15) we obtain

$$\dot{\theta} = \frac{-I_2\sum_{i=1}^{\infty}(J_i\dot{q}_i)}{\tau + \sum_{i=1}^{\infty} {q_i}^2} \tag{19}$$

where $J_i = w_i(L) - w_i(-L)$. We note that (19) is non–integrable for any truncation of the infinite series in (17).

4. ASYMPTOTIC REORIENTATION MANEUVERS

The goal is to accomplish asymptotic maneuvers, i.e. starting with $\theta(t_0) = \theta_0$, $y(s,t_0) = y_t(s,t_0) = 0$ we want to rotate the beam so that $\theta(t) \to \theta_d$, $y(s,t) \to 0$ and $y_t(s,t) \to 0$ as $t \to \infty$ for some desired angle θ_d.

Consider the periodic excitation of the beam at a single frequency ω as

$$v_j(t) = v_j^0 + v_j^\omega \cos(\omega t), j = 1, 2, \ldots, m \tag{20}$$

Since the shape space dynamics of the free–free beam is asymptotically stable, the steady-state motion of the beam is given by

$$q_i(t) = l_i + a_i \cos(\omega t + \phi_i) \tag{21}$$

where

$$l_i = \frac{1}{{\omega_i}^2} \sum_{j=1}^{m} b_{ij} v_j^0, \tag{22}$$

$$a_i = \frac{1}{\sqrt{(\omega_i - \omega^2)^2 + {c_i}^2 {\omega_i}^2}} \sum_{j=1}^{m} b_{ij} v_j^\omega, \tag{23}$$

and

$$\phi_i = -arctg\left(\frac{c_i \omega_i}{\omega_i^2 - \omega^2}\right). \tag{24}$$

The excitation function (20) should be sufficiently small so that the Euler–Bernoulli model for the shape space dynamics remains valid[6].

If $\sum_{i=1}^{\infty} q_i^2$ is small comparing with τ we can approximate

$$\frac{1}{\tau + \sum_{i=1}^{\infty} q_i^2} \approx \frac{1}{\tau}\left(1 - \frac{\sum_{i=1}^{\infty} q_i^2}{\tau}\right)$$

and thus

$$\dot{\theta} \approx -\frac{1}{\tau} \sum_{i=1}^{\infty} J_i \dot{q}_i + \frac{1}{\tau^2} \left[\sum_{i=1}^{\infty} J_i \dot{q}_i\right] \left[\sum_{j=1}^{\infty} {q_j}^2\right].$$

Integrating over one period and using equation (21) we obtain

$$\begin{aligned}
\theta(\frac{2\pi}{\omega}) - \theta(0) &= \int_0^{\frac{2\pi}{\omega}} \frac{1}{\tau^2} \left[\sum_{i=1}^{\infty} J_i \dot{q}_i\right] \left[\sum_{j=1}^{\infty} {q_j}^2\right] dt \\
&= \int_0^{\frac{2\pi}{\omega}} \frac{1}{\tau^2} \left[\sum_{i=1}^{\infty} -a_i J_i \omega \sin(\omega t + \phi_i)\right] \left[\sum_{j=1}^{\infty} (l_j + a_j \cos(\omega t + \phi_j))^2\right] dt \\
&= \frac{2\pi}{\tau^2} \sum_{i=1}^{\infty} \sum_{j=1, j\neq i}^{\infty} a_i J_i l_j a_j \sin(\phi_j - \phi_i).
\end{aligned} \tag{25}$$

Expression (25) implies that, in general, the change in angle θ in steady–state over one period is non-zero thereby proving that a periodic change in shape of the beam results in a rotation of the beam; the steady-state difference $\theta(\frac{2\pi}{\omega}) - \theta(0)$ is referred to as the geometric phase. There are cases, however, when the geometric phase turns out to be zero.

Proposition 4.1 Assume that the steady–state motion of the beam is described by equation (21). Then, $\theta(\frac{2\pi}{\omega}) - \theta(0) = 0$ if any of the following conditions hold:

1. $a_i = 0$ for all i
2. $l_i = 0$ for all i
3. $\phi_i = \phi_j$ for all i, j

The second statement of the proposition is the most important. It implies that for a non-zero geometric phase the beam should necessarily vibrate about a non-straight line reference configuration. It follows from expression (19) that following the motion $\tilde{q}_i(t) = -l_i - a_i \cos(\omega t + \phi_i)$ yields a steady–state geometric phase change negative to that of (21). Therefore, in order to rotate the beam in the opposite direction it is sufficient to reverse the signs of v_j^ω and v_j^0.

Remark 4.1 Expression (25) can be used in order to predict the sign and the value of the geometric phase. Consider a beam which has a square crossection with side size R. Assume that two actuators at $s_1 = -rL$ and $s_2 = rL$ where $0 \leq r \leq 1$ produce torques according to equation (20). Using two first modes in the series (25) yields

$$\theta(\frac{2\pi}{\omega}) - \theta(0) \approx \frac{81.5661((v_1^\omega)^2 - (v_2^\omega)^2)(v_1^0 - v_2^0)\Phi(r)\sin(\phi_2 - \phi_1)}{\rho^2 L^3 E R^7 (\frac{2}{3}(\frac{L}{R})^2 + \frac{1}{6})^2 \sqrt{(\omega_1^2 - \omega^2)^2 + c_1^2\omega^2} \sqrt{(\omega_2^2 - \omega^2)^2 + c_2^2\omega^2}} \qquad (26)$$

where

$$\Phi(r) = (\sin(2.36502r) + 0.1329\sinh(2.36502r))^2(\cos(3.9266r) - 0.0279\cosh(3.9266r)).$$

We are now in a position to formulate a specific control strategy to accomplish the asymptotic maneuver. Starting at rest with $\theta(t_0) = \theta_0$ application of control law (20) results in a nonzero rotation over a period. By repetition of cycles of motion (21) as many times as necessary the beam can be caused to rotate closer and closer to θ_d. As $\theta(t)$ approach θ_d we can reduce the amplitude of the oscillations to zero in a way so that $\theta(t) \rightarrow \theta_d$ as $t \rightarrow \infty$.

The proposed control law is of the form

$$v_j(t) = \varepsilon_k \left[\bar{v}_j^0 + \bar{v}_j^\omega \cos(\omega t)\right], j = 1, \ldots, m, \qquad (27)$$

where $\frac{2(k-1)\pi p}{\omega} \leq t - t_0 < \frac{2k\pi p}{\omega}$, $k = 1, 2, \ldots$; that is, the control excitation is an amplitude modulated function, where $\bar{v}_j^0$, $\bar{v}_j^\omega$ are constants and ε_k denotes the scalar amplitude modulation sequence that defines the control excitation on the k-th cycle. Each cycle is exactly p periods.

The constants ω, $\bar{v}_j^0$, $\bar{v}_j^\omega$ can be chosen nearly arbitrary, although one approach is to choose $\bar{v}_j^0$, $\bar{v}_j^\omega$ to maximize the geometric phase expression

$$\sum_{i=1}^{\infty} \sum_{j=1, j\neq i}^{\infty} a_i J_i l_j a_j \sin(\phi_j - \phi_i)$$

Since $|\varepsilon_k| \to 0$ then $q_i(t) \to 0$ and $\dot{q}_i \to 0$ as $t \to \infty$. By continuity $\theta(t) \to \theta^{con}$ for some constant θ^{con} as $t \to \infty$. We want to show that $\theta^{con} = \theta_d$.

By contradiction, assume that $\theta^{con} > \theta_d$. Let $\delta_3 > 0$ be sufficiently small so that $\theta^{con} - \delta_3 > \theta_d$. Choose ξ_3 so that

$$\left(\frac{\theta^{con} - \theta_d - \delta_3}{\Delta\theta^*}\right) > \xi_3^3 > 0.$$

Then, there exists an integer N_3 such that for any $k > N_3$ it follows that $|\varepsilon_k| < \xi_3$ and $|\theta_k^{ave} - \theta^{con}| < \delta_3$. Note that for any $k > N_3 + 1$ and $l > N_3 + 1$

$$r_k = \left(\frac{\theta_d - \theta_{k-1}^{ave}}{\Delta\theta^*}\right)^{\frac{1}{3}} < 0$$

and

$$|r_k| \geq \left(\frac{\theta^{con} - \delta_3 - \theta_d}{\Delta\theta^*}\right)^{\frac{1}{3}} > \xi_3 > |\varepsilon_l|.$$

Thus, we conclude from **(A2)** that for any $k, l > N_3 + 1$ it follows that $\varepsilon_k = \varepsilon_l \neq 0$. Hence, we obtain a contradiction to the convergence of the sequence ε_k to zero as $k \to \infty$. Similar arguments lead to a contradiction in case $\theta^{con} < \theta_d$. □

Finally, it follows from equations (28) and (23) that

$$\lim_{t\to\infty} \theta(t) = \theta_d, \lim_{t\to\infty} \begin{pmatrix} y(s,t) \\ y_t(s,t) \end{pmatrix} = 0, -L \leq s \leq L$$

The controller which we have constructed has two functions. Its main function is to excite the oscillations of the beam in such a way so that the beam rotates in the desired sense. Subsequently, the controller serves to suppress the vibrations previously excited so that the free–free beam comes to rest with a desired orientation. Note that control law (27) is a non–smooth feedback control law.

5. NUMERICAL EXAMPLE

Space structures can often be modeled as light and flexible beams. Consider a beam with half-length $L = 1[m]$, density per unit volume $\rho = 1400[kg/m^3]$ and square crossection with the side size $R = 0.1[m]$. Young's modulus of the beam is $E = 3.0 \times 10^6[N/m^2]$ and the damping coefficient of Kelvin-Voigt damping is $\gamma = 0.2$. Two actuators are installed near both ends of the beam at $r = 0.9$. The maximal torque each of the actuators can produce is equal $100[Nm]$. The excitation frequency $\omega = 13[Hz]$ is selected to lie between the first $10.6[Hz]$ and the second $29[Hz]$ resonant frequencies of the beam; $\bar{v}_j^0$ and $\bar{v}_j^w$, $j = 1, 2$ are chosen using expression (26) to maximize the geometric phase change over one period. For this example we choose $p = 5$ and $\gamma_1 = \gamma_2 = 0.9$. We want to rotate the beam from $\theta_0 = 0.1[rad]$ at $t = 0$ $[sec]$ to $\theta_d = 0[rad]$. The dependence of the angle $\theta(t)$ $[rad]$ on time $t[sec]$ is shown for a part of the maneuver in Fig. 2. In this case the geometric phase change over one period in steady–state predicted by expression (26) is equal to -2.7465×10^{-4} $[rad]$ whereas its actual value is equal to -3.0411×10^{-4} $[rad]$. The dependence of the control parameter ε on time is shown in Figure 3.

where $a_i, l_i, \phi_i, i = 1, \ldots,$ are related to $\bar{v}_j^0, \bar{v}_j^\omega, j = 1, \ldots, m$ according to expressions (22)–(24), and $\bar{v}_j^0, \bar{v}_j^\omega$ are constrained by

$$\sum_{j=1}^{m} (\bar{v}_j^0)^2 \leq \alpha_c, \sum_{j=1}^{m} (\bar{v}_j^\omega)^2 \leq \beta_c.$$

In terms of $\bar{v}_j^0, \bar{v}_j^\omega, j = 1, \ldots, m$ this is a constrained mathematical programming problem which is linear in $\bar{v}_j^0$ (for fixed $\bar{v}_j^\omega$) and quadratic in $\bar{v}_j^\omega$ (for fixed $\bar{v}_j^0$). We will subsequently denote the maximum value of this constrained optimization problem as $\Delta\theta^*$.

The modulation sequence ε_{k+1} is defined in terms of an "average" of $\theta(t)$, over the k-th cycle, that is

$$\theta_k^{ave} = \frac{1}{2}(\max\theta(t) + \min\theta(t)) \quad (28)$$

where the maximum and minimum are over $\frac{2(k-1)\pi p}{\omega} \leq t - t_0 \leq \frac{2k\pi p}{\omega}$. We also introduce two auxilary variables $\theta_0^{ave} = \theta_0$ and $\varepsilon_0 = \text{sign}\left(\frac{\theta_d - \theta_0}{\Delta\theta^*}\right)$. We express ε_k in terms of θ_{k-1}^{ave} and ε_{k-1} as indicated below:

(A1) Compute

$$r_k = \left(\frac{\theta_d - \theta_{k-1}^{ave}}{\Delta\theta^*}\right)^{\frac{1}{3}}.$$

(A2) In case $|r_k| \geq |\varepsilon_{k-1}|$, if r_k and ε_{k-1} have the same signs then $\varepsilon_k = |\varepsilon_{k-1}|\text{sign}(r_k)$; if r_k and ε_{k-1} have opposite signs then $\varepsilon_k = \gamma_1|\varepsilon_{k-1}|\text{sign}(r_k)$, where $0 < \gamma_1 < 1$.

(A3) If $0 < |r_k| < |\varepsilon_{k-1}|$ then $\varepsilon_k = \gamma_2 r_k$, where $0 < \gamma_2 < 1$.

(A4) If $r_k = 0$ then $\varepsilon_k = \varepsilon_{k-1}$.

Proposition 4.2 If the proposed control law is of the form (27) where ε_k is selected according to steps **(A1)–(A4)**, then

$$\lim_{k\to\infty} \theta_k^{ave} = \theta_d, \lim_{k\to\infty} \varepsilon_k = 0.$$

Sketch of the Proof. The sequence $|\varepsilon_k|$ is non-increasing and bounded on $[0, 1]$. Therefore, there exists $b \in [0, 1]$ such that $b = \inf_k |\varepsilon_k|$. We want to show that $b = 0$.

By contradiction, assume $b \neq 0$. Then, for $\xi = \frac{b(1-\max(\gamma_1,\gamma_2))}{2\max(\gamma_1,\gamma_2)}$ we can find an integer N_1 such that for all $k > N_1$ $|\varepsilon_k| - b < \xi$. From **(A2)** and **(A3)** we conclude that only two cases are possible: $\varepsilon_k = b$ for all $k > N_1$ or $\varepsilon_k = -b$ for all $k > N_1$.

Assume that the former case is true. Since the transient decays to zero and using continuity of θ with respect to q_i and $\dot{q}_i$ we assert that for $\xi_1 = \frac{1}{2}b^3\Delta\theta^*$ there exists an integer N_2 such that for any $k > N_2$

$$b^3\Delta\theta^* - \xi_1 < \theta_{k+1}^{ave} - \theta_k^{ave} < b^3\Delta\theta^* + \xi_1,$$

where $\Delta\theta^* > 0$. Note that $\frac{\theta_d - \theta_{N_2}^{ave}}{\Delta\theta^*} > b^3 > 0$. Choosing an integer l so that $l > 2\frac{\theta_d - \theta_{N_2}^{ave}}{b^3\Delta\theta^*} + 1$ we conclude that

$$0 > \frac{-(b^3\Delta\theta^* - \xi_1)l + \theta_d - \theta_{N_2}^{ave}}{\Delta\theta^*} > \frac{\theta_d - \theta_{N_2+l}^{ave}}{\Delta\theta^*} > b^3 > 0.$$

Therefore, the former case can never occur. Similarly, we can verify that the latter case also leads to a contradiction. Hence, $b = 0$.

6. CONCLUSION

In this paper the angular momentum expression for a planar free–free beam in space is derived. It is shown how the general motion of the beam can be separated into rigid and elastic motions. The change of shape of the beam is described by the Euler–Bernoulli equation with free-free boundary conditions. Angular momentum conservation leads to the nonlinear dependence of the rigid motion on the shape of the beam. As shown this dependence is non–integrable in the sense that a periodic change in shape of the beam results in a non-zero rotation of the beam over one period. Approximate relationships expressing the average rate of rotation of the beam in terms of the amplitudes and phases of periodic excitation of the beam by internal actuators are derived. Finally, a control strategy for a planar asymptotic reorientation maneuver is developed.

A general treatment of the interplay between deformations and rotations of deformable bodies is given by Shapere and Wilczek.[7] Reyhanoglu and McClamroch[8] have developed a framework for reorientation of multibody systems in space. In this paper, we have used the framework developed by Shapere and Wilczek for the specific problem of reorientation of a free-free beam in space; our results represent, in a certain sense, the limiting case of the multibody results obtained by Reyhanoglu and McClamroch when the number of bodies increases without limit.

Although our study in this paper has been concerned with the ideal case of reorientation of a free–free beam in space, we note that the same ideas are applicable to reorientation of a wide class of deformable space structures, using only actuators embedded into the structure. In this sense, smart structures technology can be used to accomplish a variety of efficient reorientation maneuvers for space structures.

7. ACKNOWLEDGMENTS

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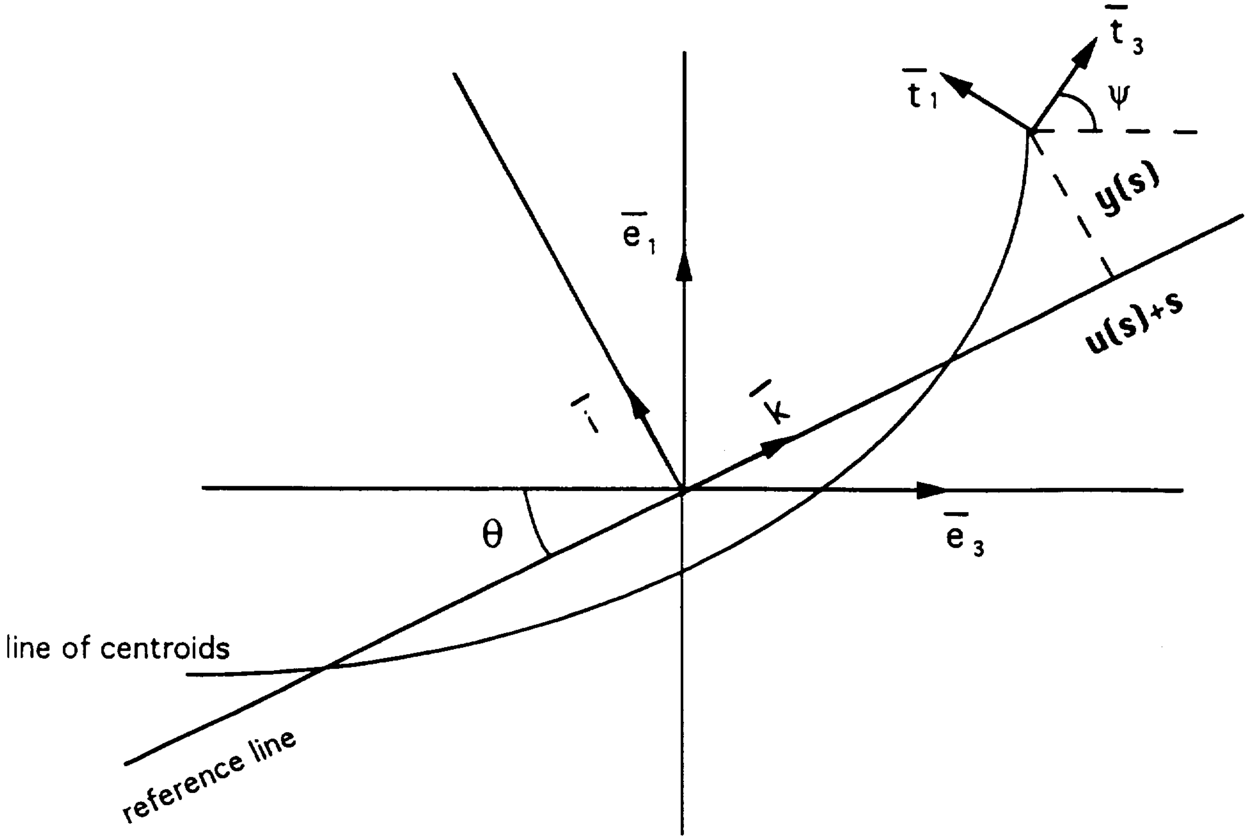
Fig 1. Inertial, Moving and Material coordinate frames.



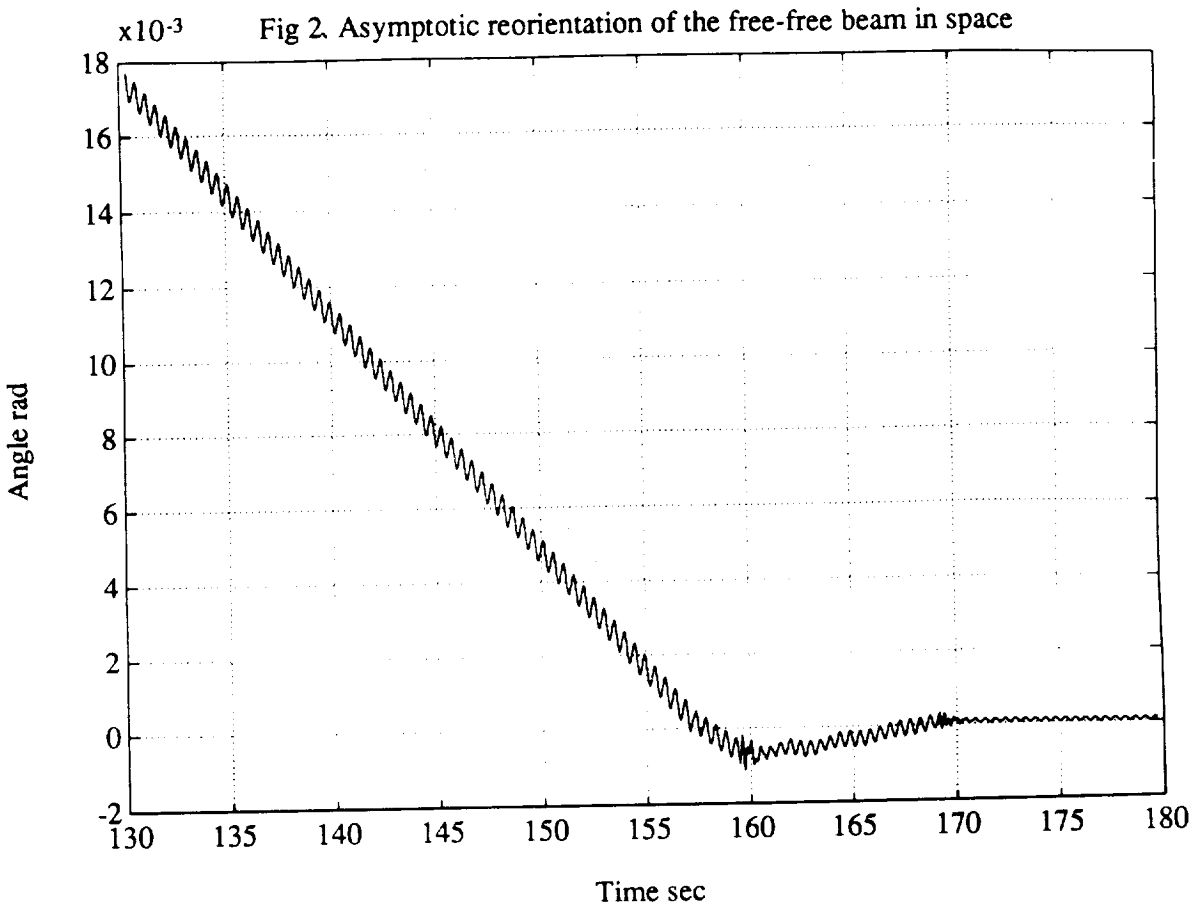

x10-3
Fig 2. Asymptotic reorientation of the free-free beam in space
18
16
14
12
10
8
6
4
2
0
-2
130
135
140
145
150
155
160
165
170
175
180
Angle rad
Time sec

Fig 3. Amplitude modulation sequence